New Ramsey Classes from Old

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Abstract

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strong amalgamation classes of finite structures, with disjoint finite signatures σ and τ. Then $\mathcal{C}_1 \wedge \mathcal{C}_2$ denotes the class of all finite $(\sigma \cup \tau)$-structures whose σ-reduct is from $\mathcal{C}_1$ and whose τ-reduct is from $\mathcal{C}_2$. We prove that when $\mathcal{C}_1$ and $\mathcal{C}_2$ are Ramsey, then $\mathcal{C}_1 \wedge \mathcal{C}_2$ is also Ramsey. We also discuss variations of this statement, and give several examples of new Ramsey classes derived from those general results.

1 Introduction and Results

A class of relational structures is a *Ramsey class* if it satisfies a strong combinatorial property that resembles the statement of Ramsey's theorem. Surprisingly many classical classes of relational structures turned out to be Ramsey classes. Nešetřil [12] asked whether one may classify all Ramsey classes that are closed under induced substructures and have the joint embedding property, and he indicated a link to the model-theoretic classification of countably infinite homogeneous structures as an approach to such a classification. This program has recently attracted attention because of a fascinating correspondence between Ramsey classes and the concept of extreme amenability in topological dynamics [10]. We would also like to mention that Ramsey classes play an important role in classifications of first-order reducts of homogeneous relational structures [5], and for complexity classification of infinite-domain constraint satisfaction [2]. Establishing that a class has the Ramsey property is often a substantial combinatorial challenge, and we are therefore interested in general transfer principles that allow to prove the Ramsey property by reducing to known Ramsey classes; this will be the topic of this text.

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For structures A and B over the same relational signature, let $\binom{B}{A}$ denote the set of all embeddings of A into B. When f is such an embedding, we write $f[A]$ for the copy of A in B that is induced by the image of A under f in B. The partition arrow $C \rightarrow (B)^A_r$ means that for every function $\chi\colon \binom{C}{A} \rightarrow \{1,\dots,r\}$ (a *colouring* with r colours) there exists $g \in \binom{C}{B}$ such that χ is constant on $\binom{g[B]}{A}$. In this case we call $g[B]$ a *monochromatic copy* of B in C. A class of finite relational structures $\mathcal{C}$ has the *Ramsey property (with respect to embeddings)*[1] if for all $A, B \in \mathcal{C}$ and $r \in \mathbb{N}$ there exists a $C \in \mathcal{C}$ such that $C \rightarrow (B)^A_r$. It is easy to see that every class $\mathcal{C}$ with the Ramsey property only contains *rigid* structures, that is, structures with only one automorphism, the identity. Note that an *ordered* structure, that is, a structure that has a strict linear order as one of its relations, is always rigid. A class of relational structures that is closed under isomorphisms and has the Ramsey property is also called a *Ramsey class*.

Examples of Ramsey classes are

- $\mathcal{LO}$, the class of all finite linear orders (this is equivalent to Ramsey's original theorem);
- the class of all ordered finite graphs (see [13]);
- the class of all ordered K_n-free graphs (see [13]);
- the class of all finite partially ordered sets with a linear extension (see [12]);
- the class of all finite tournaments with an additional linear order;
- the class of all finite convexly ordered binary branching C-relations on a finite set (see [4]; this is essentially due to [11]).

It is of major interest in combinatorics to obtain a more systematic understanding of the question which classes of structures have the Ramsey property.

Nešetřil made the important observation that Ramsey classes that are closed under taking induced substructures are linked with the concept of *amalgamation* in model theory. We say that a class of structures has the *amalgamation property* if for all $A, B_1, B_2 \in \mathcal{C}$ and embeddings $e_1\colon A \rightarrow B_1$ and $e_2\colon A \rightarrow B_2$ there exists a $C \in \mathcal{C}$ and embeddings f_i of B_i to C such that $f_1(e_1(a)) = f_2(e_2(a))$ for all $a \in A$. We call (A, B_1, B_2, e_1, e_2) the *amalgamation diagram*, and (C, f_1, f_2) an *amalgam* of the diagram (A, B_1, B_2, e_1, e_2) (in $\mathcal{C}$). If $\mathcal{C}$ has the amalgamation property for the special case that A is empty, we say that $\mathcal{C}$ has the *joint embedding property* (here, our assumption that the signature is relational becomes important). The mentioned link between Ramsey theory and amalgamation is that every class $\mathcal{C}$ of rigid finite relational structures that is closed under isomorphisms and induced substructures, and that has the joint embedding and the Ramsey property also has amalgamation property [12]. Classes of finite structures with countably many

[1] In some papers, a class $\mathcal{C}$ has the Ramsey property if and only if $\mathcal{C}$ satisfies an analogous property where the partition arrow is not about embeddings, but induced substructures. The two variants are closely related; for a discussion, see [12, 10].

non-isomorphic structures that are closed under isomorphisms, induced substructures, and have the amalgamation property are called *amalgamation classes*.

The *age* of a relational structure Γ is the class of all finite structures that embed into Γ. A structure is *homogeneous* if any isomorphism between finite induced substructures of Γ can be extended to an automorphism of Γ. When $\mathcal{C}$ is an amalgamation class, then Fraïssé's theorem shows that there exists a countable homogeneous structure Γ whose age is $\mathcal{C}$ (see e.g. [9]). The structure Γ is unique up to isomorphism, and called the *Fraïssé-limit* of $\mathcal{C}$; these homogeneous limit structures will play an important role in the proof of our main result. The significance of Nešetřil's observation is that the transition to countable homogeneous structures brings new tools for the systematic understanding of Ramsey classes; and indeed, under some additional assumptions, there are many *classification results* for homogeneous structures (such as the classification of all homogeneous directed graphs [8]).

A *strong amalgam* of an amalgamation diagram (A, B_1, B_2, e_1, e_2) is an amalgam (C, f_1, f_2) such that $f_1(e_1(A)) = f_2(e_2(A)) = f_1(B_1) \cap f_2(B_2)$. A class $\mathcal{C}$ has *strong amalgamation* if every amalgamation diagram has a strong amalgam in $\mathcal{C}$. We say that $\mathcal{C}$ is a *strong amalgamation class* if $\mathcal{C}$ is closed under isomorphisms, induced substructures, and has the strong amalgamation property. An example of a strong amalgamation class is $\mathcal{LO}$.

We write $\mathrm{Aut}(\Gamma)$ for the automorphism group of Γ. An *orbit* of Γ is meant to be an orbit of $\mathrm{Aut}(\Gamma)$, that is, a set of the form $\{\alpha(c) \mid \alpha \in \mathrm{Aut}(\Gamma)\}$ for some element c of the domain of Γ. Homogeneous structures Γ that arise as the Fraïssé-limits of strong amalgamation classes can be characterized via algebraic closure. In this context, we define the algebraic closure $\mathrm{acl}(A)$ of a finite subset $A = \{a_1, \ldots, a_n\}$ of the domain of Γ to be the set of all those elements of Γ which lie in finite orbits of the expansion $(\Gamma, a_1, \ldots, a_n)$ of Γ by the constants $a_1, \ldots, a_n$.

Proposition 1.1 (see (2.15) in [7])**.** *The age of a homogeneous structure Γ has strong amalgamation if and only if for any finite subset A of the domain of Γ, $\mathrm{acl}(A) = A$.*

Definition 1.2 ((3.9) in [7])**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strong amalgamation classes with disjoint signatures σ and τ. Then $\mathcal{C}_1 \wedge \mathcal{C}_2$ denotes the class of all finite $(\sigma \cup \tau)$-structures whose σ-reduct is from $\mathcal{C}_1$ and whose τ-reduct is from $\mathcal{C}_2$.

It is clear that $\mathcal{C}_1 \wedge \mathcal{C}_2$ also has strong amalgamation. In Section 4, we prove the following.

Theorem 1.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strong amalgamation classes with the Ramsey property, with disjoint finite signatures σ and τ. Then $\mathcal{C}_1 \wedge \mathcal{C}_2$ has the Ramsey property.*

The following is an immediate consequence of Theorem 1.3 and the previously known Ramsey results mentioned above.

Corollary 1.4. *The following classes of finite structures are Ramsey.*

1. *The class of all permutations of a finite set (represented by two linear orders);*

2. *The class of all finite sets carrying n linear orders;*
3. *The class of all finite posets with a linear extension and an additional arbitrary linear order;*
4. *The class of all finite sets carrying two posets, a linear extension of the first, and a linear extension of the second poset;*
5. *The class of all finite sets carrying a poset and a linear extension of it, and additional linear order and a graph relation.*

Item 1 in Corollary 1.4 has been obtained independently by Böttcher and Foniok [6] and by Sokić [14]. To prove the statement in item 1, Sokić developed a technique called *cross construction*; also see [15]. He also proved item 2 and 3 in Corollary 1.4. The present work has been found independently from [15], and it would be interesting to compare our approach with the approach in [15]. The other items in the list have been added mainly for illustration reasons, and it is clear that the list can be prolonged easily.

Homogeneous structures with a finite relational signature are *ω-categorical*, that is, their first-order theory has only one countable model up to isomorphism. We can weaken the assumption of having a finite signature slightly, and prove the following stronger version which captures several additional interesting classes (see Corollary 1.8).

Theorem 1.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strong amalgamation classes with ω-categorical Fraïssé-limits. If $\mathcal{C}_1$ and $\mathcal{C}_2$ have disjoint relational signatures and the Ramsey property, then $\mathcal{C}_1 \wedge \mathcal{C}_2$ is also Ramsey.*

To show the Ramsey property for even more classes, we would also like to be able to generate Ramsey classes that only have one linear order in their signature; this can be accomplished using the following proposition whose proof can be found in Section 5.

Proposition 1.6. *Let $\mathcal{C}_1$ and $\mathcal{LO} \wedge \mathcal{C}_2$ be Ramsey classes with strong amalgamation and ω-categorical Fraïssé-limits, and suppose that $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{LO}$ have pairwise disjoint relational signatures. Then $\mathcal{C}_1 \wedge \mathcal{C}_2$ has the Ramsey property.*

Proposition 1.6 is a versatile tool to construct a variety of new Ramsey classes. To state many examples, we make the following definitions.

Definition 1.7. Write

- $\mathcal{T}$ for the class of all finite tournaments,
- $\mathcal{G}$ for the class of all finite graphs,
- $\mathcal{F}_n$ for the class of all finite K_n-free graphs,
- $\vec{\mathcal{T}}$ for the class of all linearly ordered finite tournaments,
- $\vec{\mathcal{G}}$ for the class of all linearly ordered finite graphs,

- $\vec{\mathcal{F}}_n$ for the class of all linearly ordered finite K_n-free graphs,
- $\vec{\mathcal{P}}$ for the class of all linearly extended finite posets,
- $\vec{\mathcal{C}}$ for the class of all convexely ordered binary branching C-relations on a finite set,
- $\vec{\mathcal{V}}$ for the class of all finite affine vector spaces V, equipped with a 'natural order' (see [10]); the vector spaces will be represented as relational structures with an infinite signature that contains a relation symbol for every affine equation.

Corollary 1.8. *Let $\mathcal{C}_1$ be one of the classes $\vec{\mathcal{T}}, \vec{\mathcal{G}}, \vec{\mathcal{F}}_n, \vec{\mathcal{P}}, \vec{\mathcal{C}}, \vec{\mathcal{V}}$, and let $\mathcal{C}_2$ be one of the classes $\mathcal{T}, \mathcal{G}, \mathcal{F}_n$. Then $\mathcal{C}_1 \wedge \mathcal{C}_2$ has the Ramsey property.*

Also Corollary 2.4 covers examples of particular interest; for example the class $\vec{\mathcal{C}} \wedge \mathcal{T}$ will be discussed at the end of Section 2.

2 Topological Dynamics

Our combinatorial result translates nicely into a result that shows that certain intersections of extremely amenable groups are again extremely amenable, based on a connection between Ramsey theory and topological dynamics (Theorem 2.3). In fact, our presentation of the proof of Theorem 1.5 makes use of this connection, and so we briefly present it in the following.

Let us first mention that the property of ω-categoricity of a structure Γ can be characterized in terms of the automorphism group of Γ; this has been shown Engeler, Svenonius, and Ryll-Nardzewski, and we state it for easy reference. When Γ is a τ-structure with domain D, and $R \subseteq D^k$, then we say that R has a *first-order definition* in Γ if there exists a first-order formula φ with k free variables over the signature τ such that $R = \{(d_1, \ldots, d_k) \in D^k \mid \varphi(d_1, \ldots, d_n) \text{ holds in } \Gamma\}$.

Theorem 2.1 (see Theorem 6.3.1 and Corollary 6.3.3 in [9])**.** *A countable structure is ω-categorical if and only if its automorphism group is* oligomorphic*, that is, has only finitely many orbits of n-tuples, for all n. If Γ is ω-categorical, then all relations that are preserved by all automorphisms of Γ have a first-order definition in Γ.*

We say that R has a *quantifier-free definition* in Γ if φ can be chosen to be without quantifiers.

Proposition 2.2 (see (2.22) in [7])**.** *When Γ is ω-categorical, then Γ is homogeneous if and only if Γ has* quantifier-elimination*, that is, every first-order definable relation in Γ also has a quantifier-free definition.*

A topological group G is called *extremely amenable* if every continuous action of G on a compact Hausdorff space has a fixed point. We say that a homogeneous structure Γ is *Ramsey* if the class of all finite induced substructures that embed into Γ is a Ramsey class. The following is the central result from [10].

Theorem 2.3 (Theorem 4.8 in [10]). *Let* Γ *be a countable ordered homogeneous structure. Then the following are equivalent.*

- $\mathrm{Aut}(\Gamma)$ *is extremely amenable.*
- Γ *is Ramsey.*

This result and other known facts have the following consequence for the special case that Γ is additionally ω-categorical.

Corollary 2.4. *Let* Γ *be an* ω*-categorical countable homogeneous structure. Then the following are equivalent.*

1. Γ *is Ramsey.*
2. *For all finite substructures* A, B *of* Γ *it holds that* $\Gamma \to (B)^A_2$.
3. $\mathrm{Aut}(\Gamma)$ *is extremely amenable.*
4. Γ *is Ramsey, and there is a quantifier-free definition of a linear order over* Γ.

Proof. The equivalence between the first two items is based on compactness argument and a standard fact that can be found in many text-books on Ramsey theory. For the equivalence of the last two items, recall that we color *embeddings*, and not induced substructures, so the Ramsey property implies rigidity. The equivalence of rigidity and the existence of a linear order which is preserved by all automorphisms of Γ for Ramsey structures Γ is stated in Proposition 4.3 in [10]. Note that our additional assumption that Γ is ω-categorical implies that such a linear order has a first-order definition over Γ (Theorem 2.1). Finally, homogeneity and ω-categoricity of Γ imply by Proposition 2.2 that we can even find a quantifier-free definition over Γ. $\square$

We would like to comment on the consequence of Corollary 2.4 which says that Fraïssé-limits of Ramsey classes always have a definable linear order. Recall that Ramsey classes only contain rigid structures; so it is natural to ask whether more generally there is a definable linear order in every Fraïssé-limit of an amalgamation class of *rigid* structures (without the assumption that the class has the Ramsey property). This turns out to be false, and Dugald Macpherson communicated the following example to the author, an example which he credits to Peter Cameron.

Consider the class $\mathcal{C} \wedge \mathcal{T}$ (in the terminology of Definition 1.7). We claim that this class only contains rigid structures. To see this, let α be an automorphism of a structure A from this class. Note that there exists a partition $V_1 \cup V_2$ of the vertices of A such that for all $x, y \in V_1$ and $u, v \in V_2$ we have $C(x, y, u)$ and $C(u, v, y)$. Then either $\alpha(V_1) = V_1$ and $\alpha(V_2) = V_2$, or $\alpha(V_1) = V_2$ and $\alpha(V_2) = V_1$. The same argument can be applied to the structures induced in A by V_1 and by V_2. Repeating this argument, we finally obtain that $\alpha^2 = \mathrm{id}$. But then α must be the identity, for if $\alpha(x) = y$ and x and y are distinct, then either (x, y) or (y, x) is an edge in the tournament; but then $(\alpha(x), \alpha(y)) = (y, x)$ is an edge as well, which is impossible in tournaments.

On the other hand, we claim that there is no order definable in the Fraïssé-limit Γ of $\mathcal{C} \wedge \mathcal{T}$. To see this, recall that Γ has quantifier-elimination by Corollary 2.2, so it can be checked exhaustively that none of the finitely many quantifier-free formulas with variables x, y defines a linear order in Γ. As a consequence of Corollary 2.4, the class $\mathcal{C} \wedge \mathcal{T}$ is not Ramsey. But the structures in this class can be expanded by a linear order so that the resulting class of expansions has the Ramsey property: this is a consequence of Corollary 1.8, since $\vec{C} \wedge \mathcal{T}$ has the Ramsey property.

3 Model-Complete Cores

In our proofs, we make use of the concept of *model-complete cores* of ω-categorical structures. A structure Γ is called a *core* if every endomorphism[2] of Γ is an embedding. A first-order theory T is called *model-complete* if all embeddings between models of T preserve all first-order formulas. An ω-categorical structure Γ has a model-complete theory if and only if all self-embeddings e of Γ are locally generated by the automorphisms of Γ, that is, for every finite tuple t of elements from Γ there exists an automorphism α of Γ such that $e(t) = \alpha(t)$ (see e.g. Theorem 3.6.11 in [2]). In this case, we say that Γ is model-complete. Note that by the above, when Γ is homogeneous, then Γ is model-complete.

The following has been shown in [1] (also see [3]).

Theorem 3.1. *Every ω-categorical structure is homomorphically equivalent[3] to a model-complete core Δ, which is unique up to isomorphism, and again ω-categorical or finite. The expansion of Δ by all existential positive definable relations is homogeneous.*

The structure Δ in Theorem 3.1 will be called *the model-complete core of* Γ. We need the following observation.

Proposition 3.2. *The model-complete core Δ of an ω-categorical homogeneous structure Γ is homogeneous.*

Proof. Let h be a homomorphism from Γ to Δ, and let i be a homomorphism from Δ to Γ. Suppose that f is an isomorphism between two finite substructures A, A' of Δ. The restriction of i to A and to A' is an isomorphism as well, since otherwise the endomorphism $x \mapsto h(i(x))$ of Δ would not be an embedding, contradicting the assumption that Δ is a core. By homogeneity of Γ there exists an automorphism α of Γ that extends the isomorphism $i \circ f \circ i^{-1}$ between $i(A)$ and $i(A')$. The mapping $e\colon x \mapsto h(\alpha i(x))$ is an endomorphism of Δ, and therefore an embedding. Since Δ is a model-complete core, this mapping is locally generated by the automorphisms of Δ, and in particular there exists an automorphism β of Δ such that $\beta(x) = e(x) = f(x)$ for all $x \in A$. This proves homogeneity of Δ. $\square$

We now prove the following.

[2] An *endomorphism* of Γ is a homomorphism from Γ to Γ.

[3] Two structures Γ and Δ are *homomorphically equivalent* if there is a homomorphism from Γ to Δ and a homomorphism from Δ to Γ.

Theorem 3.3. *Let Γ be ω-categorical, homogeneous, and Ramsey, and let Δ be the model-complete core of Γ. Then Δ is also Ramsey.*

Proof. First note that Δ is homogeneous, by Proposition 3.2. Let h be a homomorphism from Γ to Δ, and let i be a homomorphism from Δ to Γ.

Let P, H be two finite substructures of Δ. By Corollary 2.4, it suffices to prove that $\Delta \to (H)^P_2$. Note that $i(P)$ induces in Γ a copy of P since otherwise the endomorphism $x \mapsto h(i(x))$ of Δ would not be an embedding. Moreover, for every copy Q of P in Γ we have that $h(Q)$ induces a copy of P in Δ. To see this, let α be the automorphism of Γ that maps $i(P)$ to Q; such an α exists by homogeneity of Γ. Then $e\colon x \mapsto h(\alpha i(x))$ must be an embedding, and $e(P) = h(Q)$ which proves the claim.

Let $\chi\colon \binom{\Delta}{P} \to \{1,2\}$ be arbitrary. We define a map $\xi\colon \binom{\Gamma}{P} \to \{1,2\}$ by setting $\xi(q) := \chi(h \circ q)$ for every $q \in \binom{\Gamma}{P}$. Since Γ is Ramsey, there exists a $g \in \binom{\Gamma}{H}$ such that ξ is constant on $\binom{g[H]}{P}$. Then $h \circ g \in \binom{\Delta}{H}$, and it suffices to show that χ is constant on $\binom{h \circ g[H]}{P}$. By an argument similar as given above, the restriction h' of h to $g[H]$ is an embedding, and the image of h' induces a copy M of H in Δ. Let $p_1, p_2 \in \binom{h \circ g[H]}{P}$. Then $h^{-1} \circ p_1, h^{-1} \circ p_2 \in \binom{g[H]}{P}$, and therefore $\xi(h^{-1} \circ p_1) = \xi(h^{-1} \circ p_1)$, and $\chi(p_1) = \chi(p_2)$. $\square$

4 The Full Product Structure

Let Γ_1 and Γ_2 be two structures with the same domain D and with disjoint signatures σ and τ, respectively. The *full product* $\Gamma_1 \boxtimes \Gamma_2$ of Γ_1 and Γ_2 is a $(\sigma \cup \tau)$-structure with domain D^2 defined as follows. For each k-ary $R \in \sigma$, the structure $\Gamma_1 \boxtimes \Gamma_2$ has the relation

$$R^{\Gamma_1 \boxtimes \Gamma_2} = \big\{ ((a_1,b_1),\dots,(a_k,b_k)) \mid (a_1,\dots,a_k) \in R^{\Gamma_1}, b_1,\dots,b_k \in D \big\} ,$$

and for each k-ary $R \in \tau$, it has the relation

$$R^{\Gamma_1 \boxtimes \Gamma_2} = \big\{ ((a_1,b_1),\dots,(a_k,b_k)) \mid (b_1,\dots,b_k) \in R^{\Gamma_2}, a_1,\dots,a_k \in D \big\} .$$

The proof of the following is straightforward (Proposition 3.3.13 in [2]).

Proposition 4.1. *Suppose that Γ_1 and Γ_2 are ordered, with disjoint signatures and the same domain D. Then the automorphism group of $\Gamma_1 \boxtimes \Gamma_2$ is the product action of the direct product* $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ *on D^2.*

Note that we use here that Γ_1 and Γ_2 are ordered: for $\Gamma_1 := (D; E_1)$ and $\Gamma_2 := (D; E_2)$ where both E_1 and E_2 denote the equality relation $\{(x,x) \mid x \in D\}$, the automorphism group of $\Gamma_1 \boxtimes \Gamma_2$ contains all permutations, and this is clearly not the product action of $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ on D^2.

Proposition 4.2. *Let Γ_1 and Γ_2 be ordered homogeneous structures with the same domain D and disjoint signatures. Then $\Gamma := \Gamma_1 \boxtimes \Gamma_2$ is homogeneous as well.*

Proof. Since Γ_1 and Γ_2 are ordered, the relation $\{((x,y),(u,v)) \mid x = u\}$ and the relation $\{((x,y),(u,v)) \mid y = v\}$ are preserved by isomorphisms between finite substructures of Γ. Hence, an isomorphism μ between finite substructures of Γ gives rise to isomorphisms between finite substructures of Γ_1 and Γ_2, respectively. Those can be extended to automorphisms α, β of Γ_1 and Γ_2, by homogeneity. Then $(x,y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of Γ which extends μ. $\square$

The following is also known under the name *product Ramsey theorem.*

Proposition 4.3. *Let Γ_1 and Γ_2 be ω-categorical structures with the same domain D and disjoint signatures. When $\mathrm{Aut}(\Gamma_1)$ and $\mathrm{Aut}(\Gamma_2)$ are extremely amenable, then the automorphism group of $\Gamma := \Gamma_1 \boxtimes \Gamma_2$ is oligomorphic and extremely amenable.*

Proof. It is easy to bound the number of orbits of n-tuples in Γ by the number of orbits of n-tuples of Γ_1 and Γ_2, so Γ can be seen to be ω-categorical. When G_1 and G_2 are extremely amenable groups, then $G_1 \times G_2$ is extremely amenable as well (see Lemma 6.7 in [10]). The statement follows since $\mathrm{Aut}(\Gamma_1 \boxtimes \Gamma_2)$ is the product action of $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ on D^2 by Proposition 4.1. $\square$

Strong amalgamation will be used via the following lemma. A relation is called *injective* if it only contains tuples with pairwise distinct entries.

Lemma 4.4. *Let τ be a relational signature, and let Γ be a homogeneous τ-structure such that the class of all finite τ-structures that embed into Γ has the strong amalgamation property. Suppose moreover that all relations of Γ are injective. Then every finite structure F that homomorphically maps to Γ also has an injective homomorphism to Γ.*

Proof. Let f be a homomorphism from F to Γ such that the range $f(F)$ of f is maximal. If f is injective, we are done, otherwise F has elements u and v such that $f(u) = f(v)$. Let A be the structure induced by $f(F) \setminus \{f(u)\}$ in Γ, and let B_1 and B_2 be two disjoint copies of the structure induced by $f(F)$ in Γ. Let e_1 be the embedding of A into B_1 that maps an element of $f(F) \setminus \{f(u)\}$ to its copy in B_1. Similarly, there is an embedding $e_2 \colon A \to B_2$ that maps an element of $f(F) \setminus \{f(u)\}$ to its copy in B_2. By strong amalgamation of the age of Γ, there exist embeddings $f_1 \colon B_1 \to \Gamma$ and $f_2 \colon B_2 \to \Gamma$ such that $f_1[e_1[A]] = f_2[e_2[A]] = f_1[B_1] \cap f_2[B_2]$. Then the mapping $f' \colon F \to \Gamma$ defined by $f'(w) = f_1(e_1(f(w)))$ if $w \neq u$, and defined by $f'(w) = f_2(e_2(f(w)))$ if $w \neq v$, is well-defined. To see that it is a homomorphism, note that when $R(x_1, \ldots, x_n)$ holds in F, then at most one of the x_i can be mapped to $f(u)$ since the tuples of R in Γ have only pairwise distinct entries. Since $f(x) \neq f(y)$ implies that $f'(x) \neq f'(y)$, and since moreover $f'(u) \neq f'(v)$, the function f' also has a larger range than f, a contradiction. $\square$

The following is the central lemma connecting the Fraisse-limit of $\mathcal{C}_1 \wedge \mathcal{C}_2$ with the full product of the Fraisse-limits of $\mathcal{C}_1$ and $\mathcal{C}_2$, so that we can ultimately use the product Ramsey theorem.

Lemma 4.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strong amalgamation classes of ordered structures, with disjoint signatures σ and τ, such that all relations of $\mathcal{C}_1$ and all relations of $\mathcal{C}_2$ are injective. Let Γ be the Fraïssé-limit of $\mathcal{C}_1 \wedge \mathcal{C}_2$ with domain D, and suppose that Γ is ω-categorical. Let Γ_1 and Γ_2 be the σ- and τ-reduct of Γ, respectively. If Γ_1 and Γ_2 are cores, then the following structures are isomorphic.*

1. Γ
2. *the substructure induced by* $\{(d,d) \mid d \in D\}$ *in* $\Gamma_1 \boxtimes \Gamma_2$
3. *the model-complete core of* $\Gamma_1 \boxtimes \Gamma_2$

Proof. It is straightforward to verify that $d \mapsto (d,d)$ is an isomorphism between Γ and the substructure of $\Gamma_1 \boxtimes \Gamma_2$ induced by $\{(d,d) \mid d \in D\}$.

To find an isomorphism between Γ and the model-complete core of $\Gamma_1 \boxtimes \Gamma_2$, it suffices to show that Γ is a model-complete core, and that Γ is homomorphically equivalent to $\Gamma_1 \boxtimes \Gamma_2$. We then use that the model-complete core is unique up to isomorphism (Theorem 3.1), which gives us the desired isomorphism. Model-completeness of Γ follows from homogeneity. To show that Γ is a core, let e be an endomorphism of Γ. Then e is an endomorphism of the σ-reduct Γ_1 of Γ, and an endomorphism of the τ-reduct of Γ_2 of Γ. Since both Γ_1 and Γ_2 are cores, e must be an embedding of Γ into Γ, which is what we wanted to show.

We finally show that $\Gamma_1 \boxtimes \Gamma_2$ and Γ are homomorphically equivalent. For one direction, recall that Γ maps to $\Gamma_1 \boxtimes \Gamma_2$ via the mapping $d \mapsto (d,d)$. For the other direction, it suffices to show that every finite substructure F of $\Gamma_1 \boxtimes \Gamma_2$ homomorphically maps to Γ, by a standard compactness argument and ω-categoricity of Γ (see e.g. Lemma 3.1.5 in [2]). By Lemma 4.4, there is an *injective* homomorphism h_1 from the σ-reduct of F to Γ_1 (recall here that the order of Γ_1 is by assumption strict). Similarly, there is an injective homomorphism h_2 from the τ-reduct of F into Γ_2. Let U be the $(\sigma \cup \tau)$-structure with the same domain as F, and with relations defined as follows: for each $R \in \sigma$ of arity k, a k-tuple t of elements of U is in R^U if and only if $h_1(t)$ is in R^{Γ_1}. Similarly we define R^U for relations $R \in \tau$, with h_2 taking the role of h_1 and Γ_2 taking the role of Γ_1. Clearly, h_1 is an embedding of the σ-reduct U_1 of U into Γ_1, and h_2 is an embedding of the τ-reduct U_2 of U into Γ_2. Therefore, $U_1 \in \mathcal{C}_1$ and $U_2 \in \mathcal{C}_2$. By definition of $\mathcal{C} := \mathcal{C}_1 \wedge \mathcal{C}_2$, we have that $U \in \mathcal{C}$, and there is an embedding e of U into Γ. Then e is the desired homomorphism from F to Γ. □

Proof of Theorem 1.5. Let Γ be the Fraïssé-limit of $\mathcal{C}_1 \wedge \mathcal{C}_2$, and let Γ_1 and Γ_2 be the σ- and τ-reduct of Γ, respectively. It can be shown by a straightforward back-and-forth argument that Γ_1 and Γ_2 are also homogeneous, and hence isomorphic to the Fraïssé-limit of $\mathcal{C}_1$ and $\mathcal{C}_1$, respectively. Therefore, Γ_1 and Γ_2 are ω-categorical by assumption. Moreover, since $\mathcal{C}_1$ and $\mathcal{C}_2$ have the Ramsey property, by Corollary 2.4 there are linear orders $<^{\Gamma_1}$ and $<^{\Gamma_2}$ with quantifier-free first-order definitions φ_1 and φ_2 in Γ_1 and Γ_2, respectively.

Let Γ_1^* be the structure with the same domain as Γ_1 whose relations are exactly the injective relations that are first-order definable in Γ_1. Note that this includes in particular the linear order $<^{\Gamma_1}$, and so Γ_1^* is ordered. Since Γ_1^* contains an n-ary relation for each orbit of n-tuples of distinct elements from Γ, we have that Γ_1^* is homogeneous and a core, and has the same (oligomorphic) automorphism group as Γ_1. Moreover, observe that the algebraic closure operator only depends on the automorphism group of Γ_1, and it follows by Proposition 1.1 that that also the age of Γ_1^* has strong amalgamation. We write σ^* for the signature of Γ_1^*.

Analogously we define the structure Γ_2^* from Γ_2; we choose the signature τ^* for Γ_2^* such that τ^* is disjoint from σ^*. Finally, let Γ^* be the $(\tau^* \cup \sigma^*)$-structure whose domain equals the domain of Γ_1 and Γ_2 and which is given uniquely by the requirement that it is both an expansion of Γ_1^* and an expansion of Γ_2^*. Then $\mathrm{Aut}(\Gamma^*) = \mathrm{Aut}(\Gamma)$, because an orbit of an n-tuple t in Γ is uniquely given by the orbit of t in Γ_1^* and the orbit of t in Γ_2^*. Since Γ^* contains relation symbols for tuples of pairwise distinct elements for the orbits of n-tuples in Γ_1 and in Γ_2, it is for the same reason homogeneous and therefore the Fraïssé-limit of its age. Theorem 2.1 implies that the automorphism group of Γ^* is oligomorphic because Γ is ω-categorical, and that Γ^* is ω-categorical.

The groups $\mathrm{Aut}(\Gamma_1^*) = \mathrm{Aut}(\Gamma_1)$ and $\mathrm{Aut}(\Gamma_2^*) = \mathrm{Aut}(\Gamma_2)$ are oligomorphic, and extremely amenable by Corollary 2.4. By Proposition 4.2, $\Gamma_1^* \boxtimes \Gamma_2^*$ is homogeneous, and by Proposition 4.3, $\mathrm{Aut}(\Gamma_1^* \boxtimes \Gamma_2^*)$ is extremely amenable. Then Theorem 3.3 (again in combination with Corollary 2.4) shows that the model-complete core of $\Gamma_1^* \boxtimes \Gamma_2^*$ has an extremely amenable automorphism group G. By Lemma 4.5, the model-complete core of $\Gamma_1^* \boxtimes \Gamma_2^*$ is isomorphic to Γ^*, and hence $\mathrm{Aut}(\Gamma^*) = \mathrm{Aut}(\Gamma)$ is extremely amenable. We conclude by Corollary 2.4 that $\mathcal{C}_1 \wedge \mathcal{C}_2$ has the Ramsey property. $\square$

5 Forgetting one order

We finally prove Proposition 1.6: let $\mathcal{C}_1$ and $\mathcal{LO} \wedge \mathcal{C}_2$ be Ramsey classes with strong amalgamation and ω-categorical Fraïssé-limits, and suppose that $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{LO}$ have pairwise disjoint relational signatures. We have to show that $\mathcal{C}_1 \wedge \mathcal{C}_2$ has the Ramsey property.

Proof of Proposition 1.6. We use the fact that $\mathcal{C}_3 := \mathcal{C}_1 \wedge (\mathcal{LO} \wedge \mathcal{C}_2)$ has the Ramsey property by Theorem 1.5. Let Γ be the Fraïssé-limit of $\mathcal{C}_1$. By Corollary 2.4, there is a linear order $<$ on the elements of Γ that has a quantifier-free first-order definition $\varphi(x, y)$ in Γ.

To show that $\mathcal{C}_1 \wedge \mathcal{C}_2$ has the Ramsey property, let A and B be from $\mathcal{C}_1 \wedge \mathcal{C}_2$. Let A', B' be the expansion of A, B by the relation $<$ defined by φ over A and B, respectively. Note that $A', B' \in \mathcal{C}_3$. Since $\mathcal{C}_3$ has the Ramsey property, there exists a $C' \in \mathcal{C}_3$ such that $C' \to (B')_r^{A'}$. Let C be the reduct of C' where we drop the relation $<$. We claim that $C \to (B)_r^A$. Let $\chi\colon \binom{C}{A} \to \{1, \ldots, r\}$ be arbitrary. We define a coloring $\chi'\colon \binom{C'}{A'} \to \{1, \ldots, r\}$ as follows. Let e be an arbitrary embedding of A' into C'. Since A is a reduct of A', and C is a reduct of C' with the same signature, the mapping e is also an embedding of A into

C. Therefore, e is in the range of χ, and we can define $\chi'(e) := \chi(e)$. Since $C' \to (B')^{A'}_r$, there exists an $f \in \binom{C'}{B'}$ such that χ' is constant c on $\binom{f[B']}{A'}$. By the same argument as above, f is also an embedding of B into C. We claim that χ is constant on $\binom{f[B]}{A}$. Let e be an arbitrary embedding of A into $f[B]$. Recall that A' and B' are the expansion of A and B by the relation $<$ defined by φ. Since embeddings preserve quantifier-free formulas, e preserves in particular φ. Therefore, the mapping e is an embedding of A' into the substructure $f[B']$ of C'. In particular, e is in the range of χ', and $\chi'(e) = c$. It follows that $\chi(e) = c$, which concludes the proof that χ is constant on $\binom{f[B]}{A}$. $\square$

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