

Some New Characterizations of Graph Colorability and of Blocking Sets of Projective Spaces

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Abstract

Let $G = (V, E)$ be a graph and q be an odd prime power. We prove that G possess a proper vertex coloring with q colors if and only if there exists an odd vertex labeling $x \in \mathbb{F}_q^V$ of G . Here x is called odd if there is an odd number of partitions $\pi = \{V_1, V_2, \dots, V_t\}$ of V whose blocks V_i are G -bipartite and x -balanced, i.e., such that $G|_{V_i}$ is connected and bipartite, and $\sum_{v \in V_i} x_v = 0$. Other new characterizations concern edge colorability of graphs and, on a more general level, blocking sets of projective spaces. Some of these characterizations are formulated in terms of a new switching game.

1 Introduction

The results in this paper are based on the study of a new *switching game* (Section 2). We view this game as an auxiliary tool only. Many of our results are formulated without mentioning it and provide new equivalents for the colorability of graphs, for example. Our auxiliary switching game is a single player game with complete information. The player's objective is to modify a certain *initial pattern* step by step and to reach a certain *zero pattern* eventually. The number of moves needed does not matter here. The question is whether it is possible to reach the zero pattern at all. We already studied some similar generalizations of Berlekamp's Switching Game in [Sch1, Sch2]. These games were closely

related to anti-codes (codes with restricted maximal weight) and to flows of graphs. We did also obtain some graph colorability results, some Alon-Tarsi-like results, but only by using the duality of flows and colorings. The switching game that we are going to study here is more directly related to colorings. Actually, there is only one natural step between our switching games and graph colorability, which are *blocking sets* (Section 3) of finite vector spaces $\mathcal{V} = \mathbb{F}_q^n$. As blocking sets over \mathbb{F}_q are a generalization of non- q -colorable graphs, as we will see, it is natural to study them first. We will show that a subset $B \subseteq \mathcal{V}$ is a blocking set if and only if there exist a winning strategy in our switching game. Here the set of available moves is defined with respect to B , so that the *switchability* of the initial pattern actually depends on B . We will also present some best possible strategies, i.e., general strategies that turn out to be winning strategies whenever the game actually can be won. All these strategies and the whole concept of switchability can be expressed as mere linear combinations and linear dependencies of certain vectors in $\mathbb{Z}^{\mathcal{V}}$ (or in $\mathbb{Z}_r^{\mathcal{V}}$).

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We then turn to graphs (Section 4). Graphs $G = (V, E)$ correspond to blocking sets with special properties. Hence, our results about blocking sets hold for graphs as well, and we can even simplify or improve them based on the additional properties of graphs. It turns out that some quantitative statements about certain classes of partitions of V are equivalent to the q -colorability of G . The general form of these equivalents might be technical, but in some special cases we obtain cleaner results. For example, if q is an odd prime power, we can prove that the graph G possesses a proper vertex coloring using at most q many colors if and only if there exists an odd vertex labeling $x \in \mathbb{F}_q^V$ of G . Here x is called odd if there is an odd number of partitions $\pi = \{V_1, V_2, \dots, V_t\}$ of V whose blocks V_i are G -bipartite and x -balanced, i.e., such that $G|_{V_i}$ is connected and bipartite, and $\sum_{v \in V_i} x_v = 0$. Other new characterizations concern edge colorability of graphs.

$G = (V, E)$

We also obtain some results with respect to the number of colorings in modular arithmetic, when we count modulo some $r \geq 2$. Many algebraic tools work or can be applied in modular arithmetic. In some of the applications of such tools, it can be important to know that the number of colorings of a graph is nonzero modulo r . This can be a problem. For example, one can sometimes use the Combinatorial Nullstellensatz to prove the colorability of some graphs or to generalize colorability results to list coloring results, as e.g. in [Sch3, Sec. 5]. In such approaches, one usually has to show that a certain coefficient in a certain polynomial is nonzero. Sometimes, this coefficient equals the number of colorings of the examined graph G . Hence, it seems that, in that case, one should already be satisfied with the existence of colorings. However, in modular arithmetic, having a coloring and having a nonvanishing number of colorings is not the same. The number might be zero modulo r , even though colorings exist. Therefore, it might be good to know that, with just one additional linear restriction on the colorings, one can actually force that number to be nonzero modulo r . More precisely, if $\mathcal{C}_q(G) \subseteq \mathbb{F}_q^n$ denotes the set of colorings with colors taken from \mathbb{F}_q , we show that to any divisor r of $q - 1$ there exists an $x \in \mathcal{V} = \mathbb{F}_q^n$ such that $\frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1}$ is nonzero modulo r , provided $\mathcal{C}_q(G) \neq \emptyset$. It is conceivable that one can find ways to employ this, for example in connection with the Combinatorial Nullstellensatz.

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$\mathcal{C}_q(G)$

2 Switching Games

Our single player switching game is played on the *board* $\mathcal{V} := \mathbb{F}_q^n$. A *pattern* is a function $f: \mathcal{V} \rightarrow \mathbb{Z}$. Typically, we start with the fixed *initial pattern* $f_0 := q^{n-1}\{0\}$, defined by

$$q^{n-1}\{0\}(x) := \begin{cases} q^{n-1} & \text{if } x = 0, \\ 0 & \text{if } x \in \mathcal{V} \setminus 0. \end{cases} \quad (1)$$

This initial pattern will then be modified by adding integral multiples mf of certain *available* patterns f ,

$$f_0 \mapsto f_0 + mf. \quad (2)$$

This is called making a *move*. We also say that we *switch* f exactly m many times, where negative multiplicities m are allowed. The class of available patterns f will be defined later and will vary from case to case; hence, actually we will examine several different games. We say that we can *switch off* f_0 , or that f_0 is *switchable*, if there is a finite sequence of moves that results in the zero function denoted \emptyset . For an integer $r \geq 2$, we say that we can switch off f_0 *modulo* r , or that f_0 is switchable modulo r , if we can switch it off, but with all function values of all patterns taken modulo r . One could also say that a *winning strategy* exists, or that it exists modulo r , in that case.

Before we play the game, here are some more conventions:

We identify subsets $U \subseteq \mathcal{V}$ with their characteristic functions $\mathcal{V} \rightarrow \{0, 1\}$ as 0-1 patterns:

$$U(x) := \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \in \mathcal{V} \setminus U. \end{cases} \quad (3)$$

This is used extensively. It simplifies notation but can lead to unusual expressions. For example, the singletons $\{u\} \subseteq \mathcal{V}$ are also viewed as 0-1 patterns

$$\{u\}: \mathcal{V} \rightarrow \{0, 1\}, \quad x \mapsto \{u\}(x). \quad (4)$$

These one-point sets form the standard basis of the \mathbb{Z} -module $\mathbb{Z}^{\mathcal{V}}$ of all patterns. They are usually not available as moves, only the initial pattern, as a multiple of $\{0\}$, could be considered as an unavoidable initial move.

Note that both the board $\mathcal{V} := \mathbb{F}_q^n$, as well as the set of patterns $\mathbb{Z}^{\mathcal{V}}$, carry the structure of a module. Our board \mathcal{V} is a module over \mathbb{F}_q , while $\mathbb{Z}^{\mathcal{V}}$ is a module over \mathbb{Z} . They both provide an addition. Therefore, we have to be careful with the notation. Subsets of \mathcal{V} are usually viewed as 0-1 patterns in $\mathbb{Z}^{\mathcal{V}}$ and added in $(\mathbb{Z}^{\mathcal{V}}, +)$, while elements of \mathcal{V} , or combinations of one element and one subset of \mathcal{V} , are always added in $(\mathcal{V}, +)$. For example, if $n = 2$ and we consider $(0, 1)$ and $(1, 0)$ as two points of the board $\mathcal{V} = \mathbb{F}_q^2$, we have

$$(0, 1) + (1, 0) = (1, 1) \in \mathcal{V}, \quad (5)$$

as we would view the $+$ as addition in \mathcal{V} here, and

$$(0, 1) + \{(1, 0)\} = \{(1, 1)\} \in \{0, 1\}^{\mathcal{V}} \subseteq \mathbb{Z}^{\mathcal{V}}, \quad (6)$$

again with the addition in \mathcal{V} (with $x + U = \{x + u \mid u \in U\}$), but

$$\{(0, 1)\} + \{(1, 0)\} = \{(0, 1), (1, 0)\} \in \mathbb{Z}^{\mathcal{V}}, \quad (7)$$

where the $+$ is meant as addition in $\mathbb{Z}^{\mathcal{V}}$. Similar rules are used for the two scalar multiplications; hence, for example,

$$2\{(1, 0)\} \neq \{2(1, 0)\}, \quad (8)$$

where the left scalar multiplication is meant in $\mathbb{Z}^{\mathcal{V}}$ and produces the pattern that has the value 2 in $(1, 0)$ and 0 elsewhere, while the right scalar multiplication is in $\mathcal{V} = \mathbb{F}_q^2$ and produces the pattern $\{(2, 0)\}$, which has the value 1 in $(2, 0)$ and 0 elsewhere. The reader may try to read the left side of the first equation in Theorem 3.3 with these conventions. There, we view a linear combination inside $\mathbb{Z}^{\mathcal{V}}$ of certain patterns $\langle S \rangle \in \mathbb{Z}^{\mathcal{V}}$, and evaluate it in a point x . However, the patterns $\langle S \rangle$ appear as linear spans inside the vector space \mathcal{V} first. One has to recognize that they, as subsets of \mathcal{V} , also are patterns in $\mathbb{Z}^{\mathcal{V}}$.

3 Blocking Sets

A *blocking set* in the \mathbb{F}_q -vector space $\mathcal{V} := \mathbb{F}_q^n$ is a subset B such that each hyperplane $H \triangleleft \mathcal{V}$ contains at least one element of B , $B \cap H \neq \emptyset$. Usually, this is formulated in the language of projective spaces, but we do not do it here. To characterize blocking sets $B \subseteq \mathcal{V}$ in terms of switchability, we have to recall some basic facts about the subspaces of our \mathbb{F}_q -vector space \mathcal{V} . Such enumerative facts are most elegantly denoted with q -analogs, in particular, the *q -numbers*

$$[m]_q := \frac{q^m - 1}{q - 1} = q^{m-1} + \cdots + q^1 + q^0. \quad (9)$$

Our first lemma counts hyperplanes, i.e., maximal proper subspaces H of \mathcal{V} , $H \triangleleft \mathcal{V}$. We frequently will use them, and define, for $A, B \subseteq \mathcal{V}$ and $a \in \mathcal{V}$,

$$\mathcal{H}_A^B := \{H \triangleleft \mathcal{V} \mid A \subseteq H, B \cap H = \emptyset\}, \quad \mathcal{H}_a^B := \mathcal{H}_{\{a\}}^B, \quad \mathcal{H}^B := \mathcal{H}_{\emptyset}^B, \quad \text{etc.} \quad (10)$$

We recall some basic enumerative facts about \mathcal{H}_A and translate them into our switching game notation:

Lemma 3.1. *Let U be an m -dimensional proper subspace of the n -dimensional vector space $\mathcal{V} := \mathbb{F}_q^n$ over \mathbb{F}_q , and let $x \in \mathcal{V} \setminus U$. Then*

$$|\mathcal{H}_U| = [n-m]_q \quad \text{and} \quad |\mathcal{H}_{U \cup \{x\}}| = [n-m-1]_q.$$

In particular,

$$\sum_{H \in \mathcal{H}_U} H = [n-m]_q U + [n-m-1]_q (\mathcal{V} \setminus U) = q^{n-m-1} U + [n-m-1]_q \mathcal{V},$$

and this equation also holds for the full space $U = \mathcal{V}$ if we set $[-1]_q := -q^{-1}$.

We also will need enumerative facts about \mathcal{H}^B :

Lemma 3.2. *Let $\emptyset \neq B \subseteq \mathcal{V}$. Then*

$$|\mathcal{H}^B| = \sum_{S \subseteq B} (-1)^{|S|} [n - \dim \langle S \rangle]_q = \frac{1}{q-1} \sum_{S \subseteq B} (-1)^{|S|} q^{n - \dim \langle S \rangle},$$

where the first equation also holds for $B = \emptyset$.

Proof. There are $[n - \dim \langle S \rangle]_q$ many hyperplanes containing a given subset $S \subseteq B$. Therefore, the first equation follows from the principle of inclusion exclusion. The second equation easily follows from our Definition (9) of q -numbers, using that, for $B \neq \emptyset$,

$$\sum_{S \subseteq B} (-1)^{|S|} = 0. \quad (11)$$

□

At this point, one may observe that the last formula for $|\mathcal{H}^B|$ contains a not obvious insight about the spanning subsets of blocking sets. If we read our formula only modulo q , we may omit all summands with $\dim \langle S \rangle < n$, so that the sum only runs over spanning subsets $S \subseteq B$, $\langle S \rangle = \mathcal{V}$. Hence, if B is a blocking set, $|\mathcal{H}^B| = 0$, we see that

$$q \text{ divides } \sum_{\substack{S \subseteq B \\ \langle S \rangle = \mathcal{V}}} (-1)^{|S|}. \quad (12)$$

This might be of some interest on its own, but we also need it at some point. More important, however, is the following application of the lemmas above:

Theorem 3.3. *Let $\emptyset \neq B \subseteq \mathcal{V}$ and $x \in \mathcal{V}$. Then*

$$\left(\sum_{S \subseteq B} (-1)^{|S|} q^{n - \dim \langle S \rangle} \langle S \rangle \right) (x) = q|\mathcal{H}_x^B| - |\mathcal{H}^B|.$$

Or, put another way,

$$\sum_{\substack{S \subseteq B \\ \langle S \rangle \ni x}} (-1)^{|S|} q^{n - \dim \langle S \rangle} = q|\mathcal{H}_x^B| - |\mathcal{H}^B|.$$

Proof. Using $q[m-1]_q = [m]_q - 1$, which also holds for $m = 0$, we calculate:

$$\begin{aligned} & \sum_{S \subseteq B} (-1)^{|S|} q^{n - \dim \langle S \rangle} \langle S \rangle \\ & \stackrel{3.1}{=} \sum_{S \subseteq B} (-1)^{|S|} q \left(\sum_{H \in \mathcal{H}_S} H - [n - \dim \langle S \rangle - 1]_q \mathcal{V} \right) \\ & = q \sum_{H \in \mathcal{V}} \left(\sum_{S \subseteq B \cap H} (-1)^{|S|} \right) H - \sum_{S \subseteq B} (-1)^{|S|} [n - \dim \langle S \rangle]_q \mathcal{V} + \sum_{S \subseteq B} (-1)^{|S|} \mathcal{V} \end{aligned}$$

$$\begin{aligned}
&\stackrel{3.2}{=} q \sum_{\substack{H \in \mathcal{V} \\ H \cap B = \emptyset}} H - |\mathcal{H}^B| \mathcal{V} + \emptyset \\
&= \sum_{x \in \mathcal{V}} (q|\mathcal{H}_x^B| - |\mathcal{H}^B|) \{x\}.
\end{aligned}
\quad \square$$

If B is a blocking set, $\mathcal{H}^B = \emptyset$, this theorem simplifies to the following equation:

Corollary 3.4. *If B is a blocking subset of \mathcal{V} , then*

$$\sum_{S \subseteq B} (-1)^{|S|} q^{n - \dim \langle S \rangle} \langle S \rangle = \emptyset.$$

With this, we are able to prove the following theorem:

Theorem 3.5. *Let q be a prime power and assume $r \in \mathbb{N}$ does not divide q^{n-1} . For any subset $B \subseteq \mathcal{V} := \mathbb{F}_q^n$, the following are equivalent over the board \mathcal{V} :*

- (i) *B is a blocking set.*
- (ii) *The pattern $q^{n-1}\{0\}$ can be switched off by switching each hyperplane that contains at least one $b \in B$ exactly -1 times and by switching the full space \mathcal{V} exactly $[n-1]_q$ many times.*
- (iii) *The pattern $q^{n-1}\{0\}$ can be switched off by switching for each nonempty $S \subseteq B$ the span $\langle S \rangle$ exactly $(-1)^{|S|} q^{n - \dim \langle S \rangle - 1}$ many times (which then automatically adds up to an integral number of switches for any possible span $\langle S \rangle$).*
- (iv) *The pattern $q^{n-1}\{0\}$ can be switched off modulo r by switching affine lines $u + \langle b \rangle$ with directions b taken from B .*
- (v) *Some modulo r nonvanishing multiple $\ell\{0\}$ of $\{0\}$ can be switched off modulo r by switching affine subspaces $u + U$ whose corresponding direction subspaces U contain at least one $b \in B$.*

Proof. To prove (i) \Rightarrow (ii), assume that B is a blocking set. Then any hyperplane H contains at least one element $b \in B$ and shall be switched in Statement (ii). If we actually switch each hyperplane H exactly -1 time, then the point 0 is switched $-[n]_q$ many times, while any other point $x \neq 0$ is switched $-[n-1]_q$ many times. Thus, if we also switch the full space \mathcal{V} exactly $[n-1]_q$ many times, the points $x \neq 0$ remain unswitched, while 0 is switched $-[n]_q + [n-1]_q = -q^{n-1}$ many times, canceling the initial pattern $q^{n-1}\{0\}$.

To gain insight in the implication (i) \Rightarrow (iii), we divide the equation in Corollary 3.4 by q and move the summand corresponding to $S = \emptyset$ to the other side. The equation obtained can then be viewed as a switching procedure. As mentioned, this procedure necessarily involves an integral number of switches of any $\langle S \rangle$, which easily follows from Statement (12).

The implications $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (iv)$ follow from the fact that any subspace that contains a $b \in B$ can be decomposed into parallels of the line $\langle b \rangle$.

The implication $(iv) \Rightarrow (v)$ is trivial.

In order to prove $\neg(i) \Rightarrow \neg(v)$, assume that a hyperplane H is not blocked by B . Then any legitimate move $u + U$ in Part (v) is not parallel to H , as there is a $b \in B$ with

$$U \ni b \notin H. \quad (13)$$

It intersects H , as well as any fixed parallel $h + H \neq H$ of H , in exactly $q^{\dim(U)-1}$ many points,

$$|(u + U) \cap (h + H)| = q^{\dim(U)-1} = |(u + U) \cap H|. \quad (14)$$

Applied to any pattern $f \in \mathbb{Z}^{\mathcal{V}}$, the move $u + U$ will not change the difference

$$\Delta(f) := \sum_{x \in H} f(x) - \sum_{x \in h+H} f(x). \quad (15)$$

This difference is an invariant,

$$\Delta(f + (u + U)) = \Delta(f), \quad (16)$$

where, in this expression, one should recognize the different additions in \mathcal{V} and $\mathbb{Z}^{\mathcal{V}}$. If the invariant is nonzero at the beginning, then it will stay nonzero forever. In particular, the pattern $f = \ell\{0\}$, with $\ell \not\equiv 0 \pmod{r}$, cannot be transformed into \emptyset modulo r , as this would require a change of the invariant modulo r . \square

An equivalence like this is usually more powerful if it connects weak and strong statements; thus, we tried to make the weak ones weaker and the strong ones stronger. In our theorem, Statement (iv) is relatively weak and most similar to the switching rules in the Berlekamp games described in [Sch1] and [Sch2]. The weakest switchability statement in our equivalence is Statement (v), but it could be weakened even more. If $q = p^\beta$ and \mathbb{F}_p is the prime field of \mathbb{F}_q , we may even allow switching of all affine \mathbb{F}_p -subspaces $u + U$ with $U \supseteq \langle b \rangle_{\mathbb{F}_q}$ for some $b \in B$. Our strong switchability statements (ii) and (iii) are already very narrow and precise. They describe the best possible winning strategies mentioned in the introduction. In all these different games, the reader may just wonder about the unexpected multiplier q^{n-1} in our initial pattern $q^{n-1}\{0\}$. This actually can be altered in some cases. If we play modulo r , with $r \in \mathbb{N} \setminus 0$ coprime to q (so that 1 is a multiple of q^{n-1} modulo r), then we may replace $q^{n-1}\{0\}$ by the neater $\{0\}$ as initial pattern.

The next equivalent is formulated without mentioning any games. Its equations might also be viewed as linear dependencies of some patterns in $\mathbb{Z}^{\mathcal{V}}$:

Theorem 3.6. *Let q be a prime power and $\emptyset \neq B \subseteq \mathcal{V} := \mathbb{F}_q^n$. If $r > 1$ divides $q - 1$, then the following are equivalent:*

- (i) B is a blocking set.
- (ii) $\sum_{S \subseteq B} (-1)^{|S|} q^{n - \dim \langle S \rangle} = 0$.

$$(iii) \sum_{\substack{S \subseteq B \\ \langle S \rangle \ni x}} (-1)^{|S|} q^{n-\dim \langle S \rangle} = 0 \quad \text{for all } x \in \mathcal{V}.$$

$$(iv) \sum_{\substack{S \subseteq B \\ \langle S \rangle \ni x}} (-1)^{|S|} \equiv 0 \pmod{r} \quad \text{for all } x \in \mathcal{V}.$$

Moreover, if Condition (iii) or Condition (iv) is not fulfilled in $x = 0$, then these conditions are not fulfilled in a further point $x \in \mathcal{V}$.

Proof. The equivalence (i) \Leftrightarrow (ii) is the content of Lemma 3.2.

The implication (i) \Rightarrow (iii) is the content of Corollary 3.4, and (iii) \Rightarrow (iv) is trivial.

That (iv) \Rightarrow (i), follows from the implication (v) \Rightarrow (i) in Theorem 3.5. Indeed, assuming Condition (iv), we know that the switchable pattern

$$U := \sum_{\substack{S \subseteq B \\ S \neq \emptyset}} (-1)^{|S|} q^{n-\dim \langle S \rangle} \langle S \rangle \equiv \sum_{\substack{S \subseteq B \\ S \neq \emptyset}} (-1)^{|S|} \pmod{r} \quad (17)$$

is zero modulo r in any $x \neq 0$. However, in 0 it is nonzero modulo r , as by Theorem 3.3,

$$U(0) = q|\mathcal{H}_0^B| - |\mathcal{H}^B| - q^n = (q-1)|\mathcal{H}^B| - q^n \equiv -q^n \equiv -1 \not\equiv 0 \pmod{r}. \quad (18)$$

Hence, by Theorem 3.5, B is a blocking set.

To verify the additional statement, assume that Condition (iii) or Condition (iv) is not fulfilled for $x = 0$. Then, as we have shown already, B is not a blocking set, so that no switchable pattern is a (modulo r nonvanishing) multiple of $\{0\}$. In particular, this holds for the pattern U above. Therefore, as already $U(0) \not\equiv 0 \pmod{r}$, U must differ from the pattern $U(0)\{0\}$, and be nonzero modulo r in another point x . However, for $x \neq 0$, $U(x)$ is, modulo r , exactly the value of the sums in Condition (iii) and Condition (iv); hence, both conditions are not met in that point, too. \square

We also provide the following more quantitative version:

Theorem 3.7. *Let q be a prime power, $\emptyset \neq B \subseteq \mathcal{V} := \mathbb{F}_q^n$ and assume $r > 1$ is a divisor of $q - 1$. Then*

$$\sum_{\substack{S \subseteq B \\ \langle S \rangle \ni x}} (-1)^{|S|} \equiv -|\mathcal{H}^{B \cup \{x\}}| \pmod{r} \quad \text{for all } x \in \mathcal{V},$$

and

$$|\mathcal{H}^{B \cup \{x\}}| \not\equiv 0 \pmod{r} \quad \text{for at least one } x \in \mathcal{V} \setminus 0$$

if and only if B is not a blocking set.

Proof. The first part can be deduced from Theorem 3.3 as follows:

$$\sum_{\substack{S \subseteq B \\ \langle S \rangle \ni x}} (-1)^{|S|} \equiv \sum_{\substack{S \subseteq B \\ \langle S \rangle \ni x}} (-1)^{|S|} q^{n-\dim \langle S \rangle} \equiv 1|\mathcal{H}_x^B| - |\mathcal{H}^B| = -|\mathcal{H}^B \setminus \mathcal{H}_x^B| = -|\mathcal{H}^{B \cup \{x\}}| \pmod{r}. \quad (19)$$

The second part ensues from this equality and Equivalence 3.6. \square

We close this section with an example. This example shows that, in Equivalent (v) of Theorem 3.5, we cannot ignore the point of origin $0 \in \mathcal{V}$. If this were possible, it would lead to a characterization of blocking sets through the linear independence of the corresponding available moves (as patterns over $\mathcal{V} \setminus 0$). Actually, during our research, we had the idea that this could be possible. In the similar switching game in our paper [Sch2] it was possible to ignore the point of origin (and we then played the game over the projective space $\text{PG}_{n-1}(\mathbb{F}_q)$). The example is the following:

Example 3.8. Let $\mathcal{V} := \mathbb{F}_2^3$ and $B := \{e_1, e_2, e_3 + e_2, e_3 + e_1\}$, where e_1, e_2 and e_3 denote the vectors of the standard base of \mathcal{V} . Then $\langle e_1 + e_2 \rangle^\perp$ is not blocked by B , so that we cannot switch patterns of the form $\ell\{0\} \neq \emptyset$ in any of our games. However,

$$e_1^\perp - e_2^\perp - \langle e_2 \rangle - \langle e_3 + e_2 \rangle + \langle e_1 \rangle + \langle e_3 + e_1 \rangle = \emptyset. \quad (20)$$

This linear dependence would be a proper switchability condition over the board $\mathcal{V} \setminus 0$, where $\ell\{0\} = \emptyset$. Indeed, it is easy to check that all summands here are legal moves in the game of Equivalent (v) of Theorem 3.5.

In the next section, we will connect graphs to blocking sets. Actually, with this connection, one can show that another counterexample arises from the complete tripartite graph $K_{7,7,7}$, for instance. Hence, even in the more special graph theoretical situation, the point of origin is indispensable.

4 Graph Colorings

In this section, let $G = (V, E)$ be a graph on n vertices with at least one edge. For vertices $v \in V$, let

$$\hat{v} := (?_{(u=v)})_{u \in V} \in \mathcal{V} := \mathbb{F}_q^V \quad (21)$$

be the tuple with a 1 in position v and zeros elsewhere. For edges $e = uv := \{u, v\} \in E$, we define

$$\hat{e} := \hat{u} - \hat{v} \in \mathcal{V}. \quad (22)$$

Obviously, this is not well defined, since the edge e can be written in two ways, as uv and as vu . However, \hat{e} is well defined up to the sign only, but the sign will not matter in our investigations. Therefore, we just stick to one of the two possible choices for each edge e . With that, we define to every set $S \subseteq E$ of edges a corresponding set \hat{S} of vectors,

$$\hat{S} := \{\hat{e} \mid e \in S\} \subseteq \mathcal{V}, \quad (23)$$

and a corresponding partition $\pi(S) = \pi(V, S)$ of the set of vertices V ,

$$\pi(S) := \{U \subseteq V \mid U \text{ is vertex set of a connected component of the graph } (V, S)\}. \quad (24)$$

Hence, if $\Pi(V)$ denotes the set of all partitions π of V , and $\Pi_c(G)$ denotes the set of those π whose blocks $U \in \pi$ are connected in G ,

$$\Pi_c(G) := \{\pi \in \Pi(V) \mid \forall U \in \pi: G|_U \text{ is connected}\}, \quad (25)$$

then

$$\{\pi(S) \mid S \subseteq E\} = \Pi_c(G). \quad (26)$$

Indeed, one can easily find to any $\pi \in \Pi_c(G)$ an edge set $S_\pi \subseteq E$ with $\pi(S_\pi) = \pi$, and use this fact to prove this equation. Moreover, for $x \in \mathcal{V} = \mathbb{F}_q^V$, we say that a subset $U \subseteq V$ of vertices is *x-balanced* if the restriction $x|_U: U \longrightarrow \mathbb{F}_q, u \longmapsto x_u$ has vanishing value sum,

$$\sum_{u \in U} x_u = 0. \quad (27)$$

A partition π of V is *x-balanced* if each of its blocks is *x-balanced*. We denote with $\langle \pi \rangle_q$, or just $\langle \pi \rangle$, the subspace of all vectors x for which π is *x-balanced*,

$$\langle \pi \rangle = \langle \pi \rangle_q := \left\{ x \in \mathcal{V} \mid \sum_{u \in U} x_u = 0 \text{ for all } U \in \pi \right\} \leq \mathcal{V}. \quad (28)$$

We can use our definitions to calculate the linear spans $\langle \hat{S} \rangle$ that correspond to edge sets $S \subseteq E$. In the simplest case, if (V, S) is connected,

$$\langle \hat{S} \rangle = \{x \in \mathcal{V} \mid \sum_{v \in V} x_v = 0\} = \langle \{V\} \rangle. \quad (29)$$

Here, $\{V\}$ is the trivial partition of V , and our definition of $\langle \pi \rangle$ simplifies for $\pi = \{V\}$ to the set in the middle. That this set contains \hat{S} and its linear span $\langle \hat{S} \rangle$ is obvious. Conversely, that the set in the middle is contained in $\langle \hat{S} \rangle$ follows easily from the fact that, due to the connectedness of (V, S) , all vectors $x \in \mathcal{V}$ with just one 1 and one -1 entry (apart from zeros) already belong to $\langle \hat{S} \rangle$. The general case can be reduced to this case. Obviously, for any $S \subseteq E$,

$$\langle \hat{S} \rangle = \langle \pi(S) \rangle. \quad (30)$$

In particular,

$$\dim \langle \hat{S} \rangle = \dim \langle \pi(S) \rangle = n - |\pi(S)|. \quad (31)$$

We will also need that

$$\begin{aligned} \sum_{\substack{S \subseteq E \\ \langle \pi(S) \rangle \ni x}} (-1)^{|S|} &= \sum_{\substack{\pi \in \Pi_c(G) \\ \langle \pi \rangle \ni x}} \sum_{\substack{S \subseteq E \\ \pi(S) = \pi}} (-1)^{|S|} \\ &= \sum_{\substack{\pi \in \Pi_c(G) \\ \langle \pi \rangle \ni x}} \sum_{\substack{S \subseteq E \\ \pi(S) = \pi}} \prod_{U \in \pi} (-1)^{|S|_U} \\ &= \sum_{\substack{\pi \in \Pi_c(G) \\ \langle \pi \rangle \ni x}} \prod_{U \in \pi} \sum_{\substack{S \subseteq E(G|_U) \\ \pi(U, S) = \{U\}}} (-1)^{|S|}, \end{aligned} \quad (32)$$

where, in the first equation, we replaced the condition $\langle \pi(S) \rangle \ni x$, which means that $\pi(S) \in \Pi_c(G)$ is *x-balanced*, by the condition that $\pi(S)$ is equal to a partition $\pi \in \Pi_c(G)$

that is x -balanced. In the second equation, we just decomposed the edge sets S into their connected components $S|_U$. Finally, in the third equation, we have to find (and then sum over) all edge sets $S \subseteq E$ with the blocks U of π as connected components. This is the same as, first, to find to every block U of π all edge sets S_U that connect U , and, second, to combine them in all possible ways (which the distributive law accomplishes for us automatically).

With these definitions, it should be clear that G has a proper vertex coloring with q colors if and only if \hat{E} is not a blocking set,

$\mathcal{C}_q(G)$

$$\mathcal{C}_q(G) := \{c \in \mathcal{V} \mid c \text{ is coloring of } G\} \neq \emptyset \iff \mathcal{H}^{\hat{E}} \neq \emptyset. \quad (33)$$

Indeed, if $c \in \mathcal{V}$ ($c: V \rightarrow \mathbb{F}_q$) is a vertex labeling then, for any edge $e = uv \in E$,

$$c(u) \neq c(v) \iff c(u) - c(v) \neq 0 \iff c \cdot \hat{e} \neq 0 \iff c^\perp \not\supseteq \hat{e}, \quad (34)$$

with the standard scalar product $\pm c \cdot \hat{e} = c(u) - c(v)$ in \mathcal{V} . Hence,

$$c \in \mathcal{C}_q(G) \iff c^\perp \in \mathcal{H}^{\hat{E}}. \quad (35)$$

The correspondence between hyperplanes and colorings is 1 to $q - 1$ here, as there are $q - 1$ elements in the multiplicative group of \mathbb{F}_q (and $0 \notin \mathcal{C}_q(G)$, as G contains an edge). Hence,

$$|\mathcal{C}_q(G)| = (q-1)|\mathcal{H}^{\hat{E}}|. \quad (36)$$

Our correspondence also can be restricted to $\mathcal{C}_q(G) \setminus x^\perp$ and $\mathcal{H}^{\hat{E} \cap \{x\}}$ (for any $x \in \mathcal{V}$), so that

$$|\mathcal{C}_q(G) \setminus x^\perp| = (q-1)|\mathcal{H}^{\hat{E} \cup \{x\}}|. \quad (37)$$

In particular,

$$q - 1 \text{ divides } |\mathcal{C}_q(G) \setminus x^\perp|, \quad (38)$$

which we will not mention further when we use the quotient of these two numbers. Another well known fact, is the following formula:

$$|\mathcal{C}_q(G)| = \sum_{S \subseteq E} (-1)^{|S|} q^{|\pi(S)|}. \quad (39)$$

We know this already from Lemma 3.2. However, a main difference here, and the main reason for the additional results about graphs in this section, is that this does not just hold for one initially given prime power q . In contrast to the blocking set candidates $B \subseteq \mathcal{V} := \mathbb{F}_q^V$ of the last section, the graph G is defined independently from q and \mathbb{F}_q . Accordingly, the set \hat{E} can be interpreted as a subset of \mathbb{F}_q^V for any prime power q . Hence, $\mathcal{C}_q(G)$ is defined, and the formula holds, for every prime power q . Actually, not even the restriction to prime powers is required here. We just have to extend our definition of $\mathcal{C}_q(G) \subseteq \mathcal{V} := \mathbb{F}_q^V$ slightly. In fact, our definition inside Statement (33) makes sense for any set \mathbb{F}_q of q fixed colors, if we set $\mathcal{V} := \mathbb{F}_q^V$ as before. We only need \mathbb{F}_q to be a field

when we apply the results of the previous sections, and then q necessarily has to be a prime power.

We start the transfer of results with the following specialization of Theorem 3.6, which contains one additional equivalent based on Equation (32):

Theorem 4.1. *Let q be a prime power. If $r > 1$ is a divisor of $q - 1$, then the following are equivalent:*

- (i) G has a q -coloring.
- (ii) $\sum_{S \subseteq E} (-1)^{|S|} q^{|\pi(S)|} \neq 0$.
- (iii) $\sum_{\substack{S \subseteq E \\ \langle \pi(S) \rangle \ni x}} (-1)^{|S|} q^{|\pi(S)|} \neq 0$ for an $x \in \mathcal{V}$.
- (iv) $\sum_{\substack{S \subseteq E \\ \langle \pi(S) \rangle \ni x}} (-1)^{|S|} \not\equiv 0 \pmod{r}$ for an $x \in \mathcal{V}$.
- (v) $\sum_{\substack{\pi \in \Pi_c(G) \\ \langle \pi \rangle \ni x}} \prod_{U \in \pi} \sum_{\substack{S \subseteq E(G|_U) \\ \pi(U, S) = \{U\}}} (-1)^{|S|} \not\equiv 0 \pmod{r}$ for an $x \in \mathcal{V}$.

Moreover, if Condition (iii) or (iv) or (v) is satisfied in $x = 0$, then these conditions hold true in a further point $x \in \mathcal{V}$.

From Theorem 3.7, we obtain the following:

Theorem 4.2. *Let q be a prime power, and assume $r > 1$ is a divisor of $q - 1$. Then*

$$\sum_{\substack{\pi \in \Pi_c(G) \\ \langle \pi \rangle \ni x}} \prod_{U \in \pi} \sum_{\substack{S \subseteq E(G|_U) \\ \pi(U, S) = \{U\}}} (-1)^{|S|} = \sum_{\substack{S \subseteq E \\ \langle \pi(S) \rangle \ni x}} (-1)^{|S|} \equiv -\frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1} \pmod{r} \quad \text{for all } x \in \mathcal{V},$$

and

$$\frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1} \not\equiv 0 \pmod{r} \quad \text{for at least one } x \in \mathcal{V} \setminus 0,$$

if and only if G is q -colorable.

We also mention that the expression $|\mathcal{C}_q(G) \setminus x^\perp|$ in this theorem can be replaced by $-|\mathcal{C}_q(G) \cap x^\perp|$ frequently. For example, we know that

$$q! \text{ divides } |\mathcal{C}_q(G)|, \tag{40}$$

if G is not colorable with just $q - 2$ colors, as there are $q!$ many permutations of q colors. Hence, in this case, if $r \neq q - 1$, indeed,

$$\frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1} = \frac{|\mathcal{C}_q(G)|}{q-1} - \frac{|\mathcal{C}_q(G) \cap x^\perp|}{q-1} \equiv -\frac{|\mathcal{C}_q(G) \cap x^\perp|}{q-1} \pmod{r}. \tag{41}$$

For $r = 2$, we also obtain the following corollary:

Corollary 4.3. *Let q be an odd prime power. Then*

$$|\{S \subseteq E \mid \langle \pi(S) \rangle \ni x\}| \equiv \frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1} \pmod{2} \quad \text{for any } x \in \mathcal{V},$$

and

$$|\{S \subseteq E \mid \langle \pi(S) \rangle \ni x\}| \not\equiv 0 \pmod{2} \quad \text{for one } x \in \mathcal{V} \setminus 0$$

if and only if G is q -colorable.

To achieve further simplifications in our colorability criteria, we have to examine the most inner sum in Condition (v) of Theorem 4.1. Formulated for V and G , instead of U and $G|_U$, this is

$$\sum_{\substack{S \subseteq E \\ \pi(S) = \{V\}}} (-1)^{|S|}. \quad (42)$$

For this sum, one can find some interesting interpretations in the literature. At first, one may try to pair off and cancel as many as possible oppositely signed summands. In this way, Whitney showed in [Wh] that one can restrict the summation range to subsets S that do not contain a broken circuit. Here, a broken circuit is a circuit with its biggest edge removed, i.e., biggest with respect to a given total ordering on E . Another interesting interpretation was given by Greene and Zaslavsky in [GrZa, Theorem 7.3], with a combinatorial proof in [GeSa, Theorem 1.2 & 1.3]. It states that our sum counts the acyclic orientations of G with one fixed given sink $v_0 \in V$. In our investigations, we have the additional advantage that we have to count only modulo r . This allows the following reinterpretation in terms of colorings based on Equation (39):

$$\sum_{\substack{S \subseteq E \\ \pi(S) = \{V\}}} (-1)^{|S|} = \sum_{\substack{S \subseteq E \\ |\pi(S)|=1}} (-1)^{|S|} \equiv \sum_{S \subseteq E} (-1)^{|S|} r^{|\pi(S)|-1} = \frac{1}{r} |\mathcal{C}_r(G)| \pmod{r}. \quad (43)$$

With this, the last condition in Theorem 4.1 can be rewritten:

Theorem 4.4. *Let q be a prime power. If $r > 0$ is a divisor of $q - 1$, then G is vertex colorable with q many colors if and only if there exists an $x \in \mathcal{V}$ with*

$$\sum_{\substack{\pi \in \Pi_c(G) \\ \langle \pi \rangle \ni x}} \prod_{U \in \pi} \frac{1}{r} |\mathcal{C}_r(G|_U)| \not\equiv 0 \pmod{r}.$$

The sum in this theorem simplifies even more if $r = 2$, because $\frac{1}{2} |\mathcal{C}_2(G|_U)|$ is nonzero modulo 2 only if $G|_U$ is bipartite and connected. Based on this observation, we say that a set $U \subseteq V$ of vertices is G -bipartite if the induced subgraph $G|_U$ is connected and bipartite. A partition π of V is G -bipartite if each of its blocks is G -bipartite. Finally, we say that x is *odd* if the number of x -balanced and G -bipartite partitions of V is odd. With these definitions, we see that x is odd if and only if the sum in the last theorem is nonzero modulo $r = 2$. Therefore, we obtain the following main result:

Theorem 4.5. *Let q be an odd prime power. Then a graph G admits a vertex coloring with q colors if and only if it admits an odd vertex labeling $x: V \rightarrow \mathbb{F}_q$.*

Moreover, the additional statement of Theorem 4.1 holds accordingly. If the all-zero labeling $0 \in \mathcal{V}$ is odd, then there are also other odd labelings $x \neq 0$. Our more precise congruency in Theorem 4.2 can also be specialized:

Theorem 4.6. *Let q be an odd prime power. Then, for any vertex labeling $x \in \mathcal{V} := \mathbb{F}_q^V$,*

$$\frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1} \text{ is odd} \iff x \text{ is odd.}$$

If $q > 3$ and G is not colorable with just 3 colors, then $q(q-1)(q-2)(q-3)$ divides $|\mathcal{C}_q(G)|$. Hence, $\frac{|\mathcal{C}_q(G)|}{q-1}$ is then even, and, similarly as in Equation (41),

$$\frac{|\mathcal{C}_q(G) \cap x^\perp|}{q-1} \equiv \frac{|\mathcal{C}_q(G) \setminus x^\perp|}{q-1} \pmod{2}. \quad (44)$$

In particular, (by Theorem 4.6) in this case any odd labeling $x \in \mathcal{V}$ guarantees not just a coloring c that is not orthogonal to it, but it guarantees also at least one coloring that is orthogonal to it.

Since the edge colorings of a graph are the vertex colorings of its line graph, we may translate our theorems into the language of edge colorings. This is straightforward. We just define the corresponding terms and formulate the theorem: We say that a set $F \subseteq E$ of edges is *junction-free* if the induced subgraph $G|_F$ is a path or an even cycle. If the restriction to F of an edge labeling $y: E \rightarrow \mathbb{F}_q$ has zero sum, $\sum_{e \in F} y_e = 0$, we say that a set $F \subseteq E$ of edges is *y-balanced*. We also say that a partition π of E is *y-balanced*, respectively *junction-free*, if each of its blocks is *y-balanced*, respectively *junction-free*. Finally, we say that y is *odd* if the number of *y-balanced* and *junction-free* partitions of E is odd. With these definitions, we obtain the following result:

Theorem 4.7. *If q is an odd prime power, then a graph G admits an edge coloring with q colors if and only if it admits an odd edge labeling $y: E \rightarrow \mathbb{F}_q$.*

If $\mathcal{C}'_q(G)$ denotes the set of proper edge colorings $E \rightarrow \mathbb{F}_q$, we also have the following:

Theorem 4.8. *Let q be an odd prime power. Then, for any edge labeling $y \in \mathbb{F}_q^E$,*

$$\frac{|\mathcal{C}'_q(G) \setminus y^\perp|}{q-1} \text{ is odd} \iff y \text{ is odd.}$$

We finish with some remarks about the number and the lengths of paths in junction-free edge partitions: If the number of vertices of odd degree is $2t$, then the number of unclosed paths in any junction-free partition π of E must be at least t . That is because, any vertex v of odd degree must be the end of at least one path in π , as all other paths

through v cover an even number of its edges. It follows that the average length of the paths in π is at most $|E|/t$. Hence, for q -regular graphs, this upper bound for the average length would be q .

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