# The parity of a thicket*

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**Abstract**

A thicket in a graph $G$ is defined as a set of even circuits such that every edge lies in an even number of them. If $G$ is directed, then each circuit in the thicket has a well defined directed parity. The parity of the thicket is the sum of the parities of its members, and is independent of the orientation of $G$. We study the problem of determining the parity of a thicket $\mathcal{T}$ in terms of structural properties of $\mathcal{T}$. Specifically, we reduce the problem to studying the case where the underlying graph $G$ is cubic. In this case we solve the problem if $|\mathcal{T}| = 3$ or $G$ is bipartite. Some applications to the problem of characterising Pfaffian graphs are also considered.

## 1 Introduction

In [2] necessary and sufficient conditions are found for the possibility of orienting a graph so that each even circuit has a specified directed parity. (The *directed parity* of a circuit of even length is that of the number of edges directed in agreement with a specified sense. This parity is clearly well defined.) The motivation was to shed light on the problem of characterising Pfaffian graphs. A graph is said to be *Pfaffian* if it can be oriented so that every alternating circuit (circuit which is the symmetric difference of two 1-factors) is of odd directed parity. Pfaffian graphs are of interest because their 1-factors (perfect matchings) can be easily enumerated. (See [3].) However the result in [2] cannot be applied directly to the Pfaffian problem because it requires that a directed parity be specified for every even circuit, not just for alternating circuits. In this paper we consider near bricks and study the problem of specifying a directed parity for an even circuit that is not alternating. (A matching covered non-bipartite graph is a *near brick* if its alternating space is equal to its even space. The vector spaces here are over $\mathbb{Z}_2$, the vectors are circuits

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considered as sets of edges and the operation of addition is symmetric difference. The alternating space is that spanned by the alternating circuits. Similarly the even space is spanned by the even circuits.)

Our study leads to the concept of a thicket. A *thicket* in a graph $G$ is a collection $\mathcal{T}$ of even circuits with empty sum. Each circuit in the thicket is therefore the sum of all the others. If $G$ is a directed graph, then each circuit in $\mathcal{T}$, being of even length, has a well defined directed parity. Indeed, any circuit $C$ of $\mathcal{T}$ may be assigned a sense and the directed parity of $C$ is then measured as the parity of the number of edges in $C$ that are oriented in agreement with the specified sense. As $|C|$ is even, this parity does not change if the sense assigned to $C$ is reversed. The *parity* of $\mathcal{T}$ is the sum (modulo 2) of the parities of its members. A change in the orientation of an edge $e$ of $G$ induces a change in the directed parity of any member of $\mathcal{T}$ that contains $e$. As there are an even number of such members of $\mathcal{T}$, the parity of $\mathcal{T}$ itself remains unchanged. It is therefore independent of the orientation of $G$. It is an inherent property of $\mathcal{T}$. We study the problem of relating the parity of a given thicket to other aspects of its structure.

For each edge $e$ of $G$ we denote by $C(e)$ the set of circuits of $\mathcal{T}$ that contain $e$. We say that $e$ *carries* each circuit in $C(e)$. Clearly we may assume that $G$ is induced by $\cup\mathcal{T}$, as any edge $e$ for which $C(e) = \emptyset$ is irrelevant. We say that $G$ is *induced* by $\mathcal{T}$. For each $v \in VG$, let $C(v)$ be the set of circuits in $\mathcal{T}$ that pass through $v$. Each contains just two edges incident on $v$.

We begin by investigating the contribution of each edge of $G$ to the parity of $\mathcal{T}$. Give $G$ an arbitrary orientation and assign to each circuit of $\mathcal{T}$ an arbitrary sense. The *parity* of an edge $e$ is defined to be that of the number of circuits in $C(e)$ whose sense agrees with the orientation of $e$. As $|C(e)|$ is even, this parity is independent of the orientation of $e$. The sum of the parities of the edges of $G$ is therefore independent of the orientation of $G$. As each circuit in $\mathcal{T}$ is of even length, it is also independent of the senses assigned to the members of $\mathcal{T}$. In fact it is equal to the parity of $\mathcal{T}$. Thus the parity of $\mathcal{T}$ is that of the number of odd edges of $G$. Accordingly in what follows we study the parity of the number of odd edges of $G$.

For each edge $e \in EG$ let $|C(e)| = 2n_e$. Let $G^*$ be the graph obtained from $G$ by replacing each edge $e$ with $n_e$ parallel edges having the same pair of ends as $e$. Then $G^*$ is induced by a thicket $\mathcal{T}^*$ for which each edge of $G^*$ carries just two members. This thicket is not necessarily well defined, but each possible choice of $\mathcal{T}^*$ has the same parity as $\mathcal{T}$. We may therefore assume that each edge of $G$ carries just two members of $\mathcal{T}$. In this case we refer to $\mathcal{T}$ as a *basic* thicket. It is in fact simply a circuit double cover in which the length of every circuit is even.

In Section 2 we show how to reduce the problem to the case of basic thickets in cubic graphs. We also solve the problem for basic thickets of cardinality 3 and for basic thickets in bicubic graphs. We conclude in Section 3 by indicating the potential application to the study of the Pfaffian property of near bricks.

## 2 Basic thickets

We begin our study of basic thickets by showing how to reduce the problem to the case of a cubic graph.

### 2.1 Reduction to the cubic case

First we show that the circuits of a basic thicket $\mathcal{T}$ may be assumed to be the boundaries of the faces of an embedding of $G$. Let $VG = \{v_1, v_2, \ldots, v_m\}$. For each $i$ let $H(v_i)$ be a graph whose vertices are the edges of $G$ incident on $v_i$, and let two vertices of $H(v_i)$ be joined by as many edges as the number of circuits of $\mathcal{T}$ that contain them both. (This number, of course, must be 0, 1 or 2.) Since each edge of $G$ carries just two circuits of $\mathcal{T}$, the degree of every vertex of $H(v_i)$ is 2. Therefore $H(v_i)$ is induced by a set of vertex-disjoint circuits. Let $H$ be the graph such that $VH = EG$ and any two vertices of $H$ that are adjacent edges of $G$ are joined by as many edges as the number of circuits of $\mathcal{T}$ that contain them both. This graph is also induced by a unique set of circuits, namely $\cup_{i=1}^{m} \mathcal{C}_i$ where $\mathcal{C}_i$, for each $i$, is the set of circuits in $H(v_i)$.

Now let $G'$ be a graph whose edges are the vertices of $H$ (in other words, the edges of $G$) and whose vertices are the circuits of $H$. Since the circuits of each $H(v_i)$ are vertex-disjoint, it follows that each vertex $e$ of $H$ lies on only two circuits; let these be the vertices joined by $e$ in $G'$. Thus $EG' = EG$ and $\mathcal{T}$ is also a thicket in $G'$. Each edge of $G'$ therefore has the same parity as in $G$. Consequently the parity of $\mathcal{T}$ in $G'$ is the same as its parity in $G$. Moreover, since each edge of $G'$ carries just two circuits of $\mathcal{T}$ and the vertices of $G'$ are circuits of $H$, we find that $\mathcal{T}$ is the set of face boundaries of an embedding of $G'$. Therefore we may assume that $\mathcal{T}$ is the set of face boundaries of an embedding of $G$, as claimed.

If $G$ is a graph, then we denote by $N(G)$ the number of vertices whose degree is 2 or a multiple of 4. If $\mathcal{T}$ is a thicket, then we define $P(\mathcal{T}) = 0$ if $\mathcal{T}$ is even and $P(\mathcal{T}) = 1$ otherwise.

**Theorem 2.1.** *Let $G$ be a graph induced by a basic thicket $\mathcal{T}$. Then $\mathcal{T}$ may be reduced to a thicket $\mathcal{T}'$ in a cubic graph such that*

$$P(\mathcal{T}') \equiv P(\mathcal{T}) + N(G) \pmod{2}.$$

*Proof.* As $EG$ is a union of circuits, the degree of each vertex is at least 2. Let $v$ be a vertex of degree 2 adjacent to distinct vertices $u$ and $w$. Then the two edges $a$ and $b$ incident on $v$ are of equal parity since they belong to the same two circuits $A$ and $B$ of the thicket $\mathcal{T}$. Without loss of generality we may assume that $a$ joins $v$ to $u$, so that $b$ joins $v$ to $w$. We may also assume that $A$ and $B$ have been assigned senses so that the sense of $A$ on $a$ is directed from $u$ and the sense of $B$ on $a$ is directed from $v$; then $a$ and $b$ are both odd edges. Let $G_v$ be the graph obtained from $G - \{v\}$ by replacing $v$ and its incident edges with vertices $x, y, z$ and edges $a', b', c, d, e, f$, where $a'$ joins $x$ to $u$, $b'$ joins $z$ to $w$, $c$ and $d$ both join $x$ to $y$ and $e$ and $f$ both join $y$ to $z$. (See Figure 1.) Note that

$x$ and $z$ are of degree 3 but the degree of $y$ is 4. Define

$$A' = (A - \{a, b\}) \cup \{a', c, e, b'\} = A + \{a, b, a', c, e, b'\},$$
$$B' = B + \{a, b, a', d, f, b'\},$$
$$C = \{c, d\},$$
$$D = \{e, f\}.$$

These circuits are of even length. Let

$$\mathcal{T}_v = \mathcal{T} + \{A, B, A', B', C, D\}.$$

Then $\mathcal{T}_v$ is a thicket in $G_v$ in which every edge carries just two circuits. Assign senses to $A'$ and $B'$ so that the sense of $A'$ on $a'$ is directed from $u$ and the sense of $B'$ on $a'$ is directed from $x$. Assign senses to $C$ and $D$ arbitrarily. Then the number of odd edges in the set $\{a', b', c, d, e, f\}$ is even. Hence $\mathcal{T}_v$ is of the same parity as $\mathcal{T}$. We may therefore assume that the degree of every vertex is at least 3.

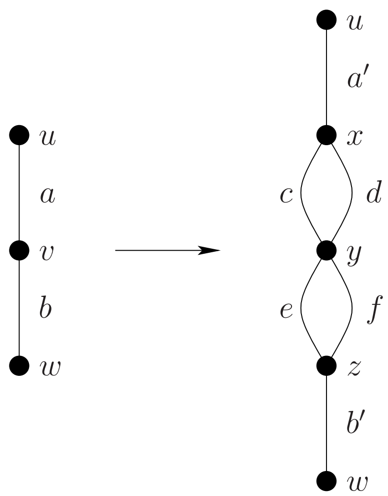


Figure 1:

Now let $v$ be a vertex of even degree greater than 3, and let $e_1, e_2, \ldots, e_{2m}$ be the edges incident on $v$; let them join $v$ to vertices $v_1, v_2, \ldots, v_{2m}$ respectively. We may assume that senses have been assigned to the circuits of $\mathcal{T}$ so that these edges are all odd. Let $C(v) = \{C_1, C_2, \ldots, C_{2m}\}$. As $\mathcal{T}$ is the set of face boundaries of an embedding of $G$, we may also assume that $\{e_i, e_{i+1}\} \subset C_i$ for all $i$, where $e_{2m+1} = e_1$. Let $G_v$ be the graph obtained from $G - \{v\}$ by replacing the vertex $v$ with vertices

$$u_1, u_2, \ldots, u_{2m}, w_1, w_2, \ldots, w_{2m}$$

and the edges incident on $v$ with edges

$$e'_1, e'_2, \ldots, e'_{2m}, x_1, x_2, \ldots, x_{2m}, y_1, y_2, \ldots, y_{2m}, z_1, z_2, \ldots, z_m$$

where, for all $i \leqslant 2m$, $e'_i$ joins $v_i$ to $u_i$, $x_i$ joins $u_i$ to $w_i$, $y_i$ joins $w_i$ to $u_{i+1}$ (where $u_{2m+1} = u_1$) and, for all $i \leqslant m$, $z_i$ joins $w_i$ to $w_{m+i}$. Note that the vertices of $VG_v - VG$ are all of degree 3. (An example where the degree of $v$ is 4 is given in Figure 2.) For each $i \leqslant 2m$ let

$$C'_i = C_i + \{e_i, e_{i+1}, e'_i, x_i, y_i, e'_{i+1}\},$$

where $e'_{2m+1} = e'_1$, and for each $i \leqslant m$ define

$$D_i = \{y_{i-1}, x_i, z_i, x_{m+i}, y_{m+i-1}, z_{i-1}\},$$

where $y_0 = y_{2m}$ and $z_0 = z_m$. Note that these circuits are all even. Let

$$\mathcal{T}_v = \mathcal{T} + C(v) + \{C'_1, C'_2, \ldots, C'_{2m}, D_1, D_2, \ldots, D_m\}.$$

Then $\mathcal{T}_v$ is a thicket in $G_v$ in which every edge carries just two circuits. For each $i \leqslant 2m$

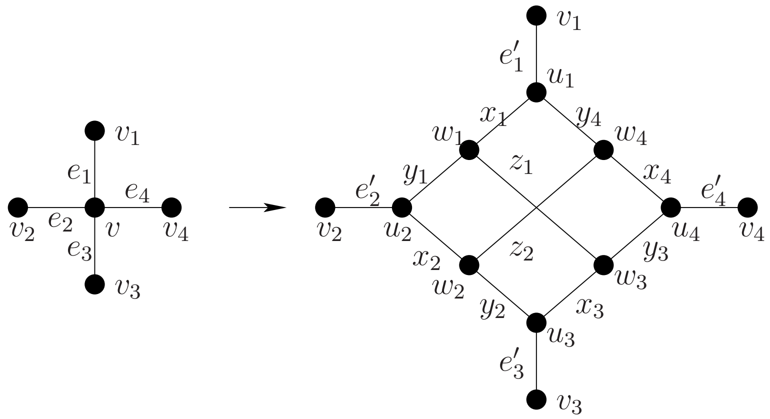


Figure 2:

orient $x_i$ from $u_i$ and assign to $C'_i$ the sense that agrees with this orientation. Similarly for each $i \leqslant m$ assign to $D_i$ the sense that agrees with the orientation of $x_i$. Then edges

$$e'_1, e'_2, \ldots, e'_{2m}, x_{m+1}, x_{m+2}, \ldots, x_{2m}, y_m, y_{m+1}, \ldots, y_{2m-1}, z_1, z_2, \ldots, z_{m-1}$$

are odd but

$$x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_{m-1}, y_{2m}, z_m$$

are even. Thus the odd edges $e_1, e_2, \ldots, e_{2m}$ in $G$ are replaced in $G_v$ by $5m-1$ odd edges. This number is even if and only if $m$ is odd. Hence the parity of $\mathcal{T}_v$ is different from that of $\mathcal{T}$ if and only if the degree of $v$ is divisible by 4.

Suppose on the other hand that $v$ is of odd degree greater than 3. Let $e_0, e_1, \ldots, e_{2m}$ be the edges incident on $v$, and let them join $v$ to vertices $v_0, v_1, \ldots, v_{2m}$ respectively. We assume as before that these edges are odd. Let $C(v) = \{C_0, C_1, \ldots, C_{2m}\}$. We may also assume that $\{e_i, e_{i+1}\} \subset C_i$ for all $i$, where this time subscripts are to be read modulo

$2m+1$. Let $G_v$ be the graph obtained from $G-\{v\}$ by replacing the vertex $v$ with vertices $u_0, u_1, \ldots, u_{2m}$ and the edges incident on $v$ with edges

$$e_0', e_1', \ldots, e_{2m}', x_0, x_1, \ldots, x_{2m}$$

where, for all $i$, $e_i'$ joins $v_i$ to $u_{2i}$ and $x_i$ joins $u_i$ to $u_{i+1}$. Again the subscripts are to be read modulo $2m+1$. As this number is odd but $2i$ is even for every $i$, each $u_i$ is adjacent to a unique $v_j$ as well as to $u_{i-1}$ and $u_{i+1}$. Hence each vertex of $VG_v - VG$ is of degree 3, as before. (An example where the degree of $v$ is 5 is given in Figure 3.) For each $i$ let

$$C_i' = C_i + \{e_i, e_{i+1}, e_i', x_{2i}, x_{2i+1}, e_{i+1}'\}.$$

Again these circuits are even. Let

$$\mathcal{T}_v = \mathcal{T} + C(v) + \{C_0', C_1', \ldots, C_{2m}'\}.$$

Then $\mathcal{T}_v$ once again is a thicket in $G_v$ in which every edge carries just two circuits. Orient

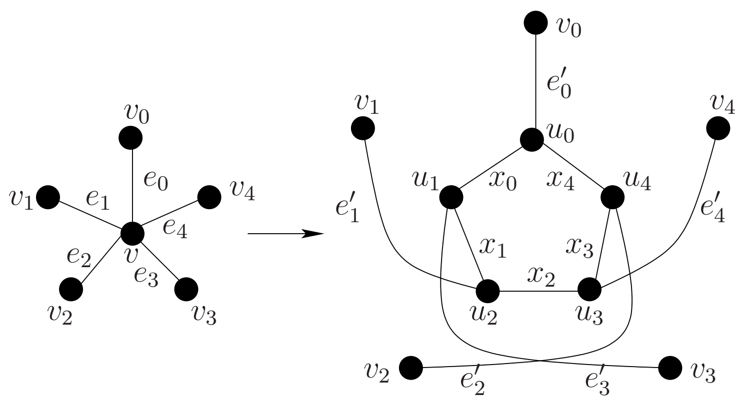


Figure 3:

each $x_i$ so that the circuit $X = \{x_0, x_1, \ldots, x_{2m}\}$ is directed, and assign to each $C_i'$ the sense that agrees with the orientations of its edges in $X$. Then edges $e_0', e_1', \ldots, e_{2m}'$ are odd but the edges of $X$ are even. We conclude that the parity of $\mathcal{T}_v$ is equal to that of $\mathcal{T}$ in this case. □

## 2.2 Basic thickets in cubic graphs

We begin our study of cubic graphs by introducing an operation—the *cancellation* of an edge. Let $G$ be a cubic graph induced by a basic thicket $\mathcal{T}$. Then any pair of adjacent edges of $G$ belong to a unique common member of $\mathcal{T}$. Let $e$ be an edge of $G$ joining vertices $u$ and $v$. Let $a$ and $b$ be the edges of $EG - \{e\}$ incident on $u$ and let $c$ and $d$ be those incident on $v$. Let $w, x, y, z$ be the ends of $a, b, c, d$, respectively, in $VG - \{u, v\}$. (See Figure 4.)

There is a unique circuit $C \in \mathcal{T}$ containing $a$ and $b$ and a unique $D \in \mathcal{T}$ containing $c$ and $d$. We shall assume that $C \neq D$, in which case we describe $e$ as a *dipole*. There is also

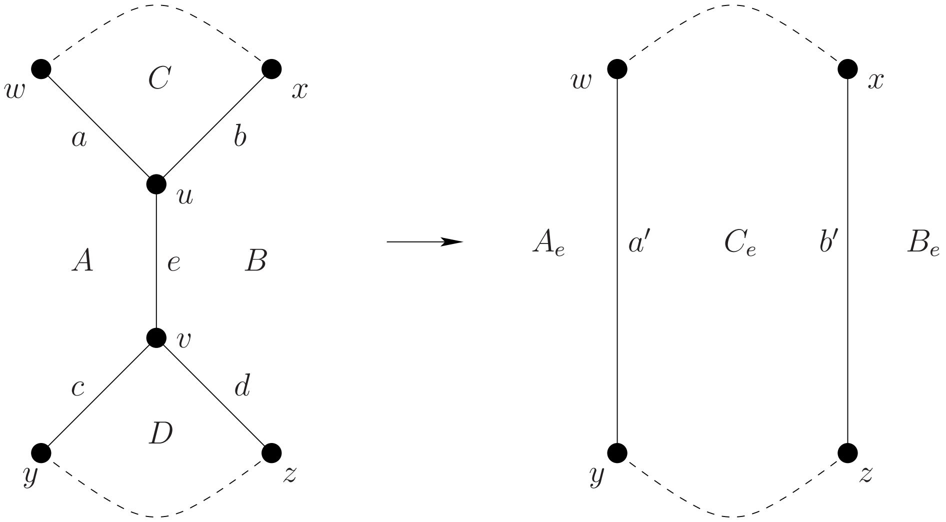


Figure 4:

a unique $A \in \mathcal{T}$ such that $\{a, e\} \subset A$ and a unique $B \in \mathcal{T}$ for which $\{b, e\} \subset B$. Without loss of generality we may assume that $c \in A$ and $d \in B$. Note that $A - \{a, e, c\} \neq \emptyset$ and $B - \{b, e, d\} \neq \emptyset$ since $|A|$ and $|B|$ are even. We may also assume that $A, B, C, D$ have been assigned senses so that edges $a, b, c, d, e$ are all odd. Let $G_e$ be the graph obtained from $G - \{u, v\}$ by replacing the edges $a, b, c, d, e$ with edges $a'$ and $b'$ joining $w$ to $y$ and $x$ to $z$ respectively. We say that $G_e$ is obtained from $G$ by *cancelling* $e$ and that $G$ is obtained from $G_e$ by *creating* $e$. Define

$$A_e = A + \{a, e, c, a'\},$$

$$B_e = B + \{b, e, d, b'\},$$

$$C_e = C + D + \{a, b, c, d, a', b'\}$$

and

$$\mathcal{T}_e = \mathcal{T} + \{A, B, C, D, A_e, B_e, C_e\}.$$

Then $\mathcal{T}_e$ is a thicket in $G_e$, and each edge of $G_e$ carries just two circuits of $\mathcal{T}_e$.

**Lemma 2.2.** *Let $G$ be a cubic graph induced by a basic thicket $\mathcal{T}$, and let $e$ be a dipole in $G$. Then the parity of $\mathcal{T}_e$ is the opposite of that of $\mathcal{T}$.*

*Proof.* We use the notation developed in the preceding discussion. We may assign senses to $A, B, C, D$ so that edges $a, b, c, d, e$ are all odd. Recalling that $A - \{a, e, c\}$ is non-empty, we may choose an edge $f \in A \cap A_e$ and assign to $A_e$ the sense that agrees on $f$ with the sense of $A$ on $f$. We assign a sense to $B_e$ in the same manner. Similarly there is a sense of $C_e$ that agrees with that of $C$ on any edge of $C - \{a, b\}$ and with that of $D$ on any edge of $D - \{c, d\}$; assign this sense to $C_e$. Then $a'$ and $b'$ are odd edges in $\mathcal{T}_e$. Hence the number of odd edges in $\mathcal{T}_e$ is 3 less than in $\mathcal{T}$, and the result follows. $\square$

**Corollary 2.3.** *The numbers of even edges in $\mathcal{T}_e$ and $\mathcal{T}$ are of equal parity.*

We now use our results to determine the parity of thickets of cardinality 3 in cubic graphs. The proof of the next theorem requires the concept of a 3-graph. Readers unfamiliar with the use of 3-graphs to model embeddings of graphs in surfaces are referred to [1].

**Theorem 2.4.** *Let $G$ be a cubic graph induced by a basic thicket $\mathcal{T}$ which can be partitioned into three cells of disjoint circuits with the property that the union of the circuits in each cell gives a spanning cycle of $G$. Then the parity of $\mathcal{T}$ is that of $|\mathcal{T}|$.*

*Proof.* Let $\{A, B, C\}$ be a partition of $\mathcal{T}$ into cells of disjoint circuits with the hypothesised property. Then $A \cap B$, $B \cap C$, $A \cap C$ are disjoint 1-factors of $G$. We conclude that $G$ has a proper edge colouring in three colours, and therefore that it is a 3-graph. It may thus be put into a canonical form by means of dipole cancellations and creations, moves which preserve the parity of the number of even edges.

If the canonical form is bipartite, then either it is $\Theta$ (the unique cubic graph with just two vertices) or it models an embedding of a graph in an orientable surface. In both cases the circuits of $\mathcal{T}$ may be assigned senses so that every edge is odd. In other words, the number of even edges is even, which is the parity of the Euler characteristic for an orientable embedding.

Suppose on the other hand that the canonical form is not bipartite. Then it models an embedding of a graph in a non-orientable surface. Let $g$ be the genus of this surface. The canonical form $G_g$ has vertices $v_0, v_1, \ldots, v_{4g-1}$ and edges

$$e_0, e_1, \ldots, e_{4g-1}, a_0, a_1, \ldots, a_{g-1}, b_0, b_1, \ldots, b_{g-1}$$

where, for all $i < 4g$, $e_i$ joins $v_i$ to $v_{i+1}$ and, for all $i < g$, $a_i$ joins $v_{4i}$ to $v_{4i+2}$ and $b_i$ joins $v_{4i+1}$ to $v_{4i+3}$. Here the subscripts are to be read modulo $4g$. (Figure 5 gives an example where $g = 2$.) The thicket which induces $G_g$ is

$$\mathcal{T}_g = \{A, B, C_0, C_1, \ldots, C_{g-1}\}$$

where

$$A = \{e_0, e_1, \ldots, e_{4g-1}\},$$

$$B = \{a_0, e_1, b_0, e_3, a_1, e_5, b_1, e_7, \ldots, a_{g-1}, e_{4g-3}, b_{g-1}, e_{4g-1}\}$$

and

$$C_i = \{a_i, e_{4i+2}, b_i, e_{4i}\}$$

for all $i < g$. Orient the edges of $A$ so that $A$ is a directed circuit and assign to the circuits of $\mathcal{T}_g$ the senses that agree with the orientations of the edges

$$e_0, e_4, \ldots, e_{4g-4}, e_{4g-1}.$$

Then edges

$$e_1, e_2, e_5, e_6, \ldots, e_{4g-3}, e_{4g-2}, a_0, a_1, \ldots, a_{g-1}$$

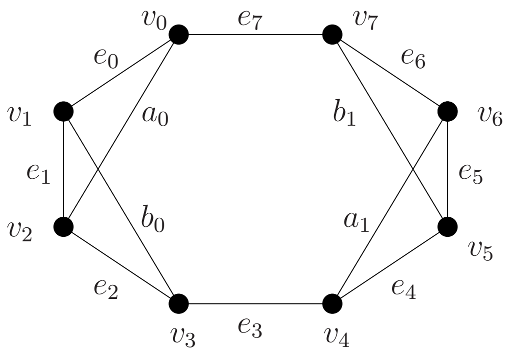


Figure 5:

are odd but the remaining $3g$ edges of $G_g$ are even. Thus the number of even edges has the same parity as $g$. But, for a non-orientable embedding, $g$ is of the same parity as the Euler characteristic.

In any case, therefore, the parity of the number of even edges is that of the Euler characteristic, $|VG| - |EG| + |\mathcal{T}|$. Moreover $|VG|$ is even since $G$ is cubic. Let $n_0$ and $n_1$ be the numbers of even and odd edges, respectively, in $G$. Then

$$n_0 \equiv |EG| + |\mathcal{T}| \equiv n_0 + n_1 + |\mathcal{T}|.$$

(All congruences in this paper are to be taken modulo 2.) Hence $n_1 + |\mathcal{T}|$ is even, and so the parity of $\mathcal{T}$ is that of $|\mathcal{T}|$. $\square$

**Corollary 2.5.** *Let $G$ be a cubic graph induced by a thicket $\mathcal{T}$ of cardinality 3. Then the parity of $\mathcal{T}$ is odd.*

*Proof.* If $\mathcal{T} = \{X, Y, Z\}$, then $X, Y, Z$ must be Hamilton circuits. Therefore we may take $\{\{X\}, \{Y\}, \{Z\}\}$ as the partition in the hypothesis of the theorem. $\square$

**Corollary 2.6.** *Let $G$ be a graph with $N(G)$ vertices of degree 2 and the rest of degree 3. Let $G$ be induced by a thicket $\mathcal{T}$ which can be partitioned into three cells of circuits with the property that the union of the circuits in each cell gives a spanning cycle of $G$. Then the parity of $\mathcal{T}$ is that of $|\mathcal{T}| + N(G)$.*

*Proof.* We use induction on the number of vertices of degree 2. The result follows from the theorem if $G$ is cubic. It is therefore enough to show that if the vertices of degree 2 are replaced by vertices of degree 3 as in the procedure described in Section 2.1, then the hypotheses of the corollary are still satisfied by the resulting thicket. Consequently it suffices to consider the case where there is a unique vertex of degree 2. Let $v$ be this vertex, and let it be joined to vertices $u$ and $w$ by edges $c_1$ and $c_2$ respectively. Let $C_1$ and $C_2$ be the circuits of $\mathcal{T}$ that contain $c_1$ and $c_2$. These circuits meet, and therefore belong to distinct cells $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, of the given partition $\mathcal{P}$ of $\mathcal{T}$ of cardinality 3. Let $\mathcal{D}$ be the third cell of this partition. Thus $\mathcal{P} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}\}$.

The reduction to the case of cubic graphs requires that a graph $G_v$ be obtained from $G - \{v\}$ by replacing $v$ with vertices

$$v_1, v_2, u_1, u_2, u_3, u_4, w_1, w_2, w_3, w_4$$

and its incident edges with edges

$$c_1', c_2', a_1, a_2, b_1, b_2, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2$$

where $c_1'$ joins $u$ to $v_1$, $c_2'$ joins $v_2$ to $w$, $a_1$ joins $v_1$ to $u_1$, $a_2$ joins $u_2$ to $v_2$, $b_1$ joins $v_1$ to $u_4$, $b_2$ joins $u_3$ to $v_2$, $x_i$ joins $u_i$ to $w_i$ for each $i$, $y_i$ joins $w_i$ to $u_{i+1}$ for each $i$ (where $u_5 = u_1$), $z_1$ joins $w_1$ to $w_3$ and $z_2$ joins $w_2$ to $w_4$. (See Figure 6.)

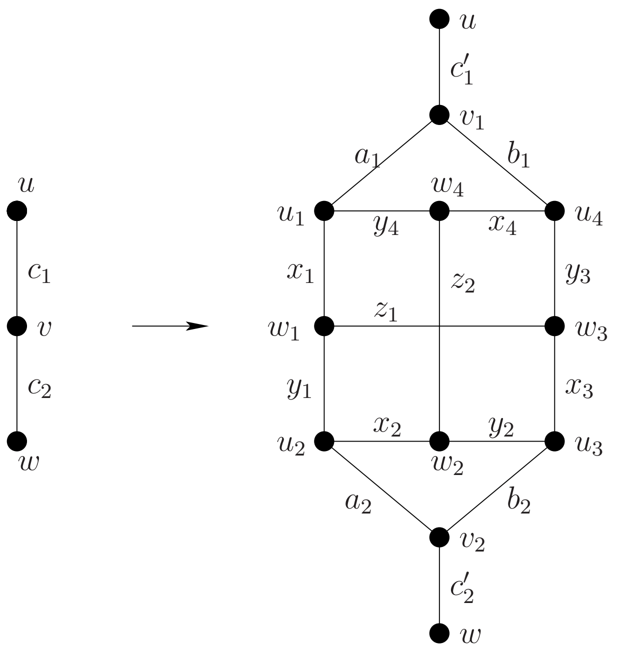


Figure 6:

Let

$$C_1' = C_1 + \{c_1, c_2, c_1', a_1, x_1, y_1, x_2, z_2, x_4, y_3, x_3, b_2, c_2'\},$$
$$C_2' = C_2 + \{c_1, c_2, c_1', b_1, x_4, y_4, x_1, z_1, x_3, y_2, x_2, a_2, c_2'\},$$
$$D = \{b_1, y_3, z_1, y_1, a_2, b_2, y_2, z_2, y_4, a_1\}$$

and

$$\mathcal{T}' = \mathcal{T} + \{C_1, C_2, C_1', C_2', D\};$$

$\mathcal{T}'$ is a thicket in $G_v$. Moreover

$$\{\mathcal{C}_1 + \{C_1, C_1'\}, \mathcal{C}_2 + \{C_2, C_2'\}, \mathcal{D} \cup \{D\}\}$$

is a partition of $\mathcal{T}'$ into three cells with the required properties. The conclusion therefore follows from Theorem 2.1. □

We can also determine the parity of a thicket that induces a bicubic graph. First we need the following lemma.

**Lemma 2.7.** *Let $G$ be a cubic graph induced by a basic thicket $\mathcal{T}$. Then the even edges form a cycle of $G$.*

*Proof.* Choose $v \in VG$. There are three edges, $a, b, c$, incident on $v$. There are circuits $A, B, C \in \mathcal{T}$ such that $\{a, b\} \subset A$, $\{b, c\} \subset B$ and $\{a, c\} \subset C$. These circuits may be assigned senses so that edges $a, b, c$ are all odd. Each of $A, B, C$ contains exactly two of $a, b, c$. Therefore an even number of $a, b, c$ are even under any assignment of senses to $A, B, C$. The result follows. □

**Theorem 2.8.** *Let $G$ be a bipartite cubic graph, induced by a thicket $\mathcal{T}$, with bipartition $\{X, Y\}$. Then the parity of $\mathcal{T}$ is that of $|X|$.*

*Proof.* Note first that, as $G$ is cubic, $|VG|$ is even; hence $|X| \equiv |Y|$.

Since the degree of every vertex is odd, we have $|EG| \equiv |X|$. As $G$ is bipartite and the set $S$ of even edges is a cycle by the lemma, it follows that $|S| \equiv 0$. Hence

$$|T| = |EG \cap T| \equiv |X|,$$

where $T = EG - S$. As $T$ is the set of odd edges of $G$, the result follows. □

# 3 Applications to the Pfaffian problem

Let $C$ be an even circuit in a near brick $G$. As the even space of $G$ is equal to the alternating space, it has a basis $\mathcal{B}$ consisting of alternating circuits. If $G$ is given a Pfaffian orientation, then the basis circuits have odd directed parity. In any case $C$ can be expressed uniquely as a sum of circuits in $\mathcal{B}$. Let $S(C)$ be the set of alternating circuits in $\mathcal{B}$ whose sum is $C$. Then $S(C) \cup \{C\}$ is a thicket $\mathcal{T}$. If its parity is known and the specified parities for the circuits in $S(C)$ are all equal to their actual odd directed parity under a given Pfaffian orientation of $G$, then the directed parity of $C$ can be calculated. We therefore specify the result of this calculation as the directed parity for $C$.

If $G$ is Pfaffian, then it can be oriented so that every even circuit has its specified directed parity. Suppose therefore that $G$ is not Pfaffian. It is still possible to specify parities so that the specified parity of every alternating circuit is odd and the specified parity for any even circuit that is not alternating is calculated from the basis $\mathcal{B}$ as in the previous paragraph, but this time $G$ cannot be oriented so that every even circuit has its specified directed parity. It follows from the main theorem in [2] that in this case there is an obstruction set of three or four even circuits that cannot all be given their specified directed parities under any orientation of $G$. Moreover, because of the way in which the specified parities have been calculated for the even circuits that are not alternating, any such obstruction set contains at least one alternating circuit not in $\mathcal{B}$.

One further conclusion can be drawn from these considerations. Two of the obstruction sets in [2] consist, respectively, of the three even circuits in an even subdivision of $K_{2,3}$

and the three even circuits in an even subdivision of the graph $K_4^*$ obtained from $K_4$ by subdividing once every edge incident on a given vertex. In these obstruction sets an even number of the even circuits are specified to have even directed parity. In every other obstruction set, an odd number of the even circuits are specified to have even directed parity. In the latter obstruction sets, therefore, at least one of the even circuits is not an alternating circuit, and we deduce that in this case such an even circuit must exist. In any matching covered graph that is not a near brick, the existence of an even circuit that is not alternating follows from the fact that the alternating space of such a graph is a proper subspace of the even space. We therefore reach the following conclusion.

**Theorem 3.1.** *Any non-Pfaffian graph contains an even subdivision of $K_{2,3}$ or $K_4^*$ or an even circuit that is not alternating.*

It is known [4] that a non-Pfaffian graph must contain a subdivision of $K_{3,3}$. Note that if each edge of $K_{3,3}$ not in a given 1-factor is subdivided once, then the resulting graph contains no even subdivision of $K_{2,3}$ or $K_4^*$.

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