A note on coloring line arrangements

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Abstract

We show that the lines of every arrangement of n lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors such that no face of the arrangement is monochromatic. This improves a bound of Bose et al. by a $\Theta(\sqrt{\log n})$ factor. Any further improvement on this bound would also improve the best known lower bound on the following problem of Erdős: estimate the maximum number of points in general position within a set of n points containing no four collinear points.

Keywords: Arrangements of lines, chromatic number, sparse hypergraphs.

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1 Introduction

Given a simple arrangement \mathcal{A} of a set L of lines in \mathbb{R}^2 (no parallel lines and no three lines going through the same point), decomposing the plane into the set C of cells (i.e. maximal connected components of $\mathbb{R}^2 \setminus L$), Bose et al. [1] defined a hypergraph $\mathcal{H}_{\text{line-cell}} = (L, C)$ with the vertex set L (the set of lines of \mathcal{A}), and each hyperedge $c \in C$ being defined by the set of lines forming the boundary of a cell of \mathcal{A} . They initiated the study of the chromatic number of $\mathcal{H}_{\text{line-cell}}$, and proved that for |L| = n, $\chi(\mathcal{H}_{\text{line-cell}}) = O(\sqrt{n})$ and $\chi(\mathcal{H}_{\text{line-cell}}) = \Omega\left(\frac{\log n}{\log \log n}\right)$. In other words, they proved that the lines of every simple arrangement of n lines can be colored with $O(\sqrt{n})$ colors so that there is no monochromatic face; furthermore, they provided an intricate construction of a simple arrangement of nlines that requires $\Omega\left(\frac{\log n}{\log \log n}\right)$ colors.

In this short note, we improve their upper bound by a $\Theta(\sqrt{\log n})$ factor, and extend it to not necessarily simple arrangements.

Theorem 1. The lines of every arrangement of n lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors so that no face of the arrangement is monochromatic.

A set of points in the plane is in general position if it does not contain three collinear points. Let $\alpha(S)$ denote the maximum number of points in general position in a set Sof points in the plane, and let $\alpha_4(n)$ be the minimum of $\alpha(S)$ taken over all sets S of npoints in the plane with no four point on a line. Erdős pointed out that $\alpha_4(n) \leq n/3$ and suggested the problem of determining or estimating $\alpha_4(n)$. Füredi [3] proved that $\Omega(\sqrt{n \log n}) \leq \alpha_4(n) \leq o(n)$.

We observe that any improvement of the bound in Theorem 1 would immediately imply a better lower bound for $\alpha_4(n)$. Indeed, suppose that $\chi(A) \leq k(n)$ for any arrangement of n lines, and let P be a set of n points, no four on a line. Let P^* be the dual arrangement of a slightly perturbed P (according to the usual point-line duality, see, e.g., [2, § 8.2]). Color P^* with k(n) colors such that no face is monochromatic, let $S^* \subseteq P^*$ be the largest color class, and let S be its dual point set. Observe that the size of S is at least n/k(n)and it does not contain three collinear points, since the three lines that correspond to any three collinear points in P bound a face of size three in P^* .

2 Proof of Theorem 1

Let \mathcal{A} be an arrangement of a set L of n lines, decomposing the plane into the set C of cells, and let $\mathcal{H}_{\text{line-cell}}$ be the corresponding hypergraph (defined as in the previous section). We show that $\chi(\mathcal{H}_{\text{line-cell}}) = O\left(\sqrt{\frac{n}{\log n}}\right)$.

An independent set in $\mathcal{H}_{\text{line-cell}}$ is a set $S \subset L$ such that for every $c \in C$, c is not a subset of S (in other words, no cell of \mathcal{A} has its boundary formed only by lines in S). The proof is based on the following fact.

Theorem 2. There is an absolute constant c > 0 such that the size $\alpha(\mathcal{H}_{line-cell})$ of the maximum independent set is at least $c\sqrt{n\log n}$.

We color the lines in \mathcal{A} so that no face is monochromatic by following the same method as in [1] (where they used the weaker version of Theorem 2 stating $\alpha(\mathcal{H}_{\text{line-cell}}) = \Omega(\sqrt{n})$). That is, we iteratively find a large independent set of lines (whose existence is guaranteed by Theorem 2), color them with the same (new) color, and remove them from \mathcal{A} .

Clearly, this algorithm produces a valid coloring. We verify, by induction on n, that at most $\frac{2}{c}\sqrt{n/\log n}$ colors are used in this coloring. We assume the bound is valid for all $n \leq 256$ (by taking sufficiently small c > 0). For n > 256, we have $\log 4 < \frac{1}{4} \log n$. Let i be the smallest integer such that after i iterations the number of remaining lines is at most n/4. Since in each of these iterations at least $c\sqrt{\frac{n}{4}\log\frac{n}{4}} \ge c\sqrt{\frac{n}{8}\log n}$ vertices (lines) are removed, $i \leq \frac{n/4}{c\sqrt{\frac{n}{8}\log n}} \leq \frac{1}{\sqrt{2c}}\sqrt{n/\log n}$. Therefore, by the induction hypothesis the number of colors that the algorithm uses is at most

$$\begin{split} i + \frac{2}{c}\sqrt{\frac{\frac{n}{4}}{\log\frac{n}{4}}} &\leqslant \frac{1}{\sqrt{2}c}\sqrt{\frac{n}{\log n}} + \frac{1}{c}\sqrt{\frac{n}{\log n - \frac{1}{4}\log n}} \\ &< \frac{1}{\sqrt{2}c}\sqrt{\frac{n}{\log n}} + \frac{\sqrt{4/3}}{c}\sqrt{\frac{n}{\log n}} < \frac{2}{c}\sqrt{\frac{n}{\log n}}. \end{split}$$

The proof of Theorem 2 is based on a result on independent sets in sparse hypergraphs. Given a hypergraph \mathcal{H} on a vertex set V, the sub-hypergraph $\mathcal{H}[X]$ induced by $X \subset V$ consists of all edges of \mathcal{H} that are contained in X. A hypergraph $\mathcal{H} = (V, E)$ is k-uniform if every edge $e \in E$ has size k. Given a k-uniform hypergraph \mathcal{H} and a set $S \subset V$ with |S| = k - 1, the co-degree of S is the number of all vertices $v \in V$ such that $S \cup \{v\} \in E$. Kostochka et al. [4] proved that if \mathcal{H} is a k-uniform hypergraph, $k \ge 3$, with all co-degrees at most d, then $\alpha(\mathcal{H}) \ge c_k \left(\frac{n}{d} \log \frac{n}{d}\right)^{\frac{1}{k-1}}$, where $c_k > 0$.

In fact, a careful look at their proof reveals the following result, that we state for 3-uniform hypergraphs, since this is the case that we need.

Lemma 2.1 ([4]). Let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph on |V| = n vertices with all co-degrees at most $d, d < n/(\log n)^{12}$. Let X be a random subset of V, obtained by choosing each vertex of V independently with probability $p = \frac{n^{-2/5}}{(d\log \log \log n)^{3/5}}$. Let Z be a set chosen uniformly at random among all the independent sets of $\mathcal{H}[X]$. Then, with high probability $|Z| = \Omega(\sqrt{n \log n})$.

With Lemma 2.1 in hand we can now prove Theorem 2.

Proof of Theorem 2: A cell of an arrangement \mathcal{A} is called an *r*-cell, if *r* lines of *L* are forming its boundary. Let $\mathcal{H}_{\Delta} \subset \mathcal{H}_{\text{line-cell}}$ be the 3-uniform hypergraph with the vertex set *L* being the set of lines, and each hyperedge defined by the triple of lines forming the boundary of a 3-cell of \mathcal{A} . Since any two lines can participate in the boundaries of at most four 3-cells of \mathcal{A} , all co-degrees of \mathcal{H} are at most d = 4. Now, as in Lemma 2.1, let *X* be a random subset of *L*, obtained by choosing each line in *L* independently with probability $p = \frac{n^{-2/5}}{(4 \log \log \log n)^{3/5}}$. Since there are $O(n^2)$ faces in \mathcal{A} and O(n) of them are 2-cells (since every line can bound at most four such faces), the expected number of 2-cells of \mathcal{A} in $\mathcal{H}_{\text{line-cell}}[X]$ is $O(p^2n) = o(\sqrt{n \log n})$, and the expected number of r-cells, $r \ge 4$, of \mathcal{A} in $\mathcal{H}_{\text{line-cell}}[X]$ is $O(p^4n^2) = o(\sqrt{n \log n})$. From Lemma 2.1 it follows that there exists a set $Z \subset X \subset L$ of size $\Omega(\sqrt{n \log n})$, that is an independent set of $\mathcal{H}_{\Delta}[X]$, and such that the number of r-cells, $r \ne 3$, of \mathcal{A} in $\mathcal{H}_{\text{line-cell}}[Z]$ is $o(\sqrt{n \log n})$. Removing from Z one vertex (line) for each such r-cell, we obtain an independent set of $\mathcal{H}_{\text{line-cell}}$ of size $\Omega(\sqrt{n \log n})$. \Box

References

- P. Bose, J. Cardinal, S. Collette, F. Hurtado, S. Langerman, M. Korman, and P. Taslakian, Coloring and guarding arrangements, *Discrete Mathematics and Theoretical Computer Science* 15 (2013), 139–154. Also in: 28th European Workshop on Computational Geometry (EuroCG), March 19–21, 2012, Assisi, Perugia, Italy, 89–92.
- [2] M. de Berg, O. Cheong, M. van Krefeld, M. Overmars, *Computational Geometry: Algorithms and Applications*, 3rd edition, Springer, 2008.
- [3] Z. Füredi, Maximal independent subsets in Steiner systems and in planar sets, SIAM J. Disc. Math. 4 (1991), 196–199.
- [4] A. Kostochka, D. Mubayi, J. Verstraëte, On independent sets in hypergraphs, Random Structures and Algorithms 44 (2014), 224–239.