Total transversals and total domination in uniform hypergraphs

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Abstract

In 2012, the first three authors established a relationship between the transversal number and the domination number of uniform hypergraphs. In this paper, we establish a relationship between the total transversal number and the total domination number of uniform hypergraphs. We prove tight asymptotic upper bounds on the total transversal number in terms of the number of vertices, the number of edges, and the edge size.

Keywords: Total domination; total transversal; hypergraph.

1 Introduction

In this paper, we explore the study of total domination in hypergraphs. We establish a relationship between the transversal number and the total domination number of of uniform hypergraphs. We introduce the concept of a total transversal in a hypergraph and prove a general upper bound on the total domination number of a uniform hypergraph in terms of its total transversal number.

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V(H), E(H))$ is a finite set $V(H)$ of elements, called *vertices*, together with a finite multiset $E(H)$ of subsets of $V(H)$, called *hyperedges* or simply *edges*. If the hypergraph H is clear from the context, we simply write $V = V(H)$ and $E = E(H)$. We shall use the notation $n_H = |V|$ (or $n(H)$) and $m_H = |E|$ (or $m(H)$), and sometimes simply n and m without subscript if the actual H need not be emphasized, to denote the *order* and *size* of H, respectively. The edge set E is often allowed to be a multiset in the literature, but in the present context we exclude multiple edges. Also, in the problems studied here, one may assume that $|V(e)| \geqslant 2$ holds for all $e \in E$. An *isolated edge* in H is an edge in H that does not intersect any other edge in H. A *linear hypergraph* is a hypergraph in which every two edges intersect in at most one vertex.

A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. The *degree* of a vertex v in H, denoted by $d_H(v)$ or $d(v)$ if H is clear from the context, is the number of edges of H which contain v. A vertex of degree k is called a *degree-k vertex*. The number of degree-1 vertices in H is denoted by $n_1(H)$. The minimum degree among the vertices of H is denoted by $\delta(H)$ and the maximum degree by $\Delta(H)$.

Two vertices x and y in H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq V(e)$. The *open neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v. We call a vertex in $N(v)$ a *neighbor* of v. Two vertices x and y in H are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components.

For a hypergraph H, the *open neighborhood hypergraph* of H, denoted by $\mathrm{ONH}(H)$, is the hypergraph with vertex set $V(H)$ and edge set $\{N_H(v) \mid v \in V(H)\}$ consisting of the open neighborhoods of vertices of $V(H)$ in H.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A *strong transversal*, often called a *2-transversal*, in H is a transversal that contains at least two vertices from every edge in H. The *strong transversal number* $\tau_s(H)$ of H is the minimum size of a strong transversal in H. Transversals in hypergraphs are well studied in the literature (see, for example, [3, 5, 7, 12, 13, 14, 15, 16, 21, 22, 23]).

We define a *total transversal* in H to be transversal T in H with the additional property

that every vertex in T has at least one neighbor in T, and we define the *total transversal number* $\tau_t(H)$ of H to be the minimum size of a total transversal in H.

For a subset $X \subset V(H)$ of vertices in H, we define $H - X$ to be the hypergraph obtained from H by deleting the vertices in X and all edges incident with X, and deleting resulting isolated vertices, if any. We note that if T' is a transversal in $H - X$, then $T' \cup X$ is a transversal in H. If $X = \{x\}$, then we write $H - X$ simply as $H - x$.

A *dominating set* in a hypergraph $H = (V, E)$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. Equivalently, every vertex $v \in V \setminus D$ is adjacent with a vertex in D. The *domination number* $\gamma(H)$ is the minimum cardinality of a dominating set in H. A vertex v in H is said to be a *dominating vertex* if it is adjacent to every other vertex in H. A *total dominating set*, abbreviated TD-set, in a hypergraph $H = (V, E)$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap (D \setminus \{v\}) \neq \emptyset$. Equivalently, D is a TD-set in H if every vertex in H is adjacent with a vertex in D. The *total domination number* $\gamma_t(H)$ is the minimum cardinality of a TD-set in H. A TD-set in H of cardinality $\gamma_t(H)$ is called a $\gamma_t(H)$-set.

While domination and total domination in graphs is very well studied in the literature (see, for example, [8, 9, 10, 17]), domination in hypergraphs was introduced relatively recently by Acharya [1] and studied further in [2, 4, 11, 18, 19] and elsewhere.

An *edge-cover* in a graph G is a set of edges such that every vertex in G is incident with at least one edge in the edge-cover. We define a *total edge-cover* in G to be an edge-cover that induces a subgraph with no isolated edge. We let $\mathrm{ec}_t(G)$ denote the minimum cardinality of a total edge-cover in G. A *2-section graph*, $(H)_2$, of a hypergraph H is defined as the graph with the same vertex set as H and in which two edges are adjacent in $(H)_2$ if and only if they belong to a common edge in H.

The interplay between total domination in graphs and transversals in hypergraphs has been studied in several papers (see, for example, [13, 14, 22]). The first three authors [4] establish a relationship between the transversal number and the domination number of uniform hypergraphs. In the present work, we establish a relationship between the total transversal number and the total domination number of uniform hypergraphs.

1.1 Key Definitions

We shall need the following definitions.

Definition 1. For an integer $k \geqslant 2$, let $\mathcal{H}_k$ be the class of all k-uniform hypergraphs containing no isolated vertices or isolated edges or multiple edges. Further, for $k \geqslant 3$ let $\mathcal{H}_k^*$ consist of all hypergraphs in $\mathcal{H}_k$ that have no two edges intersecting in $k - 1$ vertices. We note that $\mathcal{H}_k^*$ is a proper subclass of $\mathcal{H}_k$.

Definition 2. For an integer $k \geqslant 2$, let

$$b_k = \sup_{H \in \mathcal{H}_k} \frac{\tau_t(H)}{n_H + m_H}.$$

2 Main Results

We shall prove the following upper bounds on the total domination number of a uniform hypergraph in terms of its total transversal number, order and size. A proof of Theorem 3 is presented in Section 5.1.

Theorem 3. *For $k \geqslant 3$, if $H \in \mathcal{H}_k$, then $\gamma_t(H) \leqslant \left(\max \left\{ \dfrac{2}{k+1}, b_{k-1} \right\} \right) n_H$.*

In view of Theorem 3, it is of interest to determine the value of b_k for $k \geqslant 2$. A proof of Theorem 4 is presented in Section 5.2.

Theorem 4. $b_2 = \frac{2}{5}$, $b_3 = \frac{1}{3}$, *and $b_4 \leqslant \frac{1}{3}$. Further for $k \geqslant 5$, we have $b_k \leqslant \frac{2}{7}$.*

By Theorem 4, we observe that

$$b_{k-1} \leqslant \frac{2}{k+1} \qquad \text{for } k \in \{3, 4, 5, 6\}.$$

Hence as a consequence of Theorem 3 and Theorem 4, and the well-known fact (see, [6]) that if $H \in \mathcal{H}_2$, then $\gamma_t(H) \leqslant 2n_H/3$, we have the following result. The sharpness of the bound in Theorem 5 is shown in Observation 17 in Section 3.

Theorem 5. *For $k \in \{2, 3, 4, 5, 6\}$, if $H \in \mathcal{H}_k$, then $\gamma_t(H) \leqslant 2n_H/(k+1)$, and this bound is sharp.*

The following result is a strengthening of the upper bound of Theorem 3 if we restrict the edges to intersect in at most $k - 2$ vertices. A proof of Theorem 6 is presented in Section 5.3

Theorem 6. *For $k \geqslant 4$, if $H \in \mathcal{H}_k^*$, then $\gamma_t(H) \leqslant \left(\max \left\{ \dfrac{2}{k+2}, b_{k-1} \right\} \right) n_H$.*

Corollary 7. *For $k \geqslant 4$, if $H \in \mathcal{H}_k^*$, then $\gamma_t(H) \leqslant n_H/3$.*

The following result establishes a tight asymptotic bound on b_k for k sufficiently large. A proof of Theorem 8 is presented in Section 6.

Theorem 8. *For all $k \geqslant 2$, $b_k = (1 + \mathrm{o}(1))\dfrac{\ln(k)}{k}$.*

Theorem 8 implies that the inequality $b_{k-1} \leqslant 2/(k+1)$ is not valid when k is large enough. Using a result due to Alon [3], the first three authors proved the following result.

Theorem 9. ([4]) *For all $k \geqslant 3$,*

$$\sup_{H \in \mathcal{H}_k} \frac{\gamma(H)}{n_H} = (1 + \mathrm{o}(1))\frac{\ln(k-1)}{k-1}.$$

By definition, $\gamma_t(H) \geqslant \gamma(H)$ for every hypergraph H with no isolated vertex. Hence as a consequence of Theorem 3, Theorem 8 and Theorem 9, we have the following result.

Theorem 10. *For all $k \geqslant 3$,*

$$\sup_{H \in \mathcal{H}_k} \frac{\gamma_t(H)}{n_H} = (1 + \mathrm{o}(1))\frac{\ln(k-1)}{k-1} = (1 + \mathrm{o}(1))\frac{\ln(k)}{k}.$$

We remark that Theorem 10 implies that Theorem 5 is not true for large k.

3   Known Results and Observations

Cockayne et al. [6] established the following bound on the total domination number of a connected graph in terms of its order.

Theorem 11. *([6]) If G is a connected graph of order $n \geqslant 3$, then $\gamma_t(G) \leqslant 2n/3$.*

We shall need the following result due to Kelmans and Mubayi [20].

Theorem 12. *([20]) A cubic graph G contains at least $\lceil |V(G)|/4 \rceil$ vertex disjoint P_3's.*

The following result shows that the total domination number of a hypergraph H is precisely the total domination of its 2-section graph and the transversal number of its open neighborhood hypergraph.

Observation 13. *Let H be a hypergraph with no isolated vertex. Then the following holds.*
  (a) $\gamma_t(H) = \gamma_t((\mathcal{H})_2)$.
  (b) $\gamma_t(H) = \tau(\mathrm{ONH}(H))$.

Proof. (a) Part (a) follows readily from the fact that two vertices in H are adjacent in H if and only if they are adjacent in the 2-section graph $(H)_2$ of H.
  (b) On the one hand, every TD-set in H contains a vertex from the open neighborhood of each vertex in H, and is therefore a transversal in ONH, implying that $\tau(\mathrm{ONH}(H)) \leqslant \gamma_t(H)$. On the other hand, every transversal in ONH contains a vertex from the open neighborhood of each vertex of H, and is therefore a TD-set in G, implying that $\gamma_t(H) \leqslant \tau(\mathrm{ONH}(H))$. Consequently, $\gamma_t(H) = \tau(\mathrm{ONH}(H))$. $\square$

We shall need the following properties of hypergraphs in the family $\mathcal{H}_k$.

Observation 14. *For $k \geqslant 2$, if $H \in \mathcal{H}_k$, then the following hold.*
  (a) $n_H \geqslant k + 1$, $m_H \geqslant 2$ and $\Delta(H) \geqslant 2$.
  (b) $2n_H - n_1(H) \geqslant 2k$.

Proof. Part (a) is immediate from the definition of the family $\mathcal{H}_k$. To prove Part (b), let $n_{\geqslant 2}(H)$ denote the number of vertices in H of degree at least 2. Let e and f be any two intersecting edges in H and suppose they intersect in ℓ vertices, and so $|e \cup f| = 2k - \ell$. Then, $n_H \geqslant 2k - \ell \geqslant 2k - n_{\geqslant 2}(H)$, or, equivalently, $2n_H - n_1(H) = n_H + n_{\geqslant 2}(H) \geqslant 2k$. $\square$

In order to state our next result, we first define two families of hypergraphs.

Definition 15. For $k \geqslant 2$, let H be obtained from a hypergraph $F \in \mathcal{H}_k$ as follows. For each vertex v in F, add k new vertices $v_1, v_2, \ldots, v_k$ and two new k-edges $\{v, v_1, \ldots, v_{k-1}\}$ and $\{v_1, v_2, \ldots, v_k\}$. Let $\mathcal{F}_k$ denote the family of all such hypergraphs H.

Definition 16. For $k \geqslant 3$, let H be obtained from a hypergraph $F \in \mathcal{H}_k^*$ as follows. For each vertex v in F, add $k+1$ new vertices $v_1, v_2, \ldots, v_{k+1}$ and two new k-edges $\{v, v_1, v_2, \ldots, v_{k-1}\}$ and $\{v_2, v_3, \ldots, v_{k+1}\}$. Let $\mathcal{F}_k^*$ denote the family of all such hypergraphs H.

Observation 17. *The following holds.*
    (a) *For $k \geqslant 2$, if $H \in \mathcal{F}_k$, then $\gamma_t(H) = 2n_H/(k+1)$.*
    (b) *For $k \geqslant 3$, if $H \in \mathcal{F}_k^*$, then $\gamma_t(H) = 2n_H/(k+2)$.*

Proof. For $k \geqslant 2$, let $H \in \mathcal{F}_k$ be constructed as in Definition 15. Then, $H \in \mathcal{H}_k$ and $n_H = (k+1)n_F$. Every TD-set in H contains at least two vertices in $\{v, v_1, v_2, \ldots, v_k\}$, implying that $\gamma_t(H) \geqslant 2n_F$. However, the set $V(F) \cup T$, where $|T| = n_F$ and $T \subseteq V(H) \setminus V(F)$ consists of one added neighbor of each vertex in $V(F)$, is a TD-set in H, implying that $\gamma_t(H) \leqslant 2n_F$. Consequently, $\gamma_t(H) = 2n_F = 2n_H/(k+1)$. For $k \geqslant 3$, let $H \in \mathcal{F}_k^*$ be constructed as in Definition 16. Then, $H \in \mathcal{H}_k^*$ and $\gamma_t(H) = 2n_F = 2n_H/(k+2)$. $\square$

4   Preliminary Result

We show that total transversals of a 2-regular hypergraph H correspond to total edge-covers in the dual multigraph, G_H, of H, where the vertices of G_H are the edges of H and the edges of G_H correspond to the vertices of H: if a vertex of H is contained in the edges e and f of H, then the corresponding edge of the multigraph G_H joins vertices e and f of G_H.

Lemma 18. *If H is a linear 2-regular hypergraph and G_H is the dual of H, then $\tau_t(H) = \mathrm{ec}_t(G_H)$.*

Proof. By the linearity of H, the multigraph G_H is in fact a graph. Let T be a total transversal in H. Then the edges of G_H corresponding to vertices in T form an edge-cover in G_H. Further if u and v are neighbors in H that belong to T, then the edges in G_H corresponding to the vertices u and v in H have a vertex in common, namely the edge in H that contains both u and v. This implies that the edge-cover in G_H corresponding to the total transversal T in H is a total edge-cover in G_H. Similarly, every total edge-cover in G_H corresponds to a total transversal in H. Therefore, $\tau_t(H) = \mathrm{ec}_t(G_H)$. $\square$

5 Proofs of Main Results

5.1 Proof of Theorem 3

In this section, we present a proof of Theorem 3. Recall its statement.

Theorem 3. *For $k \geqslant 3$, if $H \in \mathcal{H}_k$, then $\gamma_t(H) \leqslant \left(\max \left\{ \dfrac{2}{k+1}, b_{k-1} \right\} \right) n_{_H}$.*

Proof of Theorem 3. Suppose to the contrary that the theorem is not true. Let $H \in \mathcal{H}_k$ be a counterexample with $n_{_H} + m_{_H}$ a minimum. In what follows we present a series of claims describing some structural properties of H which culminate in the implication of its non-existence.

Claim 19. *The following properties hold in the hypergraph H.*
    (a) *H is connected.*
    (b) *The deletion of any edge in H creates an isolated vertex or an isolated edge.*
    (c) *There is no dominating vertex in H.*

Proof of Claim 19. Part (a) is immediate from the minimality of H. Part (b) is also immediate since the deletion of an edge cannot decrease the total domination number. To prove Part (c), suppose that H contains a dominating vertex v. The vertex v and any one of its neighbors forms a TD-set in H, implying that $\gamma_t(H) = 2$. As $H \in \mathcal{H}_k$, there is no isolated vertex or isolated edge in H, implying that $n_{_H} \geqslant k + 1$. Hence, $\gamma_t(H) \leqslant 2n_{_H}/(k+1)$, contradicting the fact that H is a counterexample to the theorem. This proves Part (c). $_{(\square)}$

Claim 20. *Every edge in H contains at least one degree-1 vertex.*

Proof of Claim 20. Suppose to the contrary that there is an edge e that does not contain any degree-1 vertices. Thus every vertex contained in e has degree at least 2 in H. By Claim 19(b), there is therefore an edge, e_1, which would become isolated after the deletion of the edge e from H. Thus, every vertex in $e \cap e_1$ has degree 2 in H, while every vertex in $e_1 \setminus e$ has degree 1 in H. Let $v \in e \cap e_1$. Then, $d_H(v) = 2$. By Claim 19(a), H is connected and by Claim 19(c), the vertex v is not a dominating vertex of H, implying that there exists an edge, e_2, such that $v \notin e_2$ but e_2 intersects e. Since $e \neq e_2$ and $v \notin e_2$, we note that $e_1 \cap e_2 = \emptyset$. Let $u \in e \cap e_2$ and note that $u \notin e_1$.

Initially we set $T = \emptyset$ and we now construct a hypergraph H' from H as follows. We delete all edges incident with u or v or with both u and v and we delete any resulting isolated vertices. Further we add both vertices u and v to the set T. We note that the edges e and e_1 are both deleted, implying that every vertex in e_1 becomes an isolated vertex. Further since we remove all edges incident with u, the vertex u becomes an isolated vertex. We therefore delete at least $k + 1$ vertices and we add two vertices to T. If this process creates an isolated edge, then such an isolated edge necessarily contains a vertex that is adjacent to at least one of u and v (for otherwise it would be an isolated edge in H, a contradiction). From each such isolated edge f, if any, we choose one vertex that is a neighbor of u or v and add it into T, and delete the k vertices in f. Hence,

$|T| = 2 + \ell$, where $\ell \geqslant 0$ denotes the number of isolated edges created when removing u and v.

Let n' denote the number of vertices in H that are not deleted in the process (possibly, $n' = 0$). At least $k + 1 + k\ell$ vertices were deleted from H. Thus, $n' \leqslant n_H - k - 1 - k\ell$, implying that

$$
\begin{aligned}
\left(\frac{2}{k+1}\right)(n_H - n') & \geqslant \left(\frac{2}{k+1}\right)(k + 1 + k\ell) \\
& = 2 + \left(\frac{2k}{k+1}\right)\ell \\
& \geqslant 2 + \ell \\
& = |T|.
\end{aligned}
$$

If $n' = 0$, then the set T is a TD-set in H, implying that $\gamma_t(H) \leqslant |T| \leqslant 2n_H/(k+1)$, a contradiction. Hence, $n' > 0$. Let H' denote the resulting hypergraph on these n' vertices. Let H' have size m'. By construction, the hypergraph H' is in the family $\mathcal{H}_k$. In particular, we note that $n' \geqslant k + 1$. By the minimality of H, we have that

$$
\gamma_t(H') \leqslant \left(\max\left\{\frac{2}{k+1}, b_{k-1}\right\}\right)n'.
$$

Let T' be a $\gamma_t(H')$-set and note that the set $T \cup T'$ is a TD-set of H. Suppose that $2/(k+1) \geqslant b_{k-1}$. Then, $|T'| \leqslant 2n'/(k+1)$, and so

$$
\gamma_t(H) \leqslant |T \cup T'| \leqslant \left(\frac{2}{k+1}\right)(n_H - n') + \left(\frac{2}{k+1}\right)n' = \left(\frac{2}{k+1}\right)n_H,
$$

a contradiction. Hence, $2/(k+1) < b_{k-1}$. Thus, $|T'| \leqslant b_{k-1}n'$, and so

$$
\gamma_t(H) \leqslant |T \cup T'| \leqslant \left(\frac{2}{k+1}\right)(n_H - n') + b_{k-1}n' < b_{k-1}(n_H - n') + b_{k-1}n' = b_{k-1}n_H,
$$

a contradiction. This completes the proof of Claim 20. $\square$

We now return to the proof of Theorem 3. By Claim 20, every edge in H contains at least one degree-1 vertex. If there are two edges, f_1 and f_2, in H that intersect in $k - 1$ vertices, then for $j \in \{1, 2\}$, the edge f_j contains exactly one vertex, v_j say, not in f_{3-j} and this vertex has degree 1 in H. Thus if we delete the vertices v_1 and v_2 from H, then we would create a multiple edge, namely $f_1' = f_1 \setminus \{v_1\}$ and $f_2' = f_2 \setminus \{v_2\}$. Let H' be the hypergraph obtained from H by deleting exactly one degree-1 vertex from each edge and by replacing resulting multiple edges, if any, by single edges. Let H' have order n' and size m'. Then, $n' = n_H - m_H$ and $m' \leqslant m_H$. Thus, $n' + m' \leqslant n_H$.

Claim 21. $H' \in \mathcal{H}_{k-1}$ and $\tau_t(H') \leqslant b_{k-1}n_H$.

Proof of Claim 21. If H' contains an isolated edge, then every vertex in such an isolated edge would be a dominating vertex in H, contradicting Claim 19(c). Hence, H' contains no isolated edge. By construction, H' has no multiple edges and no isolated vertices. Therefore, $H' \in \mathcal{H}_{k-1}$. We note that $k - 1 \geqslant 2$. By Definition 2 we have that $\tau_t(H') \leqslant (n' + m')b_{k-1} \leqslant b_{k-1}n_H$. □

Claim 22. $\tau_t(H') = \gamma_t(H)$.

Proof of Claim 22. Among all $\gamma_t(H)$-sets, let S be chosen to contain as few vertices of degree 1 in H as possible. Suppose that S contains a degree-1 vertex, x, in H. Let e_x be the edge containing x. By the minimality of the set S, the set $S_x = S \setminus \{x\}$ is not a TD-set in H. Let y be a vertex in S that is adjacent to x in H. Then, $y \in e_x$. If y is adjacent to a vertex of S_x, then the set S_x would be a TD-set in H, a contradiction. Hence, y is adjacent to no vertex of S except for the vertex x. Since H contains no dominating vertex and since H has no isolated edge, there exists a neighbor, w say, of y that has degree at least 2 in H. But then $S_x \cup \{w\}$ is a TD-set of H of cardinality $|S| = \gamma_t(H)$ that contains fewer degree-1 vertices than does S, contradicting our choice of the set S. Therefore, S contains no vertices of degree 1, implying that $S \subseteq V(H')$. Further if S is not a transversal in H, then let e' be an edge in H not intersected by S. But since e' contains a degree-1 vertex, such a vertex would not be (totally) dominated by S in H, a contradiction. Hence, S is a transversal in H. Further since every vertex in the TD-set S has a neighbor in H that belongs to S, the set S is in fact a total transversal of H. Since $S \subseteq V(H')$, the set S is therefore also a total transversal of H', implying that $\tau_t(H') \leqslant \gamma_t(H)$. Conversely, every total transversal in H' is a TD-set in H' and therefore also in H, implying that $\gamma_t(H) \leqslant \tau_t(H')$. Consequently, $\tau_t(H') = \gamma_t(H)$. □

By Claim 21 and Claim 22, we have that $\gamma_t(H) \leqslant b_{k-1}n_H$, a contradiction. This completes the proof of Theorem 3. □

5.2 Proof of Theorem 4

In this section, we present a proof of Theorem 4. We first consider the family $\mathcal{H}_2$.

Theorem 23. *If $H \in \mathcal{H}_2$, then $\tau_t(H) \leqslant 2(n_H + m_H)/5$.*

Proof of Theorem 23. Suppose to the contrary that the theorem is not true. Let $H \in \mathcal{H}_2$ be a counterexample with $n_H + m_H$ a minimum. Clearly, H is connected. By Observation 14, we have that $n_H \geqslant 3$, $m_H \geqslant 2$ and $\Delta(H) \geqslant 2$. If $\tau_t(H) = 2$, then the result is immediate. Hence we may assume that $\tau_t(H) \geqslant 3$. Let x be a vertex of maximum degree in H. Since $\tau_t(H) \geqslant 3$, there is a neighbor y of x that is not isolated in $H - x$. We delete the vertices x and y and all edges incident with x or y, together with any resulting isolated vertices, if any, and let $T = \{x, y\}$. Further if this process creates an isolated edge, e, then such an isolated edge necessarily contains a vertex that is adjacent to x or y, for otherwise the edge e would be an isolated edge in H, a contradiction. From each such isolated edge e, if any, we choose one vertex that is a neighbor of x or y and add it to the

set T, and delete the two vertices in e. Suppose that $\ell \geqslant 0$ isolated edges were created when x and y are deleted. Then, $|T| = 2 + \ell$ and at least $2 + 2\ell$ vertices and at least $3 + \ell$ edges were deleted. Let H' denote the resulting graph. Thus, if H' has n' vertices and m' edges, then $n' + m' \leqslant n_H + m_H - (5 + 3\ell)$. Since H is a minimum counterexample, we have that $\tau_t(H') \leqslant 2(n' + m')/5$, implying that

$$
\begin{aligned}
\tau_t(H) &\leqslant \tau_t(H') + |T| \\
&\leqslant \tfrac{2}{5}(n_H + m_H - 5 - 3\ell) + 2 + \ell \\
&\leqslant \tfrac{2}{5}(n_H + m_H) - \tfrac{\ell}{5} \\
&\leqslant \tfrac{2}{5}(n_H + m_H),
\end{aligned}
$$

contradicting the fact that H is a counterexample. $\square$

As an immediate consequence of Theorem 23, we have that $b_2 \leqslant 2/5$. Taking H to be a path P_3 on three vertices, we note that $H \in \mathcal{H}_2$ and $\tau_t(H) = 2 = 2(n_H + m_H)/5$, implying that $b_2 \geqslant 2/5$. Consequently, $b_2 = 2/5$. This can also be seen by considering the cycle of order five, C_5, instead of P_3, as $\tau_t(C_5) = 4$. We state this formally as follows.

Corollary 24. $b_2 = 2/5$.

We next consider the family $\mathcal{H}_k$, where $k \geqslant 3$.

Theorem 25. *For $k \geqslant 3$, if $H \in \mathcal{H}_k$, then $\tau_t(H) \leqslant (n_H + m_H)/3$.*

Proof of Theorem 25. Suppose to the contrary that the theorem is not true. Let $H \in \mathcal{H}_k$ be a counterexample with $n_H + m_H$ a minimum. Clearly, H is connected since otherwise the theorem holds for each component of H and therefore also for H, a contradiction. By Observation 14, we have that $n_H \geqslant k + 1$, $m_H \geqslant 2$ and $\Delta(H) \geqslant 2$. In what follows we present a series of claims describing some structural properties of H which culminate in the implication of its non-existence.

Claim A. $\tau_t(H) \geqslant 3$ *and no vertex is incident with every edge in H.*

Proof of Claim A. Suppose to the contrary that $\tau_t(H) < 3$. Then, $\tau_t(H) = 2$. Since $n_H + m_H \geqslant k + 3 \geqslant 6$, we therefore have that $\tau_t(H) = 2 \leqslant (n_H + m_H)/3$, contradicting the fact that H is a counterexample. Hence, $\tau_t(H) \geqslant 3$.

If there is a vertex v incident with every edge in H, then the vertex v and one of its neighbors form a total transversal in H, implying that $\tau_t(H) = 2$, a contradiction. Hence, no vertex is incident with every edge in H. $\square$

Claim B. *If X is a set of vertices in H, such that every vertex in X is adjacent to some other vertex of X, then $n(H - X) + m(H - X) > n_H + m_H - 3|X|$.*

Proof of Claim B. Suppose to the contrary that there exists a subset $X \subset V(H)$ such that every vertex in X is adjacent to some other vertex of X but $n(H-X)+m(H-X) \leqslant n_H + m_H - 3|X|$. Let $H' = H - X$. By supposition, $n(H') + m(H') \leqslant n_H + m_H - 3|X|$.

Let $e_1, \ldots, e_\ell$, where $\ell \geqslant 0$, be the isolated edges in H'. Since H contains no isolated edge, each isolated edge in H' contains a vertex of degree at least 2 in H. For each $i = 1, \ldots, \ell$, let $z_i \in e_i$ be chosen so that $d_H(z_i) \geqslant 2$, and let $X^* = X \cup \{z_1, \ldots, z_\ell\}$. We note that every vertex in X^* is adjacent to a vertex in $X \subseteq X^*$.

Let $H^* = H - X^*$. By construction, $H^* \in \mathcal{H}_k$. Moreover, $n(H^*) = n(H') - k\ell$ and $m(H^*) = m(H') - \ell$. By the minimality of H, we have that $\tau_t(H^*) \leqslant (n(H^*) + m(H^*))/3$. Since every $\tau_t(H^*)$-set can be extended to a total transversal of H by adding to it the set X^*, and since $k \geqslant 3$, we have that

$$
\begin{aligned}
\tau_t(H) &\leqslant \tau_t(H^*) + |X^*| \\
&\leqslant \frac{1}{3}(n(H^*) + m(H^*)) + |X| + \ell \\
&= \frac{1}{3}(n(H') - k\ell + m(H') - \ell) + |X| + \ell \\
&\leqslant \frac{1}{3}(n_H + m_H - 3|X| - k\ell - \ell) + |X| + \ell \\
&\leqslant \frac{1}{3}(n_H + m_H),
\end{aligned}
$$

contradicting the fact that H is a counterexample. $\square$

Claim C. $\Delta(H) = 2$.

Proof of Claim C. Suppose to the contrary that $\Delta(H) \geqslant 3$. Let x be a vertex of maximum degree in H. By Claim A, the vertex x is not incident with every edge in H. Hence since H is connected, there exists an edge, e, that contains a neighbor, y, of x but does not contain x. Let $X = \{x, y\}$ and note that $n(H - X) \leqslant n_H - 2$ and $m(H - X) \leqslant m_H - 4$. As x and y are adjacent in H, we obtain a contradiction to Claim B. $\square$

Claim D. H *is 2-regular.*

Proof of Claim D. Suppose that there exists a vertex v_1 of degree 1 in H. Let e_1 be the edge incident with v_1. Since H has no isolated edge, let e_2 be an edge intersecting e_1, and let $v_2 \in e_1 \cap e_2$. By Claim A, the vertex v_2 is not incident with every edge in H. Hence there exists an edge, e_3, not containing v_2 that intersects e_1 or e_2 in a vertex v_3. Let $X = \{v_2, v_3\}$ and note that the vertices v_1, v_2, v_3 and the edges e_1, e_2, e_3 are removed from H in order to create $H - X$. Therefore, $n(H - X) \leqslant n_H - 3$ and $m(H - X) \leqslant m_H - 3$, which as v_2 and v_3 are adjacent in H, contradicts Claim B. $\square$

Claim E. H *is a linear hypergraph.*

Proof of Claim E. By Claim D, H is a 2-regular k-uniform hypergraph. Suppose that there are two edges e and f having two or more vertices in common. Let v be a vertex in

e that does not belong to $e \cap f$. Since H is 2-regular, there is an edge g which contains v but is different from e or f. Let u be a vertex in $e \cap f$. Since u and v belong to the common edge e, they are neighbors in H. Let $X = \{u, v\}$ and note that the vertices in $\{v\} \cup (e \cap f)$ and the edges e, f, g are removed from H in order to create $H - X$. Therefore, $n(H - X) \leqslant n_H - 3$ and $m(H - X) \leqslant m_H - 3$, which contradicts Claim B. $\square$

By Claim D and Claim E, H is a 2-regular k-uniform linear connected hypergraph.

Claim F. $k = 3$

Proof of Claim F. Suppose to the contrary that $k \geqslant 4$. Then, $n_H = km_H/2 \geqslant 2m_H$. We now consider the dual, G_H, of the hypergraph H. By the 2-regularity and the linearity of H, the dual G_H is a graph. Since H is k-uniform, the graph G_H is k-regular. Further since H is connected, so too is G_H. By construction, G_H has order $n(G_H) = m_H$ and size $m(G_H) = n_H$. Let T be a spanning tree in G_H. Since the set $E(T)$ of edges of T forms a total edge-cover in G_H and since $n_H \geqslant 2m_H$, we have by Lemma 18 that $\tau_t(H) = \mathrm{ec}_t(G_H) \leqslant |E(T)| = n(G_H) - 1 = m_H - 1 < \frac{1}{3}(n_H + m_H)$, a contradiction. $\square$

By Claim D, E and F, we have that H is a 2-regular 3-uniform linear connected hypergraph. We now consider the dual, G_H, of the hypergraph H. We note that the dual, G_H, is a connected, cubic graph. Applying Theorem 12 to the cubic graph G_H, there exist at least $\lceil n(G_H)/4 \rceil$ vertex disjoint P_3's in G_H. Let $G_1, G_2, \ldots, G_\ell$ denote vertex disjoint subgraphs in G_H each of which is isomorphic to P_3, such that $\ell \geqslant \lceil n(G_H)/4 \rceil \geqslant m_H/4$. If some vertex does not belong to one of these subgraphs $G_1, G_2, \ldots, G_\ell$, then the connectivity of G_H implies that there is an edge, e, joining a vertex in $V(G_i)$ for some i, $1 \leqslant i \leqslant \ell$, and a vertex, x, not belonging to any subgraph $G_1, G_2, \ldots, G_\ell$. We now add the vertex x and edge e to the subgraph G_i. We continue this process until all vertices in G_H belong to exactly one of the resulting subgraphs $G_1, G_2, \ldots, G_\ell$. The subgraph of G_H induced by the edges in these ℓ subgraphs is a spanning forest, F, of G_H, that contains $\ell \geqslant m_H/4$ components each of which contains at least three vertices.

Since every component of F has order at least 3, the set $E(F)$ of edges of F forms a total edge-cover in G_H. Since $n(G_H) = m_H$ and $\ell \geqslant m_H/4$, we have that $|E(F)| = n(G_H) - \ell \leqslant 3m_H/4$. Therefore, recalling that $n_H = 3m_H/2$, we have by Lemma 18 that

$$\tau_t(H) = \mathrm{ec}_t(G_H) \leqslant |E(F)| \leqslant \frac{3}{4} m_H \leqslant \frac{1}{3}(n_H + m_H),$$

a contradiction. This completes the proof of Theorem 25. $\square$

As an immediate consequence of Theorem 25, we have that $b_k \leqslant 1/3$ for all $k \geqslant 3$. Taking H to be the hypergraph of order $n_H = 4$ and size $m_H = 2$ where the two edges of H intersect in two vertices, we note that $H \in \mathcal{H}_3$ and $\tau_t(H) = 2 = (n_H + m_H)/3$, implying that $b_3 \geqslant 1/3$. Consequently, $b_3 = 1/3$. As observed earlier, $b_4 \leqslant 1/3$. We state this formally as follows.

Corollary 26. $b_3 = \frac{1}{3}$ and $b_4 \leqslant \frac{1}{3}$.

We remark that the result of Theorem 25 can be strengthened slightly when $k \geqslant 4$, as the following result shows. We omit the proof (which is similar, but simpler, to the proof of Theorem 28 presented below).

Theorem 27. *For $k \geqslant 4$, if $H \in \mathcal{H}_k$, then $6\tau_t(H) \leqslant 2n_H + 2m_H - n_1(H)$.*

We next consider the family $\mathcal{H}_k$, where $k \geqslant 5$.

Theorem 28. *For $k \geqslant 5$, if $H \in \mathcal{H}_k$, then $7\tau_t(H) \leqslant 2n_H + 2m_H - n_1(H)$.*

Proof of Theorem 28. For $k \geqslant 5$ and all hypergraphs $H \in \mathcal{H}_k$, let

$$\Theta(H) = 2n_H + 2m_H - n_1(H).$$

We wish to show that $7\tau_t(H) \leqslant \Theta(H)$. Suppose to the contrary that the theorem is not true. Let $H \in \mathcal{H}_k$ be a counterexample with minimum $\Theta(H)$. Clearly, H is connected since otherwise the theorem holds for each component of H and therefore also for H, a contradiction. By Observation 14(a), we have that $n_H \geqslant k + 1$, $m_H \geqslant 2$ and $\Delta(H) \geqslant 2$. By Observation 14(b), we have that $2n_H - n_1(H) \geqslant 2k$. In what follows we present a series of claims describing some structural properties of H which culminate in the implication of its non-existence.

Claim I. $\tau_t(H) \geqslant 3$.

Proof of Claim I. Suppose that $\tau_t(H) < 3$. Then, $\tau_t(H) = 2$. Since $2n_H - n_1(H) \geqslant 2k$ and $m_H \geqslant 2$, we therefore have that $7\tau_t(H) = 14 \leqslant 2k + 4 \leqslant \Theta(H)$, contradicting the fact that H is a counterexample. □

Claim II. *If X is a set of vertices in H, such that every vertex in X is adjacent to some other vertex of X, then $\Theta(H - X) > \Theta(H) - 7|X|$.*

Proof of Claim II. Suppose to the contrary that exists a subset $X \subset V(H)$ such that every vertex in X is adjacent to some other vertex of X but $\Theta(H - X) \leqslant \Theta(H) - 7|X|$. Let $H' = H - X$. Let $e_1, \ldots, e_\ell$, where $\ell \geqslant 0$, be the isolated edges in H'. Since H contains no isolated edge, every isolated edge in H' contains a vertex of degree at least 2 in H that is adjacent to a vertex of X in H. For each $i = 1, \ldots, \ell$, let $z_i \in e_i$ be chosen so that $d_H(z_i) \geqslant 2$, and let $X^* = \{z_1, \ldots, z_\ell\}$. We note that every vertex in $X \cup X^*$ is adjacent to some other vertex of X. We now consider the hypergraph $H^* = H' - X^*$.

We note that $H^* \in \mathcal{H}_k$. When constructing H^* from H' we deleted all $k\ell$ vertices from the ℓ isolated edges in H' and we deleted all ℓ isolated edges. Since each such deleted vertex has degree 1 in H', the contribution of the $k\ell$ deleted vertices from H' to the sum $2n(H') - n_1(H')$ is $k\ell$. The contribution of the ℓ deleted edges to the sum $2m(H')$ is 2ℓ. By supposition, $\Theta(H') \leqslant \Theta(H) - 7|X|$. Since $k \geqslant 5$, we therefore have that

$$
\begin{aligned}
\Theta(H^*) &= \Theta(H') - \ell(k + 2) \\
&\leqslant \Theta(H') - 7\ell \\
&\leqslant (\Theta(H) - 7|X|) - 7\ell \\
&= \Theta(H) - 7|X| - 7|X^*|.
\end{aligned}
$$

By the minimality of $\Theta(H)$, we have that $7\tau_t(H^*) \leqslant \Theta(H^*)$. Every (minimum) total transversal in H^* can be extended to a total transversal in H by adding to the set $X \cup X^*$, implying that $\tau_t(H) \leqslant \tau_t(H^*) + |X| + |X^*|$. Hence,

$$
\begin{aligned}
7\tau_t(H) & \leqslant 7\tau_t(H^*) + 7|X| + 7|X^*| \\
& \leqslant \Theta(H^*) + 7|X| + 7|X^*| \\
& \leqslant \Theta(H),
\end{aligned}
$$

a contradiction. $\square$

Claim III. $\Delta(H) \leqslant 3$.

Proof of Claim III. Suppose to the contrary that $\Delta(H) \geqslant 4$. Let x be a vertex of maximum degree in H. Since $\tau_t(H) \geqslant 3$ by Claim I, and since H is connected, there exists an edge, e, that contains a neighbor, y, of x but does not contain x. Let $X = \{x, y\}$ and consider the hypergraph $H - X$. Since $d_H(x) \geqslant 4$ and $d_H(y) \geqslant 2$, the vertices x and y both contribute 2 to the sum $2n(H) - n_1(H)$. Further since at least five distinct edges are deleted from H when constructing $H - X$, the contribution of the deleted edges to the sum $2m(H)$ is at least 10. Hence, $\Theta(H - X) \leqslant \Theta(H) - 14 = \Theta(H) - 7|X|$, contradicting Claim II. $\square$

Claim IV. $\Delta(H) = 2$.

Proof of Claim IV. As observed earlier, $\Delta(H) \geqslant 2$. By Claim III, $\Delta(H) \leqslant 3$. Suppose to the contrary that $\Delta(H) = 3$. Let x be a vertex with $d_H(x) = 3$ and consider the hypergraph $H' = H - x$. Suppose that $d_{H'}(y) \geqslant 2$ for some $y \in N_H(x)$. Let $X = \{x, y\}$ and consider the hypergraph $H - X$. Since $d_H(x) = 3$ and $d_H(y) = 3$, the vertices x and y both contribute 2 to the sum $2n(H) - n_1(H)$. Further since five distinct edges are deleted from H when constructing $H - X$, the contribution of the deleted edges to the sum $2m(H)$ is 10. Hence, $\Theta(H - X) \leqslant \Theta(H) - 14 = \Theta(H) - 7|X|$, contradicting Claim II. Therefore, $d_{H'}(y) \leqslant 1$ for every vertex $y \in N_H(x)$.

Since $\tau_t(H) \geqslant 3$ by Claim I, and since H is connected, there exists a neighbor, y^*, of x that has degree at least 1 in H'. Let $X^* = \{x, y^*\}$ and consider the hypergraph $H^* = H - X^*$. Since $d_H(x) = 3$ and $d_H(y^*) \geqslant 2$, the vertices x and y^* both contribute 2 to the sum $2n(H) - n_1(H)$. Further since four distinct edges are deleted from H when constructing H^*, the contribution of these deleted edges to the sum $2m(H)$ is 8.

Let $z \in N_H(x) \setminus \{y^*\}$. We note that if $z \in V(H^*)$, then $d_{H^*}(z) \leqslant d_{H'}(z) \leqslant 1$, implying that $d_{H^*}(z) = 1$ and that z contributes 2 to the sum $2n(H) - n_1(H)$ and 1 to the sum $2n(H^*) - n_1(H^*)$. If $z \notin V(H^*)$, then z is isolated when the vertices in X^* are deleted from H together with all edges incident with X^*, implying that in this case, z contributes at least 1 to the sum $2n(H) - n_1(H)$ and 0 to the sum $2n(H^*) - n_1(H^*)$. In both cases the contribution of z to $\Theta(H^*)$ is at least one less than its contribution to $\Theta(H)$. This is true for every vertex in $N_H(x) \setminus \{y^*\}$. Hence the total contribution of the neighbors of x different from y^* to $\Theta(H) - \Theta(H^*)$ is at least $|N_H(x) \setminus \{y^*\}| = |N_H(x)| - 1 \geqslant k - 1 \geqslant 4$.

Together with our earlier observation that the vertices x and y^*, together with the four edges incident with x or y^* in H, contribute 12 to $\Theta(H)$, this implies that $\Theta(H^*) \leqslant \Theta(H) - 12 - 4 < \Theta(H) - 14 = \Theta(H) - 7|X^*|$, contradicting Claim II. $_{(\square)}$

We now return to the proof of Theorem 28. By Claim IV, $\Delta(H) = 2$. Let x be a vertex in H with $d_H(x) = 2$. Since $\tau_t(H) \geqslant 3$ by Claim I, and since H is connected, there exists an edge, e, that contains a neighbor, y, of x but does not contain x. Let $X = \{x, y\}$ and consider the hypergraph $H - X$. Since $d_H(x) = 2$ and $d_H(y) = 2$, the vertices x and y both contribute 2 to the sum $2n(H) - n_1(H)$. Further the three edges incident with x or y contribute 6 to the sum $2m(H)$. Furthermore, each vertex in $N_H(x) \setminus \{y\}$ either does not belong to $H - X$ or has degree 1 in $H - X$ and therefore contributes at least 1 to $\Theta(H) - \Theta(H - X)$. This implies that $\Theta(H - X) \leqslant \Theta(H) - 10 - (|N_H(x)| - 1) \leqslant \Theta(H) - 10 - k + 1 \leqslant \Theta(H) - 14 = \Theta(H) - 7|X|$, contradicting Claim II. This completes the proof of Theorem 28. $\square$

As an immediate consequence of Theorem 28, we have the following results.

Corollary 29. *For $k \geqslant 5$, if $H \in \mathcal{H}_k$, then $7\tau_t(H) \leqslant 2n_H + 2m_H$.*

Corollary 30. *For all $k \geqslant 5$, we have $b_k \leqslant \frac{2}{7}$.*

Theorem 4 follows from Corollary 24, Corollary 26 and Corollary 29.

5.3 Proof of Theorem 6

In this section, we present a proof of Theorem 6. Recall its statement.

Theorem 6. *For $k \geqslant 4$, if $H \in \mathcal{H}_k^*$, then $\gamma_t(H) \leqslant \left(\max \left\{ \dfrac{2}{k+2}, b_{k-1} \right\} \right) n_H$.*

Proof of Theorem 6. Suppose to the contrary that the theorem is not true. Let $H \in \mathcal{H}_k^*$ be a counterexample with $n_H + m_H$ a minimum. We proceed in a similar manner as in the proof of Theorem 3.

Claim I. *The following properties hold in the hypergraph H.*
  (a) *H is connected.*
  (b) *The deletion of any edge in H creates an isolated vertex or an isolated edge.*
  (c) *There is no dominating vertex in H.*

Proof of Claim I. Parts (a) and (b) follows from the minimality of H and the observation that the deletion of an edge cannot decrease the total domination number. To prove Part (c), suppose that H contains a dominating vertex v. The vertex v and any one of its neighbors forms a TD-set in H, implying that $\gamma_t(H) = 2$. By the definition of $\mathcal{H}_k^*$, the hypergraph H contains no isolated vertex or isolated edge. Since no two edges of H intersect in $k-1$ vertices, we therefore have that $n_H \geqslant k+2$. Hence, $\gamma_t(H) \leqslant 2n_H/(k+2)$, contradicting the definition of H. This proves Part (c). $_{(\square)}$

Claim II. *Every edge in H contains at least one degree-1 vertex.*

Proof of Claim II. We proceed as in the proof of Claim 20. Let u, v, e, e_1 and e_2 be defined as in the proof of Claim 20. If the edge e_2 contains a degree-1 vertex, then at least one vertex in addition to the vertices in $e_1 \cup \{u\}$ becomes an isolated vertex when we delete all edges incident with u or v. Thus in this case we delete at least $k+2$ vertices and we add two vertices to T, and we proceed as in the 2nd paragraph of the proof of Claim 20. In this case, $|T| = 2 + \ell$, where $\ell \geqslant 0$ denotes the number of isolated edges created when removing u and v, and at least $k + 2 + k\ell$ vertices are deleted from H. Thus if n' denotes the number of vertices in H that are not deleted in the process, then $n' \leqslant n_H - k - 2 - k\ell$, implying that

$$
\begin{aligned}
\left(\frac{2}{k+2} \right) (n_H - n') &\geqslant \left(\frac{2}{k+2} \right) (k + 2 + k\ell) \\
&= 2 + \left(\frac{2k}{k+2} \right) \ell \\
&\geqslant 2 + \ell \\
&= |T|.
\end{aligned}
$$

Suppose that the edge e_2 does not contain any degree-1 vertices. Then there is an edge, e_3, which would become isolated after the deletion of the edge e_2 from H_2. We note that neither u nor v belong to the edge e_3 and therefore that $e_3 \notin \{e, e_1, e_2\}$. Let $w \in e_2 \cap e_3$. We now delete all edges incident with a vertex in the set $\{u, v, w\}$ and we delete any resulting isolated vertices. Further we add the three vertices u, v and w to the set T. We note that every vertex in $e_1 \cup e_3 \cup \{u\}$ becomes an isolated vertex. We therefore delete at least $2k + 1$ vertices and we add three vertices to T. If this process creates an isolated edge, then from each such isolated edge f, if any, we choose one vertex that is a neighbor of a vertex in T and add it into T, and delete the k vertices in f. Hence in this case, $|T| = 3 + \ell$, where $\ell \geqslant 0$ denotes the number of isolated edges created when removing u, v and w, and at least $2k + 1 + k\ell$ vertices are deleted from H. Thus if n' denotes the number of vertices in H that are not deleted in the process, then $n' \leqslant n_H - 2k - 1 - k\ell$. Since $k \geqslant 4$, we note that $2(2k+1)/(k+2) \geqslant 3$ and $2k/(k+2) > 1$, implying that

$$
\begin{aligned}
\left(\frac{2}{k+2} \right) (n_H - n') &\geqslant \left(\frac{2}{k+2} \right) (2k + 1 + k\ell) \\
&= \left(\frac{2(2k+1)}{k+2} \right) + \left(\frac{2k}{k+2} \right) \ell \\
&\geqslant 3 + \ell \\
&= |T|.
\end{aligned}
$$

In both cases, we therefore have that $|T| \leqslant 2(n_H - n')/(k+2)$. If $n' = 0$, then the set T is a TD-set in H, implying that $\gamma_t(H) \leqslant |T| \leqslant 2n_H/(k+2)$, a contradiction. Hence,

$n' > 0$. Let H' denote the resulting hypergraph on these n' vertices. Let H' have size m'. By construction, the hypergraph H' is in the family $\mathcal{H}_k^*$. In particular, we note that $n' \geqslant k + 2$. By the minimality of H, we have that

$$\gamma_t(H') \leqslant \left(\max \left\{ \frac{2}{k+2}, b_{k-1} \right\} \right) n'.$$

Let T' be a $\gamma_t(H')$-set and note that the set $T \cup T'$ is a TD-set of H. Suppose that $2/(k+2) \geqslant b_{k-1}$. Then, $|T'| \leqslant 2n'/(k+2)$, and so

$$\gamma_t(H) \leqslant |T \cup T'| \leqslant \left(\frac{2}{k+2} \right) (n_H - n') + \left(\frac{2}{k+2} \right) n' = \left(\frac{2}{k+2} \right) n_H,$$

a contradiction. Hence, $2/(k+2) < b_{k-1}$. Thus, $|T'| \leqslant b_{k-1} n'$, and so

$$\gamma_t(H) \leqslant |T \cup T'| \leqslant \left(\frac{2}{k+2} \right) (n_H - n') + b_{k-1} n' < b_{k-1}(n_H - n') + b_{k-1} n' = b_{k-1} n_H,$$

a contradiction. This completes the proof of Claim II. $\scriptstyle(\square)$

We now return to the proof of Theorem 6. By Claim II, every edge in H contains at least one degree-1 vertex. Let H' be the hypergraph obtained from H by deleting exactly one degree-1 vertex from each edge. Since $H \in \mathcal{H}_k^*$, we note that no multiple edges are created. Further, H' contains no isolated edge and no isolated vertices, and so $H' \in \mathcal{H}_{k-1}$. Let H' have order n' and size m'. Then, $n' = n_H - m_H$ and $m' = m_H$. Thus, $n' + m' = n_H$. We note that $k - 1 \geqslant 3$. By Definition 2 we have that $\tau_t(H') \leqslant (n' + m')b_{k-1} \leqslant b_{k-1} n_H$. An identical proof as in the proof of Claim 22 of Theorem 3 shows that $\gamma_t(H) = \tau_t(H')$, implying that $\gamma_t(H) \leqslant b_{k-1} n_H$, a contradiction. This completes the proof of Theorem 6. $\square$

6   Tight Asymptotic Bounds

In this section we prove Theorem 8 which establishes a tight asymptotic upper bound on b_k for k sufficiently large. Since every strong transversal in a hypergraph, H, is a total transversal in H, and since every total transversal in H is a transversal in H, we have the following observation.

Observation 31. *For every hypergraph H, we have $\tau(H) \leqslant \tau_t(H) \leqslant \tau_s(H)$.*

Using probabilistic arguments, Alon [3] established the following result.

Theorem 32. ([3]) *For every $\varepsilon > 0$ and sufficiently large k there exist k-uniform hypergraphs, H, satisfying*

$$\tau(H) \geqslant \left(\frac{(1-\varepsilon)\ln(k)}{k} \right) (n_H + m_H)$$

The following result establishes a tight asymptotic upper bound on the strong transversal number of a k-uniform hypergraph for k sufficiently large.

Theorem 33. *For every constant $c > 1$ and every k-uniform hypergraph H, we have*

$$\tau_s(H) \leqslant \left(\frac{\ln(k) + \ln(c)}{k-1}\right) n_{_H} + \left(\frac{\ln(k) + \ln(c)}{c(k-1)}\right) m_{_H} + \left(\frac{2}{ck}\right) m_{_H}.$$

Proof. Let $H = (V, E)$ and let $p = \ln(ck)/(k-1)$. Let X_1 be a random subset of $V(H)$ where a vertex x is chosen to be in X_1 with probability $\Pr(x \in X_1) = p$, independently of the choice for any other vertex. For every edge $e \in E$ that does not intersect X_1, select two vertices from e and let $X_2 \subseteq V$ be the resulting set of all such selected vertices. For every edge $e \in E$ such that $|e \cap X_1| = 1$, select one vertex from $e \setminus X_1$ and let $X_3 \subseteq V$ be the resulting set of all such selected vertices. The resulting set $X_1 \cup X_2 \cup X_3$ is a strong transversal in H. The expected value of the cardinality of the set X_1 is

$$\mathbb{E}(|X_1|) = pn_{_H} = \left(\frac{\ln(k) + \ln(c)}{k-1}\right) n_{_H}.$$

Using the inequality $1 - x \leqslant e^{-x}$ for $x \in \mathbb{R}$, the expected value of the cardinality of the set X_2 is given by

$$
\begin{aligned}
\mathbb{E}(|X_2|) \quad &\leqslant \quad (1-p)^k \cdot m_{_H} \cdot 2 \\
&= \quad \left(1 - \frac{\ln(ck)}{k-1}\right)^k \cdot 2m_{_H} \\
&= \quad \left(\left(1 - \frac{\ln(ck)}{k-1}\right)^{\frac{k-1}{\ln(ck)}}\right)^{\frac{k}{k-1}\ln(ck)} \cdot 2m_{_H} \\
&< \quad e^{-\frac{k}{k-1}\ln(ck)} \cdot 2m_{_H} \\
&\leqslant \quad \frac{2}{ck} \cdot m_{_H}.
\end{aligned}
$$

The expected value of the set X_3 is given by

$$
\begin{aligned}
\mathbb{E}(|X_3|) \quad &\leqslant \quad m_{_H} \cdot k \cdot p \cdot (1-p)^{k-1} \\
&= \quad k\left(\frac{\ln(ck)}{k-1}\right)\left(1 - \frac{\ln(ck)}{k-1}\right)^{k-1} \cdot m_{_H} \\
&= \quad k\left(\frac{\ln(ck)}{k-1}\right)\left(\left(1 - \frac{\ln(ck)}{k-1}\right)^{\frac{k-1}{\ln(ck)}}\right)^{\ln(ck)} \cdot m_{_H} \\
&< \quad k\left(\frac{\ln(ck)}{k-1}\right) e^{-\ln(ck)} \cdot m_{_H} \\
&= \quad \left(\frac{\ln(k) + \ln(c)}{c(k-1)}\right) m_{_H}.
\end{aligned}
$$

By linearity of expectation, we have that $\mathbb{E}(|X_1 \cup X_2 \cup X_3|) \leqslant \mathbb{E}(|X_1|) + \mathbb{E}(|X_2|) + \mathbb{E}(|X_3|)$, yielding the desired upper bound. $\square$

As a consequence of Theorem 33, we have the following results.

Corollary 34. *Given any $\varepsilon > 0$, if H is a k-uniform hypergraph with k sufficiently large, then*

$$\tau_s(H) < \left((1 + \varepsilon) \frac{\ln(k)}{k} \right) (n_H + m_H).$$

Proof. For a constant $c > 1$, we note that the functions,

$$\frac{\ln(k) + \ln(c)}{k - 1} \quad \text{and} \quad \frac{\ln(k) + \ln(c)}{c(k - 1)} + \frac{2}{ck},$$

tend to $\ln(k)/(k - 1)$ and $\ln(k)/(c(k - 1)) < \ln(k)/(k - 1)$, respectively, when k tends to infinity. Hence for k sufficiently large, we have that

$$\max \left\{ \frac{\ln(k) + \ln(c)}{k - 1}, \frac{\ln(k) + \ln(c)}{c(k - 1)} + \frac{2}{ck} \right\} < (1 + \varepsilon) \frac{\ln(k)}{k}.$$

Therefore for k sufficiently large, we have that

$$\left(\frac{\ln(k) + \ln(c)}{k - 1} \right) n_H + \left(\frac{\ln(k) + \ln(c)}{c(k - 1)} \right) m_H + \left(\frac{2}{ck} \right) m_H < \left((1 + \varepsilon) \frac{\ln(k)}{k} \right) (n_H + m_H).$$

The desired result now follows from Theorem 33. $\square$

We are now in a position to prove Theorem 8. Recall its statement.

Theorem 8. $b_k = (1 + \mathrm{o}(1)) \dfrac{\ln(k)}{k}.$

Proof of Theorem 8. It suffices for us to prove that for $\varepsilon > 0$ and for k sufficiently large, we have

$$(1 - \varepsilon) \frac{\ln(k)}{k} \leqslant b_k \leqslant (1 + \varepsilon) \frac{\ln(k)}{k}.$$

The upper bound on b_k follows from Observation 31 and Corollary 34. For the lower bound let $\varepsilon > 0$ and let k be sufficiently large, such that a k-uniform hypergraph, H, exists with $\tau(H) \geqslant [(1 - \varepsilon) \ln(k)/k](n_H + m_H)$ (which exists by Theorem 32). Assume that H contains n_0 isolated vertices and e_0 isolated edges. Let H' be obtained from H by deleting all isolated vertices and isolated edges and the vertices belonging to isolated edges. Then, $H' \in \mathcal{H}_k$. Further, $n(H') = n_H - n_0 - ke_0$ and $m(H') = m_H - e_0$. As $n_0 \geqslant 0$ and $(1 - \varepsilon) \ln(k)(k + 1)/k > 1$ when k is sufficiently large, we have that

$$
\begin{aligned}
\tau_t(H') &\geqslant \tau(H') \\
&= \tau(H) - e_0 \\
&\geqslant \left(\frac{(1-\varepsilon)\ln(k)}{k} \right)(n_H + m_H) - e_0 \\
&\geqslant \left(\frac{(1-\varepsilon)\ln(k)}{k} \right)(n(H') + m(H') + n_0 + ke_0 + e_0) - e_0 \\
&= \left(\frac{(1-\varepsilon)\ln(k)}{k} \right)(n(H') + m(H')) \\
&\quad + \left(\frac{(1-\varepsilon)\ln(k)}{k} \right)(n_0 + (k+1)e_0) - e_0. \\
&\geqslant \left(\frac{(1-\varepsilon)\ln(k)}{k} \right)(n(H') + m(H')).
\end{aligned}
$$

This implies that

$$
b_k \geqslant \left(\frac{(1-\varepsilon)\ln(k)}{k} \right),
$$

which establishes the desired lower bound on b_k and completes the proof of Theorem 8. $\square$

7 Closing Remarks and Open Problem

In view of Theorem 3, it is of interest to determine the value of b_k for $k \geqslant 2$. In Theorem 4 we show that $b_2 = \frac{2}{5}$ and $b_3 = \frac{1}{3}$, and we show that $b_{k-1} \leqslant 2/(k+1)$ for $k \in \{3,4,5,6\}$. In Theorem 8, we establish a tight asymptotic bound on b_k for k sufficiently large which shows that is not true that $b_{k-1} \leqslant 2/(k+1)$ when k is large enough. We pose the following problems that still remain to be settled.

Problem 35. Determine the exact value of b_k for $k \geqslant 4$.

Problem 36. Determine the smallest value of k for which $b_{k-1} > 2/(k+1)$.

Acknowledgements

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