On a general q-identity

Aimin Xu*

Institute of Mathematics
Zhejiang Wanli University
Ningbo 315100, China

xuaimin1009@hotmail.com; xuaimin@zwu.edu.cn

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Abstract

In this paper, by means of the q-Rice formula we obtain a general q-identity which is a unified generalization of three kinds of identities. Some known results are special cases of ours. Meanwhile, some identities on q-generalized harmonic numbers are also derived.

Keywords: q-Rice formula; q-identity; q-generalized harmonic number; Cauchy's integral formula; Faà di Bruno's formula

1 Introduction

Three kinds of identities will be introduced in this paper.

In the paper [21], Van Hamme gave the following identity

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{\binom{k+1}{2}}}{1-q^k}=\sum_{k=1}^{n}\frac{q^k}{1-q^k}. \tag{1.1}$$

One of the generalizations of (1.1) was given by Dilcher [6]:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{\binom{k}{2}+k\lambda}}{(1-q^k)^{\lambda}}=\sum_{1\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_\lambda\leqslant n}\prod_{j=1}^{\lambda}\frac{q^{\alpha_j}}{1-q^{\alpha_j}}. \tag{1.2}$$

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Prodinger [16] gave another generalization of (1.1):

$$\sum_{k=0,\neq m}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1-q^{k-m}} = (-1)^m q^{\binom{m+1}{2}} \sum_{k=0,\neq m}^{n} \frac{q^k}{1-q^{k-m}}, \tag{1.3}$$

where $0 \leqslant m \leqslant n$. Many works have been devoted to the study of the generalizations of these identities. See for example [8, 9, 17, 23]. Recently, Guo and Zhang [12] made use of the Lagrange interpolation formula to give a generalization of Prodinger's identity (1.3). They also gave a generalization of Dilcher's identity (1.2). See Theorems 1.1 and 1.2 in [12], respectively. Ismail and Stanton used the theory of basic hypergeometric functions to generalize Dilcher's identity. See Theorem 2.2 in [13].

In the paper[5], Díaz-Barrero et al. obtained two identities involving rational sums:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x+k}{k}^{-1} \sum_{1\leqslant\alpha\leqslant\beta\leqslant k} \frac{1}{x^2+(\alpha+\beta)x+\alpha\beta} = \frac{n}{(x+n)^3},$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x+k}{k}^{-1} \left\{ \sum_{\alpha=1}^{k} \frac{1}{(x+\alpha)^3} + \sum_{1\leqslant\alpha\leqslant\beta\leqslant k} \frac{1}{(x+\alpha)(x+\beta)(2x+\alpha+\beta)} + \sum_{1\leqslant\alpha<\beta<\gamma\leqslant k} \frac{1}{(x+\alpha)(x+\beta)(x+\gamma)} \right\} = \frac{n}{(x+n)^4}.$$

Recently, Prodinger [18] made use of partial fraction decomposition and inverse pairs to present a more general formula:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x+k}{k}^{-1} \sum_{c_1+2c_2+\cdots=\lambda} \prod_{j\geqslant 1} \frac{s_{k,j}^{c_j}}{c_j!j^{c_j}} = \frac{n}{(x+n)^{\lambda+1}}, \tag{1.4}$$

where $s_{k,j} = \sum_{\alpha=1}^{k}(x+\alpha)^{-j}$. Almost at the same time, Chu and Yan [2] presented a generalization with multiple λ-fold sum:

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{x+k}{k}^{-1} \sum_{0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_\lambda\leqslant k} \prod_{j=1}^{\lambda} \frac{1}{x+\alpha_j} = \frac{x}{(x+n)^{\lambda+1}}. \tag{1.5}$$

A direct proof of (1.5) can be found in Chu [1]. More recently, Mansour et al. [15] established a q-analog for the rational sum identity (1.4):

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}-k(n-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} x+k \\ k \end{bmatrix}_q^{-1} \sum_{c_1+2c_2+\cdots=\lambda} \prod_{j\geqslant 1} \frac{s_{k,j}(q)^{c_j}}{c_j!j^{c_j}} = \frac{q^{n\lambda}[n]_q}{[x+n]_q^{\lambda+1}}, \tag{1.6}$$

where $s_{k,j}(q) = \sum_{\alpha=1}^{k} q^{j\alpha}[x+\alpha]_q^{-j}$. In particular, they gave a very nice bijective proof for the case $\lambda = 1$.

In the recent paper [19], Prodinger established an interesting identity involving harmonic numbers:

$$\sum_{k=0,\neq m}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{1}{(k-m)^{\lambda}}$$
$$=(-1)^m\binom{n}{m}\binom{n+m}{n}\sum_{c_1+2c_2+\cdots=\lambda}\frac{1}{c_1!c_2!\cdots}\prod_{j=1}^{\lambda}\left(\frac{\mathcal{H}_j}{j}\right)^{c_j}, \tag{1.7}$$

where

$$\mathcal{H}_j=(-1)^{j-1}\left(H_{m+n}^{(j)}-2H_m^{(j)}\right)+H_{n-m}^{(j)},$$

and $H_n^{(r)}$ are the generalized harmonic numbers defined by

$$H_0^{(r)}=0,\quad H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^r}\quad for\quad n,r=1,2,\ldots.$$

Mansour [14] obtained a general rational sum to generalize this identity. He also obtained a q-analog of this result involving q-harmonic numbers.

Motivated by these interesting work, by means of the q-Rice formula used in [16, 17], we will establish a general q-identity which is a common generalization of those three kinds of identities introduced before.

Theorem 1.1. *Let λ be any positive integer. For $0\leqslant m\leqslant n$ and $0\leqslant l\leqslant n+\lambda-1$, there holds*

$$\sum_{k=0,\neq m}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{(\lambda-1)k+m}(1-q^{k-m})(q/z;q)_k(zq^{-l};q)_{n-k+\lambda-1}}{(1-xq^{k-m})^{\lambda+1}}z^k$$
$$=-\frac{(q;q)_n(zq^{-l};q)_l(zxq^{-m};q)_{n-l+\lambda-1}}{(xq^{-m};q)_m(xq;q)_{n-m}}\sum_{\|\vec{c}\|=\lambda}\frac{1}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{u_j}{j}\right)^{c_j}, \tag{1.8}$$

where $\vec{c}!=c_1!c_2!\cdots c_\lambda!$, $\|\vec{c}\|=c_1+2c_2+\cdots+\lambda c_\lambda$ and

$$u_j=-\sum_{k=0}^{n-l+\lambda-2}\left(\frac{zq^k}{1-zxq^{k-m}}\right)^j+\sum_{k=0,\neq m}^{n}\left(\frac{q^k}{1-xq^{k-m}}\right)^j.$$

This is a very general q-series sum identity involving five parameters λ, l, m, x and z. It contains several known identities by choosing different parameters, which will be shown in the third section. By means of our identity, we will also obtain some identities on q-generalized harmonic numbers.

Throughout this paper, we will use the standard notation. For any real number x and any integer m, define

$$[x]_q=\frac{1-q^x}{1-q},\quad (x;q)_\infty=\prod_{k=0}^{\infty}(1-xq^k),\quad (x;q)_m=\frac{(x;q)_\infty}{(xq^m;q)_\infty}.$$

For any nonnegative integer n, define

$$[n]_q! = [1]_q[2]_q\cdots[n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

2 Proof of Theorem 1.1

In the very interesting paper [16], Prodinger introduced the following formula

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{-k}) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{(q;q)_n}{(t;q)_{n+1}}f(t)dt,$$

where $\mathcal{C}$ encircles the poles q^{-1}, q^{-2},…, q^{-n} and no other. It is a q-analog of Rice's formula [7, 20]:

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k f(k) = \frac{(-1)^n}{2\pi i}\int_{\mathcal{C}}\frac{n!}{t(t-1)\cdots(t-n)}f(t)dt,$$

where $\mathcal{C}$ encircles the poles 1, 2,…, n and no other. Indeed, by Cauchy's integral formula one is not hard to find that for any integer $m \in \{0, 1, \ldots, n\}$ there holds

$$\sum_{k=0,\neq m}^{n}(-1)^{k-1}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{-k}) = (-1)^{n-1}\frac{(q;q)_n}{q^{\binom{n+1}{2}}}\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(t)dt}{\prod_{j=0}^{n}(t-q^{-j})}, \tag{2.1}$$

where $\mathcal{C}$ encircles the poles q^{-j}, $j \in \{0, 1, \ldots, n\} - \{m\}$ and no other. Prodinger first applied the q-analog of Rice's formula to prove many identities such as the identities of Van Hamme, Uchimura, Dilcher, Andrews-Crippa-Simon, and Fu-Lascoux, see [16, 17] and references therein. It was shown that this formula is a very powerful and useful tool. Now, in this section we will use this important formula and present a proof of Theorem 1.1.

Proof of Theorem 1.1. By simple calculations we have

$$\begin{aligned}
&\sum_{k=0,\neq m}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{(\lambda-1)k}(1-q^{k-m})(q/z;q)_k(zq^{-l};q)_{n-k+\lambda-1}}{(1-xq^{k-m})^{\lambda+1}}z^k \\
=&\sum_{k=0,\neq m}^{n}(-1)^k q^{\binom{k}{2}+k\lambda}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-q^{k-m}}{(1-xq^{k-m})^{\lambda+1}}(zq^{-k};q)_k(zq^{-l};q)_{n-k+\lambda-1} \\
=&\sum_{k=0,\neq m}^{n}(-1)^k q^{\binom{k}{2}+k\lambda}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-q^{k-m}}{(1-xq^{k-m})^{\lambda+1}}\frac{(zq^{-k};q)_\infty}{(z;q)_\infty}\frac{(zq^{-l};q)_\infty}{(zq^{n+\lambda-1-l-k};q)_\infty} \\
=&(zq^{-l};q)_l\sum_{k=0,\neq m}^{n}(-1)^k q^{\binom{k}{2}+k\lambda}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1-q^{k-m}}{(1-xq^{k-m})^{\lambda+1}}(zq^{-k};q)_{n-l+\lambda-1}.
\end{aligned}$$

Thus, by the q-Rice formula (2.1) there holds

$$\sum_{k=0,\neq m}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{(\lambda-1)k}(1-q^{k-m})(q/z;q)_k(zq^{-l};q)_{n-k+\lambda-1}}{(1-xq^{k-m})^{\lambda+1}} z^k$$
$$=(-1)^n(zq^{-l};q)_l \frac{(q;q)_n}{q^{\binom{n+1}{2}}} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(zt;q)_{n-l+\lambda-1}dt}{(t-xq^{-m})^{\lambda+1}\prod_{k=0,\neq m}^{n}(t-q^{-k})}, \tag{2.2}$$

where $\mathcal{C}$ (positively oriented) encloses the poles q^{-j}, $j \in \{0,1,\ldots,n\}-\{m\}$ and no other. It is obvious that

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(zt;q)_{n-l+\lambda-1}dt}{(t-xq^{-m})^{\lambda+1}\prod_{k=0,\neq m}^{n}(t-q^{-k})} = -\frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{(zt;q)_{n-l+\lambda-1}dt}{(t-xq^{-m})^{\lambda+1}\prod_{k=0,\neq m}^{n}(t-q^{-k})}, \tag{2.3}$$

where $\mathcal{C}'$ (positively oriented) encloses the pole xq^{-m}. By Cauchy's integral formula, there holds

$$\frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{(zt;q)_{n-l+\lambda-1}}{(t-xq^{-m})^{\lambda+1}\prod_{k=0,\neq m}^{n}(t-q^{-k})} = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda} \frac{(zt;q)_{n-l+\lambda-1}}{\prod_{k=0,\neq m}^{n}(t-q^{-k})}\bigg|_{t=xq^{-m}}. \tag{2.4}$$

Applying Faà di Bruno's formula [4] yields

$$\frac{d^\lambda}{dt^\lambda} \frac{(zt;q)_{n-l+\lambda-1}}{\prod_{k=0,\neq m}^{n}(t-q^{-k})}\bigg|_{t=xq^{-m}} = \frac{d^\lambda}{dt^\lambda} e^{\sum_{k=0}^{n-l+\lambda-2}\log(1-ztq^k)-\sum_{k=0,\neq m}^{n}\log(t-q^{-k})}\bigg|_{t=xq^{-m}}$$
$$= \frac{(zxq^{-m};q)_{n-l+\lambda-1}}{\prod_{k=0,\neq m}^{n}(xq^{-m}-q^{-k})} \sum_{\|\vec{c}\|=\lambda} \frac{1}{\vec{c}!} \prod_{j=1}^{\lambda} \left(\frac{u_j}{j}\right)^{c_j}. \tag{2.5}$$

From (2.2), (2.3),(2.4) and (2.5), the desired result is obtained.

Remark 2.1. *Actually, careful checking the proof of Theorem* 1.1*, one can find that Theorem* 1.1 *still holds for* $\lambda = 0$ *if in this case we assume the sum of the right hand side of* (1.8) *is equal to* 1*. This implies that for* $0 \leqslant l \leqslant n-1$ *there holds*

$$\sum_{k=0,\neq m}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1-q^{m-k})(q/z;q)_k(zq^{-l};q)_{n-k-1}}{1-xq^{k-m}} z^k = \frac{(q;q)_n(zq^{-l};q)_l(zxq^{-m};q)_{n-l-1}}{(xq^{-m};q)_m(xq;q)_{n-m}}.$$

3 Consequences of Theorem 1.1

Theorem 1.1 can help us to find some new identities or retrieve some well known identities.

Let $\lambda = 1$ and $x = 1$. (1.8) reduces to the following identity.

Corollary 3.1. *For $0 \leqslant m \leqslant n$ and $0 \leqslant l \leqslant n$, there holds*

$$\sum_{k=0,\neq m}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q/z;q)_k (zq^{-l};q)_{n-k}}{1-q^{k-m}} z^k$$
$$= (-1)^{m-1} q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (zq^{-l};q)_l (zq^{-m};q)_{n-l} \left\{ -\sum_{k=0}^{n-l-1} \frac{zq^k}{1-zq^{k-m}} + \sum_{k=0,\neq m}^{n} \frac{q^k}{1-q^{k-m}} \right\}. \tag{3.1}$$

Guo and Zhang [12] made use of Lagrange interpolation formula to obtain this identity which generalizes the identity (1.3) due to Prodinger. It is obvious that (3.1) reduces to (1.3) when $l=0$ and $z \to 0$.

Let $x=1$, $l=\lambda-1$ and $z=q^{-n}$ in (1.8). We have

Corollary 3.2. *Let λ be any nonnegative integer. For $0 \leqslant m \leqslant n$, there holds*

$$\sum_{k=0,\neq m}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2}+(\lambda-n)k}}{(1-q^{k-m})^\lambda}$$
$$= (-1)^m q^{\binom{m}{2}-nm} \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n+m \\ n \end{bmatrix}_q \sum_{\|\vec{c}\|=\lambda} \frac{1}{\vec{c}!} \prod_{j=1}^{\lambda} \left(\frac{\mathcal{H}_j(q)}{j} \right)^{c_j},$$

where

$$\mathcal{H}_j(q) = -\sum_{k=0}^{n-1} \left(\frac{q^{k-n}}{1-q^{k-m-n}} \right)^j + \sum_{k=0,\neq m}^{n} \left(\frac{q^k}{1-q^{k-m}} \right)^j.$$

This identity is a q-analog of Prodinger's identity (1.7). An alternative form of this q-identity was presented in [14].

For $m=0$, $l=0$ and $z \to 0$ in (1.8), the following identity is true.

Corollary 3.3. *Let λ be any nonnegative integer. There holds*

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} q^{\binom{k}{2}+\lambda k} \frac{1-q^k}{(1-xq^k)^{\lambda+1}} = \frac{(q;q)_n}{(xq;q)_n} \sum_{\|\vec{c}\|=\lambda} \frac{1}{\vec{c}!} \prod_{j=1}^{\lambda} \left(\frac{1}{j} \sum_{k=1}^{n} \frac{q^{jk}}{(1-xq^k)^j} \right)^{c_j}. \tag{3.2}$$

It is clear that

$$\prod_{j=1}^{n} \frac{1}{1-x_j t} = \prod_{j=1}^{n} \sum_{k \geqslant 0} (x_j t)^k = \sum_{\lambda \geqslant 0} t^\lambda \sum_{1 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_\lambda \leqslant n} \prod_{j=1}^{\lambda} x_{\alpha_j}. \tag{3.3}$$

Since

$$\frac{d^\lambda}{dt^\lambda} \prod_{j=1}^{n} \frac{1}{1-x_j t} \bigg|_{t=0} = \frac{d^\lambda}{dt^\lambda} e^{-\sum_{j=1}^{n} \log(1-x_j t)} \bigg|_{t=0},$$

we apply Faà di Bruno's formula [4] to obtain

$$\frac{d^\lambda}{dt^\lambda}\prod_{j=1}^{n}\frac{1}{1-x_jt}\bigg|_{t=0}=\sum_{\|\vec{c}\|=\lambda}\frac{\lambda!}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{\sum_{k=1}^{n}x_k^j}{j}\right)^{c_j}. \tag{3.4}$$

Comparing (3.3) with (3.4), there holds

$$\sum_{1\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_\lambda\leqslant n}\prod_{j=1}^{\lambda}x_{\alpha_j}=\sum_{\|\vec{c}\|=\lambda}\frac{1}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{\sum_{k=1}^{n}x_k^j}{j}\right)^{c_j}.$$

Therefore, (3.2) can be rewritten as

$$\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{k-1}q^{\binom{k}{2}+\lambda k}\frac{1-q^k}{(1-xq^k)^{\lambda+1}}=\frac{(q;q)_n}{(xq;q)_n}\sum_{1\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_\lambda\leqslant n}\prod_{j=1}^{\lambda}\frac{q^{\alpha_j}}{1-xq^{\alpha_j}},$$

or

$$\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{k-1}q^{\binom{k}{2}+\lambda k}\frac{1-q^k}{(1-xq^k)^{\lambda+1}}=\frac{(q;q)_n}{(xq;q)_n}\sum_{|\vec{b}|=\lambda}\prod_{j=1}^{n}\left(\frac{q^j}{1-xq^j}\right)^{b_j}, \tag{3.5}$$

where $|\vec{b}|=b_1+b_2+\cdots+b_n$. By the theory of basic hypergeometric functions Ismail and Stanton [13] found Eq. (3.5) which reduces to the Dilcher identity [6] when $x=1$.

In fact, it has been recently pointed out in [11] that the Ismail-Stanton result (3.5) is the $i=1$ (with $m=\lambda+1$) case of following formula due to Zeng [23]:

$$\sum_{k=i}^{n}(-1)^{k-i}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\i\end{bmatrix}_q\frac{q^{\binom{k-i}{2}+km}}{(1-zq^k)^m}=\frac{q^i(q;q)_{i-1}(q;q)_n}{(q;q)_i(zq;q)_n}h_{m-1}\left(\frac{q^i}{1-zq^i},\ldots,\frac{q^n}{1-zq^n}\right),$$

where $1\leqslant i\leqslant n$ and $h_k(x_1,\ldots,x_n)$ is the kth homogeneous symmetric polynomial in $x_1,x_2,\ldots,x_n$ defined by

$$h_k(x_1,\ldots,x_n)=\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n}x_{i_1}\cdots x_{i_k}=\sum_{|\vec{b}|=k}x_1^{b_1}\cdots x_n^{b_n}.$$

This more general formula can not follow from Theorem 1.1 and it can be viewed as a different generalization of the Ismail-Stanton result (3.5).

Since

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\frac{1-q^n}{1-q^k},$$

Eq. (3.2) can be rewritten as

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q q^{\binom{k}{2}+\lambda k}\frac{1}{(1-xq^k)^{\lambda+1}}=\frac{(q;q)_{n-1}}{(xq;q)_n}\sum_{\|\vec{c}\|=\lambda}\frac{1}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{1}{j}\sum_{\alpha=1}^{n}\frac{q^{j\alpha}}{(1-xq^\alpha)^j}\right)^{c_j}.$$

Using the q-inverse pair formula [10]

$$f_n=\sum_{k=1}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q g_k \Leftrightarrow g_n=\sum_{k=1}^{n}(-1)^k q^{\binom{k}{2}-k(n-1)}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q f_k,$$

we obtain the inverse of (3.2)

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}-k(n-1)}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q \frac{(q;q)_{k-1}}{(xq;q)_k} \times \sum_{\|\vec{c}\|=\lambda}\frac{1}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{1}{j}\sum_{\alpha=1}^{k}\frac{q^{j\alpha}}{(1-xq^{\alpha})^j}\right)^{c_j}=\frac{q^{n\lambda}}{(1-xq^n)^{\lambda+1}}.$$

Replacing x by q^x, we rediscover an identity due to Mansour et al. [15]:

Corollary 3.4. *Let λ be any nonnegative integer. There holds*

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}-k(n-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \begin{bmatrix}x+k\\k\end{bmatrix}_q^{-1} \times \sum_{\|\vec{c}\|=\lambda}\frac{1}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{1}{j}\sum_{\alpha=1}^{k}\frac{q^{j\alpha}}{[x+\alpha]_q^j}\right)^{c_j}=\frac{q^{n\lambda}[n]_q}{[x+n]_q^{\lambda+1}}. \tag{3.6}$$

This identity is a q-analog for the rational sum identity (1.4) due to Prodinger. If we further replace n by $n+1$ and x by $x-1$ in (3.6), then a q-analog of Chu-Yan's identity (1.5) is derived:

Corollary 3.5. *Let λ be any nonnegative integer. There holds*

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-kn}\begin{bmatrix}n\\k\end{bmatrix}_q \begin{bmatrix}x+k\\k\end{bmatrix}_q^{-1}\sum_{0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_\lambda\leqslant k}\prod_{j=1}^{\lambda}\frac{q^{\alpha_j}}{[x+\alpha_j]_q}=\frac{q^{n(\lambda+1)}[x]_q}{[x+n]_q^{\lambda+1}}.$$

Let the generalized q-harmonic numbers

$$H_0^{(r)}(q)=0,\quad H_n^{(r)}(q)=\sum_{k=1}^{n}q^{rk}[k]^{-r},\quad n\geqslant 1.$$

Recently, the q-generalized harmonic number sums have been useful in studying Feynman diagram contributions an relations among special functions [3]. Taking $x=0$ in (3.6), we have the following identities on q-generalized harmonic numbers:

Corollary 3.6. *For $\lambda\geqslant 1$, there holds*

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-(k+\lambda)n}\begin{bmatrix}n\\k\end{bmatrix}_q\sum_{\|\vec{c}\|=\lambda}\frac{1}{\vec{c}!}\prod_{j=1}^{\lambda}\left(\frac{H_k^{(j)}(q)}{j}\right)^{c_j}=-\frac{1}{[n]_q^{\lambda}}.$$

The first few cases are listed as follows.

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-(k+1)n} {n \brack k}_q H_k(q) = -\frac{1}{[n]_q},$$

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-(k+2)n} {n \brack k}_q \Big((H_k(q))^2 + H_k^{(2)}(q)\Big) = -\frac{2}{[n]_q^2},$$

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-(k+3)n} {n \brack k}_q \Big((H_k(q))^3 + 3H_k(q)H_k^{(2)}(q) + 2H_k^{(3)}(q)\Big) = -\frac{6}{[n]_q^3},$$

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-(k+4)n} {n \brack k}_q \Bigg((H_k(q))^4 + 6\left(H_k(q)\right)^2 H_k^{(2)}(q) + 3\left(H_k^{(2)}(q)\right)^2 + 8H_k(q)H_k^{(3)}(q) + 6H_k^{(4)}(q)\Bigg) = -\frac{24}{[n]_q^4},$$

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-(k+5)n} {n \brack k}_q \Bigg((H_k(q))^5 + 10\left(H_k(q)\right)^3 H_k^{(2)}(q) + 15H_k(q)\left(H_k^{(2)}(q)\right)^2 + 20\left(H_k(q)\right)^2 H_k^{(3)}(q) + 20H_k^{(2)}(q)H_k^{(3)}(q) + 30H_k(q)H_k^{(4)}(q) + 24H_k^{(5)}(q)\Bigg) = -\frac{120}{[n]_q^5}$$

These identities are q-analogs of generalized harmonic number identities which were presented in [22]:

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} H_k = -\frac{1}{n}, \tag{3.7}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_k^2 + H_k^{(2)}\right) = -\frac{2}{n^2}, \tag{3.8}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(3)}\right) = -\frac{6}{n^3}, \tag{3.9}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_k^4 + 6H_k^2H_k^{(2)} + 3\left(H_k^{(2)}\right)^2 + 8H_kH_k^{(3)} + 6H_k^{(4)}\right) = -\frac{24}{n^4}, \tag{3.10}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \Bigg(H_k^5 + 10H_k^3H_k^{(2)} + 15H_k\left(H_k^{(2)}\right)^2 + 20H_k^2H_k^{(3)} + 20H_k^{(2)}H_k^{(3)} + 30H_kH_k^{(4)} + 24H_k^{(5)}\Bigg) = -\frac{120}{n^5} \tag{3.11}$$

It is worth noticing that starting from

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-kn} {n \brack k}_q \frac{(q;q)_k}{(xq;q)_k} = \frac{q^n(1-x)}{1-xq^n}$$

and taking the jth derivative of both sides at $x = 1$, we can also arrive at Corollary 3.6. Wang and Jia [22] applied the Newton-Andrews method to some well known identities and found many interesting identities on harmonic numbers which include the identities from (3.7) to (3.11).

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