Hyperoctahedral Eulerian Idempotents, Hodge Decompositions, and Signed Graph Coloring Complexes

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Abstract

Phil Hanlon proved that the coefficients of the chromatic polynomial of a graph G are equal (up to sign) to the dimensions of the summands in a Hodge-type decomposition of the top homology of the coloring complex for G. We prove a type B analogue of this result for chromatic polynomials of signed graphs using hyperoctahedral Eulerian idempotents.

Keywords: Chromatic polynomial; signed graph; Hodge decomposition; Eulerian idempotent; coloring complex

1 Introduction

Let G denote a finite graph and $\chi_G(\lambda)$ its chromatic polynomial. The coloring complex Δ_G was defined by Einar Steingrímsson [18] in order to provide a Hilbert-polynomial interpretation of $\chi_G(\lambda)$. While Steingrímsson's original definition of Δ_G was motivated by algebraic considerations, the coloring complex can also be obtained as the link complex for a hyperplane arrangement, using techniques developed by Jürgen Herzog, Vic Reiner, and Volkmar Welker [11]. Coloring complexes have many interesting properties. Jakob Jonsson proved [13] that Δ_G is homotopy equivalent to a wedge of spheres in fixed dimension, with the number of spheres being one less than the number of acyclic orientations of G. Axel Hultman [12] proved that Δ_G, and in general any link complex for a

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sub-arrangement of the type A or type B Coxeter arrangement, is shellable. Further, Δ_G admits a convex ear decomposition, as shown by Patricia Hersh and Ed Swartz [10].

A fascinating result due to Phil Hanlon [8] is that (up to sign) the j-th coefficient of $\chi_G(\lambda)$ is equal to the dimension of the j-th summand in a Hodge-type decomposition of the top homology of Δ_G. Hanlon's Hodge decomposition is obtained using the Eulerian idempotents in the group algebra of the symmetric group S_n. These are the elements $e_n^{(j)} \in \mathbb{C}[S_n]$ defined by

$$e_n(x) = \sum_{j=1}^{n} x^j e_n^{(j)} = \sum_{\pi \in S_n} \binom{x+n-\mathrm{des}(\pi)-1}{n} \mathrm{sgn}(\pi)\pi \,.$$

These first arose in work of various authors in the 1980's. Murray Gerstenhaber and Samuel Schack [6] proved that all splitting sequences for Hochschild homology arise as linear combinations of Eulerian idempotents, which in certain cases coincides with Hodge decompositions for smooth compact complex varieties; similar work was independently introduced by Jean-Louis Loday [14]. A nice introduction to these results can be found in another paper due to Hanlon [7, Section 1]. The chain complex defining Hochschild homology is quite similar to the chain complex for Δ_G, and thus Hanlon was able to adapt the Eulerian idempotent splitting techniques in Hochschild homology to produce a similar decomposition for the top homology of Δ_G.

The coloring complex construction can be extended to hypergraphs, and Eulerian idempotents continue to play a role in this setting. Combinatorial and topological properties of hypergraph coloring complexes were investigated by Felix Breuer, Aaron Dall, and Martina Kubitzke [3], who found that many of the nicest properties of graph coloring complexes are lost in the transition to hypergraphs, e.g. Cohen-Macaulayness, partitionability, being a wedge of spheres, etc. Hypergraph coloring complexes were also considered by Jane Long and the second author [15]. They show that the homology of hypergraph coloring complexes admits a Hodge decomposition induced by Eulerian idempotents, and that the coefficients of the chromatic polynomial of a hypergraph are essentially the Euler characteristics of the Hodge subcomplexes, up to sign. The second author [17] investigated the special case of k-uniform hypergraphs, showing that their coloring complexes are shellable and that their cyclic coloring complexes have a certain homology group whose dimension is given by a binomial coefficient.

The Eulerian idempotents play key roles in other contexts as well. Adriano Garsia [5] and Christophe Reutenauer [16] studied Eulerian idempotents in their work on free Lie algebras. Persi Diaconis and Jason Fulman [4] show that the Eulerian idempotents are (up to the sign involution) eigenvectors of an "amazing" matrix arising from the study of "carries" in addition algorithms. They also show that this matrix is related to the Veronese construction in commutative algebra. Phil Hanlon and Patricia Hersh [9] prove that the homology of the complex of injective words admits a Hodge decomposition, where the dimension of the k-th Hodge summand is equal to the number of derangements with exactly k cycles. These and other results are all-the-more fascinating due to their type B extensions. The type B Eulerian idempotents, defined in Section 4, were originally defined by François Bergeron and Nantel Bergeron [2]. They proved type B extensions of several of the type A results given above. The Eulerian idempotents in types A and B also play an interesting role in shuffling problems, as discussed in several of the papers just referenced.

Given the variety of interesting applications of Eulerian idempotents, we believe that a type B version of Hanlon's result regarding $\chi_G(\lambda)$ is of interest. The goal of this paper is to prove Theorem 26, which provides the desired extension in the setting of signed graph chromatic polynomials. Section 2 contains a review of basic properties of signed graphs and their chromatic polynomials. Section 3 discusses signed graph coloring complexes and hyperoctahedral group actions on them. In Section 4, we prove that the type B Eulerian idempotents induce a Hodge-type decomposition on the top homology of each signed graph coloring complex. In Section 5 we prove our main result.

2 Signed graphs and chromatic polynomials

This section is based on Zaslavsky's papers [19, 20].

Definition 1. A *signed graph* G on the vertex set $[n]$ is a multiset E of one-element subsets of $[n]$, called *half-edges*, and two-element subsets of $[n]$, called *edges*, together with a *sign map* $\sigma : E \cap \binom{[n]}{2} \to \{1, -1\}$ such that $\sigma^{-1}(1)$ and $\sigma^{-1}(-1)$ are each the edge set of a simple graph on $[n]$. For an edge $e \in E$, if $\sigma(e) = 1$, then e is called a *positive edge* of G. If $\sigma(e) = -1$, then e is called a *negative edge* of G.

Example 2. Let G have vertex set $\{1, 2, 3\}$, positive edge $\{1, 2\}$, negative edges $\{1, 2\}$ and $\{2, 3\}$, and half-edge $\{1\}$. We schematically represent G using solid half-lines and lines for half-edges and positive edges, respectively, and using dotted lines for negative edges, as demonstrated in Figure 1.

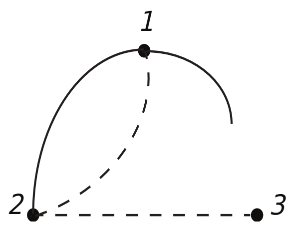

Figure 1:

Definition 3. Let G be a signed graph on $[n]$. A *c-coloring* of G is a map

$$\phi : [n] \to \{-c, -c + 1, \dots, -1, 0, 1, \dots, c\}.$$

A c-coloring ϕ is *proper* if $\phi(i) \neq \phi(j)$ for all positive edges $\{i, j\}$ in G, $\phi(i) \neq -\phi(j)$ for all negative edges $\{i, j\}$ in G, and $\phi(i) \neq 0$ for all half-edges $i \in E(G)$. Denote by $\chi_G(2c + 1)$ the number of proper c-colorings of G.

Theorem 4 (Zaslavsky [20])**.** *For G a signed graph on $[n]$, the function $\chi_G(2c+1)$ is given by a polynomial of degree n.*

Example 5. For G as in Example 2, we have

$$\chi_G(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 1) = \lambda^3 - 4\lambda^2 + 5\lambda - 2\,,$$

where λ is the number of colors in a set of colors containing the color 0. Note that evaluating $\chi_G(\lambda)$ at $\lambda = 2c + 1$ yields the number of proper c-colorings of G.

The key to proving polynomiality of $\chi_G(2c+1)$ is the relation between signed graphs, contractions, and deletions, which we will need subsequently. Contraction/deletion for signed graphs relies upon the idea of switching a signed graph at a vertex.

Definition 6. Let G be a signed graph with sign map σ, and let v be a vertex of G. We say that the signed graph G' is obtained by *switching G at v* if the vertex and edge sets for G' are $V(G)$ and $E(G)$, while the sign map σ' for G' is given by

$$\sigma'(\{i,j\}) := \begin{cases} -\sigma(\{i,j\}) & \text{if } v = i \text{ or } v = j \\ \sigma(\{i,j\}) & \text{else} \end{cases}.$$

If H is obtained from G by a finite sequence of switches, we say that G and H are *switching equivalent*.

Thus, one switches from G to G' at v by negating the sign on all edges in G incident with v. The following proposition, demonstrating the role played by switching, is simple to prove.

Proposition 7. *If G and H are switching equivalent, then*

$$\chi_H(2c+1) = \chi_G(2c+1)\,.$$

Definition 8. Let G be a signed graph with an edge $e = \{i,j\} \in E(G)$ with sign $\sigma(e)$. The *deletion* of G by e, denoted $G \setminus e$, is the signed graph obtained by removing e from $E(G)$. A *contraction* of G by e, denoted G/e, is a signed graph in the same switching class as the graph obtained from G by the following process (which is well-defined up to switching equivalence).

- If $\sigma(e) = 1$, then delete e from $E(G)$ and contract as in the ordinary graph case by identifying i and j in $V(G)$. When $\{i,j\}$ is also present in $E(G)$ as a negative edge, add a half-edge at the vertex given by $i = j$ after contracting (if this half-edge is not already present).

- If $\sigma(e) = -1$, then first switch at an endpoint of e so that $\sigma(e) = 1$ and proceed as in the positive edge contraction case.

Given a half-edge $i \in E(G)$, the *deletion* of G by i, denoted $G \setminus i$, is the signed graph obtained by removing i from $E(G)$. The *contraction* of G by i, denoted G/i, is the signed graph with vertex set $V(G) \setminus \{i\}$ and edge set $\{e \setminus \{i\} \mid e \in E(G)\}$.

Note that $E(G)$ is a multiset, thus it is possible that two copies of $\{i, j\}$ are contained in $E(G)$ with different signs. If this is the case, then only one copy of $\{j\}$ is retained in the edge set of the deletion and contraction. The key property of contraction/deletion, and what makes it relevant for the proof of Theorem 4, is given next.

Proposition 9. *Given a signed graph G with positive edge e,*

$$\chi_G(2c+1) = \chi_{G\setminus e}(2c+1) - \chi_{G/e}(2c+1)\,.$$

A final fact we need is that when the chromatic polynomial is expressed as

$$\chi_G(\lambda) = \sum_{j=1}^{n-1} (-1)^{n-j} c_j(G)\lambda^j + \lambda^n\,, \tag{1}$$

the c_j's are non-negative integers. This can be seen in several ways, e.g. by recognizing $\chi_G(\lambda)$ as the characteristic polynomial of the arrangement $\mathcal{B}_G$ defined in the next section.

3 Signed graphic arrangements, coloring complexes, and group actions

Our construction of signed graph coloring complexes involves the following hyperplane arrangement.

Definition 10. The *type B braid arrangement* is the collection of hyperplanes

$$\mathcal{B}_n := \{H_{ij}^{+1} \mid 1 \leqslant i < j \leqslant n\} \cup \{H_{ij}^{-1} \mid 1 \leqslant i < j \leqslant n\} \cup \{H_i \mid 1 \leqslant i \leqslant n\}$$

where $H_{ij}^{+1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$, $H_{ij}^{-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = -x_j\}$, and $H_i = \{(x_1, \ldots, x_n) \mid x_i = 0\}$.

The arrangement $\mathcal{B}_n$ induces a regular cell decomposition $\Delta_{\mathcal{B}_n}$ of the sphere S^{n-1}, which we describe using the choice of $\partial[-1,1]^n$ as our preferred representation of S^{n-1}. $\mathcal{B}_n$ induces a triangulation of $\partial[-1,1]^n$ where each vertex $v = (v_1, \ldots, v_n)$ of the resulting triangulation can be identified with a nonempty subset of $[n] \cup -[n]$ by the rule that for each v_i, $\pm i$ is included in the subset if $v_i = \pm 1$, respectively, and i is not included in the subset if $v_i = 0$. The faces of the triangulation are given by collections of vertices corresponding to chains (with respect to inclusion) of subsets of this type. Alternatively, given such a chain

$$C := Q_1 \subset Q_2 \subset Q_3 \subset \cdots \subset Q_r\,,$$

we associate to C the ordered set partition of $[n] \cup -[n]$ given by

$$Q_1 \,|\, Q_2 \setminus Q_1 \,|\, Q_3 \setminus Q_2 \,|\, \cdots \,|\, Q_r \setminus Q_{r-1} \,|\, [n] \cup [-n] \setminus Q_r\,.$$

It is clear that C may be fully recovered from its associated partition.

Example 11. The triangulation of $\partial[-1,1]^3$ induced by $\mathcal{B}_3$ is shown in Figure 2. The chamber marked G has vertices $(0,1,0)$, $(0,1,-1)$, and $(-1,1,-1)$; thus, G is identified with the chain

$$\{2\} \subset \{2,-3\} \subset \{-1,2,-3\}$$

with associated partition $(2|-3|-1|1,-2,3)$. Similarly, the chamber marked B has vertices $(1,0,0)$, $(1,-1,0)$, and $(1,-1,1)$, hence corresponds to the chain

$$\{1\} \subset \{1,-2\} \subset \{1,-2,3\}$$

with associated partition $(1|-2|3|-1,2,-3)$.

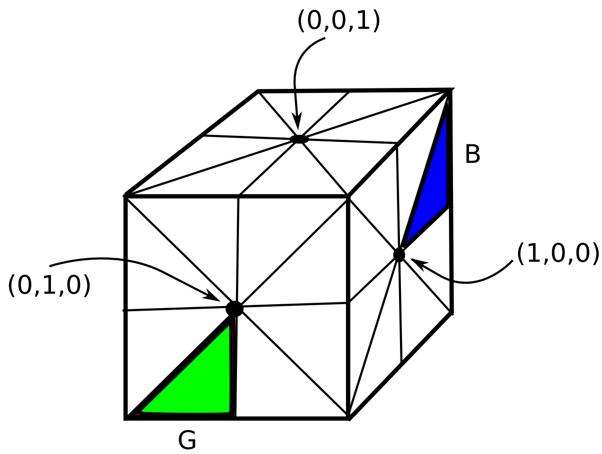

Figure 2:

Definition 12. Given a signed graph G with sign map σ, the *(signed) graphic arrangement* corresponding to G is the subarrangement $\mathcal{B}_G$ of $\mathcal{B}_n$ defined by

$$\mathcal{B}_G = \{H_{ij}^{\sigma(\{i,j\})} \mid \{i,j\} \in E(G)\} \cup \{H_i : \{i\} \in E(G)\}\,.$$

Example 13. Continuing with the signed graph of Example 2, we see that

$$\mathcal{B}_G = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \text{ either } x_1 = 0,\ x_1 = x_2,\ x_1 = -x_2 \text{ or } x_2 = -x_3\}\,.$$

Before discussing the geometric manifestation of coloring complexes using $\mathcal{B}_G$, we will first define the coloring complex of a signed graph in a purely combinatorial manner, using the viewpoint of ordered set partitions developed above. For G a signed graph on $[n]$, we say a subset $A \subset [n] \cup -[n]$ *contains an edge* of G if one of the following two cases hold for some pair $\{a,b\} \subset A$.

- $\{a,b\}$ is a positive edge in G, or
- $a \in [n]$ and $b \in -[n]$, and $\{a,-b\}$ is a negative edge in G.

Definition 14. Given a signed graph G, the coloring complex Δ_G is the simplicial complex whose facets are ordered set partitions $P_1|P_2|\cdots|P_n$ of $[n] \cup -[n]$ such that

1. $P_n = ([n] \cup -[n]) \setminus \cup_{j=1}^{n-1} P_j$,
2. for each pair $\{j, -j\}$ with $j \in [n]$, either j or $-j$ is contained in P_n, and
3. either there exists a unique non-singleton block P_i with $1 \leqslant i \leqslant n-1$ that contains an edge of G, or
4. for some half-edge $i \in E(G)$, $\{i, -i\} \subset P_n$.

The faces of Δ_G are formed by merging adjacent blocks in the partitions defining the facets. Thus, the vertices of Δ_G are given by partitions $P_1|\,([n] \cup -[n]) \setminus P_1$ where P_1 is obtained by merging an initial segment of blocks in one of the facets of Δ_G described above. The r-dimensional faces of Δ_G are the ordered set partitions with $r+2$ blocks. Note that the role of the empty set is taken by the trivial partition $[n] \cup -[n]$. As in the case for $\Delta_{\mathcal{B}_n}$, each partition $P_1|P_2|\cdots|P_n$ corresponds uniquely to a chain in $2^{[n]\cup-[n]}$ of the form

$$\emptyset \subset P_1 \subset P_1 \cup P_2 \subset \cdots \subset \cup_{i=1}^{j} P_i \subset \cdots \subset [n] \cup -[n]\,.$$

Geometrically, the coloring complex arises as $\Delta_G = \mathcal{B}_G \cap \partial[-1,1]^n$. The space Δ_G inherits the simplicial triangulation described above via the restriction of $\Delta_{\mathcal{B}_n}$ to Δ_G. The connection to the triangulation of $\partial[-1,1]^n$ induced by $\mathcal{B}_n$ is immediate from our previous discussion. We will freely use the notation Δ_G to denote both the topological space $\mathcal{B}_G \cap \partial[-1,1]^n$ and the abstract simplicial complex obtained after intersecting with $\Delta_{\mathcal{B}_n}$. Given this geometric observation, it follows that Δ_G is an example of a link complex of a subspace arrangement, resulting in the following theorem.

Theorem 15. (Hultman, [12, Theorem 4.2]) *For any signed graph G, Δ_G is shellable, hence is homotopy equivalent to a wedge of spheres of dimension $n-2$.*

One reason the complex Δ_G is important is that it provides a path through which we can interpret chromatic polynomials as Hilbert polynomials of graded algebras.

Theorem 16. (Hultman, [12, Corollary 5.8]) *Let $k[\mathrm{cone}(\Delta_G)]$ be the Stanley-Reisner ring of the cone over Δ_G. The Hilbert polynomial of $k[\mathrm{cone}(\Delta_G)]$ is given by*

$$F(k[\mathrm{cone}(\Delta_G)]; c) = (2c+1)^n - \chi_G(2c+1)\,.$$

Example 17. Let G be as in Example 2, hence $\mathcal{B}_G$ as in Example 13. The complex Δ_G, arising as a subcomplex of the triangulation of $\partial[-1,1]^3$ shown in Figure 2, is given in Figure 3. It is straightforward to verify that $\Delta_G \simeq \vee_{i=1}^{11} S^1$.

We will later need to undertake a detailed analysis of the boundary operators for Δ_G, for which the following notation is needed. Let d_i denote the map on the r-faces of Δ_G defined by:

$$d_i(P_1 \mid \cdots \mid P_i \mid P_{i+1} \mid \cdots \mid P_{r+2}) = (P_1 \mid \cdots \mid P_{i-1} \mid P_i \cup P_{i+1} \mid P_{i+2} \mid \cdots \mid P_{r+2}).$$

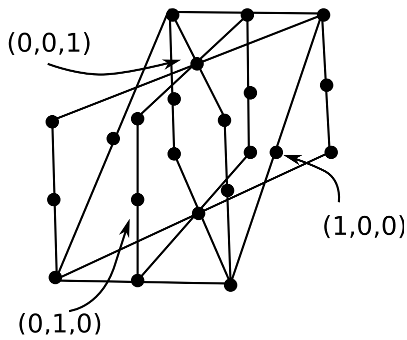

Figure 3:

The boundary operator on the r-chains of Δ_G is then defined by

$$\partial_r = \sum_{i=1}^{r+1} (-1)^{i-1} d_i.$$

Note that since Δ_G is homotopic to a wedge of spheres of dimension $n-2$, the reduced homology of Δ_G is non-vanishing only in dimension $n-2$.

Another reason that signed graph coloring complexes are interesting is that their chain spaces admit a family of actions by hyperoctahedral groups, which we will now introduce. All relevant background regarding hyperoctahedral groups can be found in [2, Section 8.1]. Let B_n denote the n-th hyperoctahedral group, i.e., B_n is the set of all permutations of $[-n, n]$ such that $\pi(-i) = -\pi(i)$ for all $i \in \{0, 1, \ldots, n\}$. Note that $\pi(0) := 0$ for all $\pi \in B_n$. For an element $\pi \in B_n$, we say the *window* for π is

$$\pi = [\pi(1)\, \pi(2)\, \cdots\, \pi(n)]\,;$$

thus, the window for π is analogous to one-line notation for the symmetric group. Recall that for an element $\pi \in B_n$, the *sign* of π is

$$\mathrm{sgn}(\pi) := (-1)^{\ell(\pi)}$$

where $\ell(\pi)$ is the length of π. Recall also that the *descent statistic* for $\pi \in B_n$ is

$$\mathrm{des}(\pi) := \#\{i \in [0, \ldots, n-1] \mid \pi(i) > \pi(i+1)\}\,.$$

Set $P_{-i} := -P_i$. For any signed graph G, an element $\pi \in B_{r+1}$ acts on $C_r(\Delta_G; \mathbb{C})$, the space of r-chains of the coloring complex for G, by extending linearly the action on basis elements given by

$$\pi(P_1 \mid P_2 \mid \cdots \mid P_{r+1} \mid P_{r+2}) = (P_{\pi^{-1}(1)} \mid \cdots \mid P_{\pi^{-1}(r+1)} \mid P'_{r+2})\,,$$

where P'_{r+2} is P_{r+2} with $j \in P_{r+2}$ changed to $-j \in P'_{r+2}$ if π changed the sign on the block containing j. Informally, π acts by permuting the first $r+1$ blocks in the partitions defining the r-faces of Δ_G by π^{-1}, changing the signs of all elements in blocks where a sign change occurs on the index, and subsequently modifying P_{r+2} to account for those sign changes.

Example 18. Let $\pi = [2\ -1\ -3] \in B_3$, hence $\pi^{-1} = [-2\ \ 1\ -3]$. For

$$\tau := (1,3|-2,5|6|-1,2,-3,4,-4,-5,-6) \in C_2(\Delta_{B_6})\,,$$

we have that

$$\pi(\tau) = (2,-5|1,3|-6|-1,-2,-3,4,-4,5,6)\,.$$

4 Eulerian idempotents and type B Hodge decompositions

Our goal in this section is to prove Theorem 21, stating that $H_{n-2}(\Delta_G)$ admits a particular type of direct sum decomposition called a *type B Hodge decomposition*. In order to state the theorem, we must begin with a discussion of the following family of idempotents defined by F. Bergeron and N. Bergeron [1].

Definition 19. The *Eulerian idempotents* of type B are elements $\tilde{\rho}_n^{(j)} \in \mathbb{C}[B_n]$ defined by

$$\tilde{\rho}_n(x) = \sum_{j=0}^{n} x^j \tilde{\rho}_n^{(j)} := \sum_{\pi \in B_n} \left[\frac{\prod_{k=1}^{n}(x - 2\mathrm{des}(\pi) + 2k - 1)}{2^n n!} \right] \mathrm{sgn}(\pi)\pi\,. \tag{2}$$

We will need two key properties of the type B Eulerian idempotents. First, Bergeron and Bergeron [1, Section 4] proved that

$$\tilde{\rho}_n^{(r)} \tilde{\rho}_n^{(s)} = \delta_{r,s} \tilde{\rho}_n^{(r)}\,. \tag{3}$$

Second, evaluating (2) at $x = 1$ yields

$$\sum_{j=0}^{n} \tilde{\rho}_n^{(j)} = \mathrm{id} \tag{4}$$

after observing that

$$\frac{\prod_{k=1}^{n}(1 - 2\mathrm{des}(\pi) + 2k - 1)}{2^n n!} = \binom{-\mathrm{des}(\pi) + n}{n} = \begin{cases} 1 & \text{if } \mathrm{des}(\pi) = 0 \\ 0 & \text{else} \end{cases}\,.$$

Thus, the $\tilde{\rho}_n^{(j)}$ form a family of orthogonal idempotents in $\mathbb{C}[B_n]$ that sum to the identity element.

Remark 20. It is not clear from Definition 19 why these elements should be orthogonal idempotents, nor that they should have any other special properties. In the opinion of the authors, the Eulerian idempotents are rather mysterious in general, though they exhibit many remarkable properties as mentioned in Section 1.

The Eulerian idempotents do admit the following concrete definition, which will be useful in this section. Set

$$\tilde{l}_n^{(j)} := (-1)^{j-1} \sum_{\substack{\pi \in B_n \\ \mathrm{des}(\pi)=j}} \mathrm{sgn}(\pi)\pi \qquad \text{and} \qquad \tilde{\lambda}_n^{(j)} = \sum_{i=0}^{j} (-1)^i \binom{n+i}{i} \tilde{l}_n^{(j-i)} .$$

It is straightforward, though tedious, to verify that

$$\tilde{\lambda}_n^{(j)} = (-1)^{j-1} \tilde{\rho}_n(2j+1) \tag{5}$$

by evaluating (2) at $2j+1$. Due to this relation, it follows that the $\tilde{\lambda}_{r+1}^{(j)}$ can be used to define the $\tilde{\rho}_{r+1}^{(j)}$ via an invertible Vandermonde matrix.

We are now able to give our main result of this section, which states that the chain complex for the signed graph coloring complex admits a direct sum decomposition. As in the case with Hanlon's type A result discussed in Section 1, we call this direct sum decomposition a type B Hodge decomposition.

Theorem 21. *Let $C_r^{(j)}(\Delta_G) = \tilde{\rho}_r^{(j)} C_r(\Delta_G)$. The chain complex for Δ_G decomposes as*

$$(C_*(\Delta_G), \partial_*) = \bigoplus_{j \geqslant 0} (C_*^{(j)}(\Delta_G), \partial_*) ,$$

thus

$$H_r(\Delta_G) = \bigoplus_{j \geqslant 0} H_r^{(j)}(\Delta_G).$$

Proof. We will push all of the hard work for the proof to Lemma 23 and Lemma 24. The proof follows easily from from these combined with (3) and (4). □

Remark 22. To prove our structural result regarding the $\tilde{\rho}_n^{(j)}$'s, it is easier to first work with the elements $\tilde{l}_n^{(j)}$ and $\tilde{\lambda}_n^{(j)}$ and then apply (5) to convert, as these lemmas and their proofs illustrate. As with the type A Eulerian idempotents discussed in Section 1, the type B Eulerian idempotents can be used to produce a Hodge-type decomposition for commutative hyperoctahedral algebra homology [1]. The chain complex defining this homology is quite similar to the chain complex for the signed graph coloring complex, thus the proofs for Lemmas 23 and 24 are similar to the proof of [1, Proposition 5.1] due to Bergeron and Bergeron. For the sake of completeness, and because [1] contains a few confusing typographical errors, we include reasonably detailed proofs of these results.

Lemma 23. *Let G be a signed graph with vertex set $[n]$, with coloring complex Δ_G having chain complex $(C_*(\Delta_G), \partial_*)$. For each r such that $0 \leqslant r \leqslant n-2$ and for each j such that $1 \leqslant j \leqslant r+1$,*

$$\partial_r \tilde{l}_{r+1}^{(j)} = \left(\tilde{l}_r^{(j)} + \tilde{l}_r^{(j-1)} \right) \partial_r .$$

Proof. Setting

$$L_{r+1}^{j} := \{\pi \in B_{r+1} \mid \mathrm{des}(\pi) = j\},$$

the strategy is to consider

$$d_i \tilde{l}_{r+1}^{(j)} = d_i(-1)^{j-1} \sum_{\pi \in L_{r+1}^{j}} \mathrm{sgn}(\pi)\pi = (-1)^{j-1} \sum_{\pi \in L_{r+1}^{j}} \mathrm{sgn}(\pi) d_i \pi \tag{6}$$

for each $i = 1, \ldots, r+1$. For some of the elements $d_i\pi \in d_i L_{r+1}^{j}$, we will produce $\sigma \in L_{r+1}^{j}$ such that $d_i \mathrm{sgn}(\pi)\pi = -d_i \mathrm{sgn}(\sigma)\sigma$, and thus these terms will cancel pairwise in (6). For all other elements, we will bijectively map each element in $d_i L_{r+1}^{j}$ to an element appearing as a summand of

$$\begin{aligned}\left(\tilde{l}_r^{(j)} + \tilde{l}_r^{(j-1)}\right)\partial_r &= \sum_{i=1}^{r+1} (-1)^{i-1} \left((-1)^{j-1} \sum_{\sigma \in L_r^{j}} \mathrm{sgn}(\sigma)\sigma + (-1)^{j-2} \sum_{\sigma \in L_r^{j-1}} \mathrm{sgn}(\sigma)\sigma \right) d_i \\ &= \sum_{i=1}^{r+1} (-1)^{i-1} \left((-1)^{j-1} \sum_{\sigma \in L_r^{j}} \mathrm{sgn}(\sigma)\sigma d_i + (-1)^{j-2} \sum_{\sigma \in L_r^{j-1}} \mathrm{sgn}(\sigma)\sigma d_i \right),\end{aligned}$$

showing that $d_i\pi$ corresponds to a term (with correct sign) σd_s in a bijection

$$\bigcup_{i=1}^{r+1} d_i L_{r+1}^{j} \setminus \{\text{pairwise canceling elements}\} \longleftrightarrow \left(\bigcup_{i=1}^{r+1} L_r^{j} d_i\right) \bigcup \left(\bigcup_{i=1}^{r+1} L_r^{j-1} d_i\right). \tag{7}$$

Doing so will yield our desired equality. As this becomes a lengthy exercise of case-by-case analysis, we will completely prove some of the cases and provide only the setup for the rest.

Case: $i = r+1$.

Let I_{r+1} denote the element of B_{r+1} that sends $r+1$ to $-(r+1)$ and fixes all other elements; note that $\ell(I_{r+1})$ is odd, as

$$I_{r+1} = (r, r+1) \cdots (2,3)(1,2)s_0(1,2)(2,3) \cdots (r, r+1),$$

where s_0 is the generator of B_{r+1} sending 1 to -1 and fixing all other elements. Next observe that $d_{r+1}I_{r+1} = d_{r+1}$, because when the $(r+1)$-st and $(r+2)$-nd blocks of an ordered partition are merged, all the elements of the $(r+1)$-st block appear in the merged block with all possible signs.

Let $\pi \in L_{r+1}^{j}$ and let $\sigma = I_{r+1}\pi$; it follows from the length of I_{r+1} being odd that $\mathrm{sgn}(\pi) = -\mathrm{sgn}(\sigma)$, by [1, Proposition 1.4.2 (ii)]. Note that σ and π differ only in that the $\pm(r+1)$ appearing in π is negated, but does not change position. If $\pi_{r+1} \neq \pm(r+1)$, then $\sigma = I_{r+1}\pi$ has the same number of descents as π, since if $r+1$ is in position j, then π is forced to have an ascent in position $j-1$ and a descent in position j, while if $-(r+1)$ is in position j, then π is forced to have a descent in position $j-1$ and an ascent in position j. Hence $d_{r+1}\text{sgn}(\sigma)\sigma = d_{r+1}I_{r+1}(-\text{sgn}(\pi))\pi = -d_{r+1}\text{sgn}(\pi)\pi$, and the pair of terms $d_{r+1}\text{sgn}(\sigma)\sigma$ and $d_{r+1}\text{sgn}(\pi)\pi$ cancel each other in (6).

Suppose now that $\pi_{r+1} = r+1$. Setting $\sigma = \pi_1\pi_2\cdots\pi_r$, it is straightforward to verify that

- $\sigma \in B_r$,
- $d_{r+1}\pi = \sigma d_{r+1}$,
- $\text{des}(\sigma) = \text{des}(\pi)$, and
- $\text{sgn}(\pi) = \text{sgn}(\sigma)$.

On the other hand, if $\pi_{r+1} = -(r+1)$, then setting $\sigma = \pi_1\pi_2\cdots\pi_r$ it is again straightforward to verify that

- $\sigma \in B_r$,
- $d_{r+1}\pi = \sigma d_{r+1}$,
- $\text{des}(\sigma) = \text{des}(\pi) - 1$, and
- $\text{sgn}(\sigma) = -\text{sgn}(\pi)$.

Mapping $d_{r+1}\pi$ to σd_{r+1} in our correspondence yields a bijection (with correct signs) pairing an element $d_{r+1}\pi \in d_{r+1}L^j_{r+1}$ satisfying $\pi_{r+1} = \pm(r+1)$ with an element of $L^j_r d_{r+1} \cup L^{j-1}_r d_{r+1}$, leading to the equality

$$d_{r+1}\tilde{l}^{(j)}_{r+1} = \left(\tilde{l}^{(j)}_r + \tilde{l}^{(j-1)}_r\right) d_{r+1}\,.$$

Case: $1 \leqslant i \leqslant r$.

For each i and each $\pi \in L^j_{r+1}$, the relative position of $\pm i$ and $\pm(i+1)$ in the window for π determines how π is handled. There are five situations that can occur:

- $\pi^{-1}(i+1) = \pi^{-1}(i) + 1 > 0$, implying that $\pi = [\cdots i\,(i+1)\cdots]$
- $\pi^{-1}(i+1) = \pi^{-1}(i) + 1 < 0$, implying that $\pi = [\cdots -(i+1)\ -i\cdots]$
- $\pi^{-1}(i+1) = \pi^{-1}(i) - 1 > 0$, implying that $\pi = [\cdots (i+1)\ i\cdots]$
- $\pi^{-1}(i+1) = \pi^{-1}(i) - 1 < 0$, implying that $\pi = [\cdots -i\ -(i+1)\cdots]$
- $\pi^{-1}(i+1) \neq \pi^{-1}(i) \pm 1$, containing all remaining cases.

We sketch below how to assign to each $d_i\pi$ a unique σd_s in each of these cases, and provide at the end a proof that these assignments are bijective as claimed in (7).

Subcase: Suppose $\pi^{-1}(i+1) = \pi^{-1}(i) + 1 > 0$, implying that $\pi = [\cdots i\,(i+1)\cdots]$.

Define s as the index such that $\pi_s = i$. We want to associate to π a unique $\sigma \in L^j_r$ with the following properties:

- $\mathrm{des}(\sigma) = \mathrm{des}(\pi)$,
- $\mathrm{sgn}(\sigma) = (-1)^{i-s}\mathrm{sgn}(\pi)$, and
- $d_i\pi = \sigma d_s$.

We claim that these properties are satisfied by $\sigma = \sigma_1 \dots \sigma_s \sigma_{s+2} \dots \sigma_{r+1}$ where

- $\sigma_m = \pi_m$ if $|\pi_m| < i+1$
- $|\sigma_m| = |\pi_m| - 1$ if $|\pi_m| > i+1$, and
- the sign pattern for σ is the same as that for π, i.e. if $\pi_j < 0$, then σ_j is also negative.

To prove that $\mathrm{des}(\sigma) = \mathrm{des}(\pi)$, note that all pairwise inequality relationships are preserved between π and σ; thus, the only possible position of an additional descent in π that does not occur in σ is between $\pi_s = i$ and $\pi_{s+1} = i+1$, where no descent occurs.

To show that $\mathrm{sgn}(\sigma) = (-1)^{i-s}\mathrm{sgn}(\pi)$, observe that there exist $i-s$ adjacent transpositions $t_1, \dots, t_{i-s}$ in B_{r+1} such that $\pi t_1 \cdots t_{i-s}$ has $i+1$ appearing in window position $i+1$; we do so by exchanging adjacent entries in the window for π repeatedly to bring $i+1$ from position $s+1$ to position $i+1$. Then, σ is obtained from $\pi t_1 \cdots t_{i-s}$ by deleting position $i+1$ and lowering the label for all window elements greater than $i+1$. Thus, both σ and $\pi t_1 \cdots t_{i-s}$ can be expressed as a product of the same number of adjacent transpositions and, for each negative element appearing in the window for π, an odd number of hyperoctahedral group Coxeter generators. Thus, the length of these two elements have the same parity, and our result follows.

Finally, to show $d_i\pi = \sigma d_s$ it suffices to show that the m-th block in the image of $(P_1 \mid \cdots \mid P_{r+1} \mid P_{r+2})$ is the same under $d_i\pi$ and σd_s. First, suppose that $|\pi_m| = k < i$. Then $d_i\pi$ will map P_m to the k-th block location in the image if $\pi_m > 0$, or to P_{-m} in the k-th block location in the image if $\pi_m < 0$. If $m < s$, then P_m will still be the m-th block in the image after d_s is applied. Since $k < i$, $\sigma_m = \pi_m$, so σ will map P_m to the k-th block location in the image if $\pi_m > 0$, or to P_{-m} in the k-th block location in the image if $\pi_m < 0$. If $m > s+1$, then P_m will be in the $(m-1)$-st location after d_s is applied. Notice though that by the definition of σ, this implies that σ_m is in the $(m-1)$-st position of σ. Since $k < i$, $\sigma_m = \pi_m$, and thus σ will map P_m to the k-th block location in the image if $\pi_m > 0$, or to P_{-m} in the k-th block location in the image if $\pi_m < 0$.

Now suppose that $|\pi_m| = k > i+1$. Then $d_i\pi$ will map P_m to the $(k-1)$-st block in the image if $\pi_m > 0$ or to P_{-m} in the $(k-1)$-st block in the image if $\pi_m < 0$. If $m < s$, then P_m will still be the m-th block in the image after d_s is applied. Since $k > i+1$, $|\sigma_m| = |\pi_m| - 1$, so σ will map P_m to the $(k-1)$-st block location in the image if $\pi_m > 0$, or to P_{-m} in the $(k-1)$-st block location in the image if $\pi_m < 0$. If $m > s+1$, then P_m will be in the $(m-1)$-st location after d_s is applied. Notice though that by the definition of σ, this implies that σ_m is in the $(m-1)$-st position of σ. Since $k > i+1$, $|\sigma_m| = |\pi_m| - 1$, and thus σ will map P_m to the $(k-1)$-st block location in the image if $\pi_m > 0$, or to P_{-m} in the $(k-1)$-st block location in the image if $\pi_m < 0$.

Now suppose that $m = s$. Then $d_i\pi$ will map P_s to $P_s \cup P_{s+1}$ in the i-th location. d_s will map P_s to $P_s \cup P_{s+1}$, and since $P_s \cup P_{s+1}$ is in the s-th block, σd_s also maps P_s to $P_s \cup P_{s+1}$ in the i-th block.

Subcase: Suppose $\pi^{-1}(i+1) = \pi^{-1}(i)+1 < 0$, implying that $\pi = [\cdots -(i+1)\ -i \cdots]$.

Let s be defined as the index such that $\pi_s = -i-1$. It suffices to show that σ defined as follows satisfies $\text{des}(\sigma) = \text{des}(\pi)$, $\text{sgn}(\sigma) = (-1)^{i-s}\text{sgn}(\pi)$, and $d_i\pi = \sigma d_s$. Let $\sigma = \sigma_1 \dots \sigma_s \sigma_{s+2} \dots \sigma_{r+1}$ where

- $\sigma_m = \pi_m$ if $|\pi_m| < i$
- $|\sigma_m| = |\pi_m| - 1$ if $|\pi_m| > i$, and
- the sign pattern for σ is the same as that for π, i.e. if $\pi_j < 0$, then σ_j is also negative.

Thus, σd_s is the element in $L_r^j d_s$ uniquely paired with $d_i\pi$. This argument is similar to the previous subcase.

Subcase: Suppose $\pi^{-1}(i+1) = \pi^{-1}(i) - 1 > 0$, implying that $\pi = [\cdots (i+1)\ i \cdots]$.

Let s be defined as the index such that $\pi_s = i+1$. It suffices to show that σ defined as follows satisfies, $\text{des}(\sigma) = \text{des}(\pi) - 1$, $\text{sgn}(\sigma) = (-1)^{i-s+1}\text{sgn}(\pi)$, and $d_i\pi = \sigma d_s$. Let $\sigma = \sigma_1 \dots \sigma_{s-1}\sigma_{s+1} \dots \sigma_{r+1}$ where

- $\sigma_m = \pi_m$ if $|\pi_m| < i+1$,
- $|\sigma_m| = |\pi_m| - 1$ if $|\pi_m| > i+1$, and
- the sign pattern for σ is the same as that for π, i.e. if $\pi_j < 0$, then σ_j is also negative.

Thus, σd_s is the element of $L_r^{j-1} d_s$ uniquely paired with $d_i\pi$. This argument is similar to the previous subcase. Note that the property $\text{sgn}(\sigma) = (-1)^{i-s+1}\text{sgn}(\pi)$ is necessary because while $\pi \in L_{r+1}^j$, we have that $\sigma \in L_r^{j-1}$ and thus in $\tilde{l}_r^{(j-1)}$ we have that $\text{sgn}(\sigma)\sigma$ is multiplied by $(-1)^{j-1}$ rather than $(-1)^j$.

Subcase: Suppose $\pi^{-1}(i+1) = \pi^{-1}(i)-1 < 0$, implying that $\pi = [\cdots -i\ -(i+1) \cdots]$.

Let s be the index such that $\pi_s = -i$. It suffices to show that σ defined as follows satisfies $\text{des}(\sigma) = \text{des}(\pi) - 1$, $\text{sgn}(\sigma) = (-1)^{i-s+1}\text{sgn}(\pi)$, and $d_i\pi = \sigma d_s$. Let $\sigma = \sigma_1 \dots \sigma_{s-1}\sigma_{s+1} \dots \sigma_{r+1}$ where

- $\sigma_m = \pi_m$ if $|\pi_m| < i$,
- $|\sigma_m| = |\pi_m| - 1$ if $|\pi_m| > i$, and
- the sign pattern for σ is the same as that for π, i.e. if $\pi_j < 0$, then σ_j is also negative.

Thus, σd_s is the element of $L_r^{j-1} d_s$ uniquely paired with $d_i \pi$. This argument is similar to the previous subcase, and the same comment as in the previous subcase about $\mathrm{sgn}(\sigma) = (-1)^{i-s+1}\mathrm{sgn}(\pi)$ applies.

Subcase: Suppose $\pi^{-1}(i+1) \neq \pi^{-1}(i) \pm 1$.

Setting $\sigma = (i, i+1)\pi$, one can show that $\mathrm{sgn}(\pi) = -\mathrm{sgn}(\sigma)$, $\mathrm{des}(\pi) = \mathrm{des}(\sigma)$, and that $d_i\pi = d_i\sigma$. Thus, the terms corresponding to $d_i\pi$ and $d_i\sigma$ cancel in (6).

Unique bijection: To see that our correspondences above are bijective, consider an element $\sigma d_s \in L_r^j d_s$, where $\sigma = \sigma_1 \cdots \sigma_{s-1}\sigma_s\sigma_{s+1} \cdots \sigma_r$. Then σd_s is obtainable from some $d_j \pi$ via our above process if σ was obtained by deleting the s-th or $(s+1)$-st element from π, i.e.

$$\text{Case A: } \sigma = [\sigma_1 \cdots \sigma_s \underbrace{\quad}_{\substack{\pi_{s+1}\\ \text{dropped}\\ \text{here}}} \sigma_{s+1} \cdots \sigma_r] \text{ or Case B: } \sigma = [\sigma_1 \cdots \sigma_{s-1} \underbrace{\quad}_{\substack{\pi_{s}\\ \text{dropped}\\ \text{here}}} \sigma_s \cdots \sigma_r]\,.$$

Considering our claimed bijective map described above, in both Case A and Case B it is the element σ_s that determined which element of π was dropped. Suppose that $\sigma_s = \pm i$. For each of Case A and Case B, a fixed parity for σ_s yields a unique π, j such that $d_j\pi$ maps to σd_s under our map, and hence our claimed bijection (7) is established. □

Lemma 24.

$$\partial_r \tilde{\lambda}_{r+1}^{(j)} = \tilde{\lambda}_r^{(j)} \partial_r \quad \textit{and} \quad \partial_r \tilde{\rho}_{r+1}^{(j)} = \tilde{\rho}_r^{(j)} \partial_r$$

Proof. Since

$$\tilde{\lambda}_{r+1}^{(j)} = \sum_{i=0}^{j} (-1)^i \binom{r+1+i}{i} \tilde{l}_{r+1}^{(j-i)}\,,$$

it follows from Lemma 23 that

$$\begin{aligned}
\partial_r \tilde{\lambda}_{r+1}^{(j)} &= \partial_r \sum_{i=0}^{j} (-1)^i \binom{r+1+i}{i} \tilde{l}_{r+1}^{(j-i)} \\
&= \sum_{i=0}^{j} (-1)^i \binom{r+1+i}{i} \left(\tilde{l}_r^{(j-i)} + \tilde{l}_r^{(j-i-1)} \right) \partial_r \\
&= \Bigg[(-1)^0 \binom{r+1}{0} \left(\tilde{l}_r^{(j)} + \tilde{l}_r^{(j-1)} \right) + (-1)^1 \binom{r+2}{1} \left(\tilde{l}_r^{(j-1)} + \tilde{l}_r^{(j-2)} \right) + \cdots \\
&\qquad + (-1)^{j-1} \binom{r+1+j-1}{j-1} \left(\tilde{l}_r^{(1)} + \tilde{l}_r^{(0)} \right) + (-1)^j \binom{r+1+j}{j} \tilde{l}_r^{(0)} \Bigg] \partial_r \\
&= \Bigg[(-1)^0 \binom{r+1}{0} \tilde{l}_r^{(j)} + \left((-1)^0 \binom{r+1}{0} + (-1)^1 \binom{r+2}{1} \right) \tilde{l}_r^{(j-1)} + \cdots \\
&\qquad + \left((-1)^{j-1} \binom{r+1+j-1}{j-1} + (-1)^j \binom{r+1}{j} \right) \tilde{l}_r^{0} \Bigg] \partial_r
\end{aligned}$$

$$= \sum_{i=0}^{j} (-1)^i \binom{r+1}{i} \tilde{l}_r^{(j-i)} \partial_r$$
$$= \tilde{\lambda}_r^{(j)} \partial_r \,,$$

which establishes the first claim.

For the second claim, note that due to the relation

$$\tilde{\lambda}_{r+1}^{(j)} = (-1)^j \tilde{\rho}_{r+1}(2j+1) \,,$$

it follows that the $\tilde{\lambda}_{r+1}^{(j)}$'s and the $\tilde{\rho}_{r+1}^{(j)}$'s are related by an invertible Vandermonde matrix. Changing basis in this manner from the first to the second set of elements establishes the second claim. □

5 Chromatic polynomial coefficients and Hodge decompositions

In this section we establish that the coefficients of $\chi_G(\lambda)$ encode (up to sign) the dimensions of the Hodge components for $H_{n-2}(\Delta_G)$. First we must establish that Hodge decompositions are preserved by switching.

Lemma 25. *If two signed graphs G and H are switching equivalent, then there is a chain complex isomorphism between $C_*(\Delta_G)$ and $C_*(\Delta_H)$ respecting the Hodge decompositions.*

Proof. Suppose that H is obtained from G by switching at vertex i. It is straightforward to check that the map $f_i : C_*(\Delta_G) \to C_*(\Delta_H)$ obtained by exchanging i and $-i$ in every face of Δ_G is a chain complex isomorphism; one way to see this is to recognize that $\Delta_G = B_G \cap \partial[-1,1]^n$ is taken to $\Delta_H = B_H \cap \partial[-1,1]^n$ by the map $x_i \to -x_i$, and this induces the map f_i at the level of chain complexes. Using the combinatorial description of coloring complexes given in Definition 14 and the action of B_{r+1} on C_r defined for any coloring complex, it is immediate that for any $\pi \in B_{r+1}$ we have

$$\pi \circ f_i = f_i \circ \pi \,,$$

hence

$$\tilde{\rho}_{r+1}^{(j)} \circ f_i = f_i \circ \tilde{\rho}_{r+1}^{(j)} \,.$$

Our lemma follows by combining this with the fact that f_i is a chain complex isomorphism. □

Theorem 26. *Let G be a signed graph on $[n]$ with at least one edge or half-edge. Writing*

$$\chi_G(\lambda) = \lambda^n + \sum_{j=0}^{n-1} (-1)^{n-j} c_j \lambda^j \,,$$

we have $\dim H^{(j)}_{n-2}(\Delta_G) = c_j$. *Equivalently,*

$$(-1)^n \left[\chi_G(-\lambda) - (-\lambda)^n\right] = \sum_{j=0}^{n} c_j \lambda^j = \sum_{j=0}^{n} \dim H^{(j)}_{n-2}(\Delta_G)\lambda^j \,.$$

Proof. We go by induction on n, similar to the proof given by Hanlon [8, Theorem 4.1].

Base Case: Suppose first that E consists of a single half-edge; without loss of generality, we can consider this half-edge to be $\{n\}$. Then $\Delta_G \cong S^{n-2}$, so $\dim H_{n-2}(\Delta_G) = 1$. Let $\gamma = (1|2|\cdots|n-1|-1\ -2\ \cdots -n\, n)$; let

$$\Gamma := \left[\frac{1}{2^{n-1}(n-1)!}\sum_{\sigma\in B_{n-1}} \mathrm{sgn}(\sigma)\sigma\right]\gamma \,.$$

Claim: $\partial\Gamma = 0$. Considering the application of each d_i independently, we obtain

$$\sum_{i=1}^{n-1}(-1)^{i-1}d_i\cdot\Gamma = \sum_{i=1}^{n-1}\frac{1}{2^{n-1}(n-1)!}\sum_{\sigma\in B_{n-1}}(-1)^{i-1}d_i\mathrm{sgn}(\sigma)\sigma\gamma$$

$$= \sum_{i=1}^{n-1}\frac{1}{2^{n-1}(n-1)!}\sum_{\sigma\in B_{n-1}}(-1)^{i-1}d_i\mathrm{sgn}(\sigma)(\gamma_{\sigma^{-1}(1)}|\cdots|\gamma_{\sigma^{-1}(n-1)}|\gamma'_n)$$

$$= \sum_{i=1}^{n-1}\frac{1}{2^{n-1}(n-1)!}(-1)^{i-1}\sum_{\sigma\in B_{n-1}}\mathrm{sgn}(\sigma)(\gamma_{\sigma^{-1}(1)}|\cdots|\gamma_{\sigma^{-1}(i)}\cup\gamma_{\sigma^{-1}(i+1)}|\cdots|\gamma_{\sigma^{-1}(n-1)}|\gamma'_n)\,.$$

For $i \neq n-1$, on the terms of the sum $\sum_{\sigma\in B_{n-1}}\mathrm{sgn}(\sigma)d_i\sigma$, consider the involution $\sigma \to (i,i+1)\sigma$. This yields a sign-reversing involution on the summands in the final displayed line above.

On the terms of the corresponding sum for d_{n-1}, consider the involution $\sigma \to I_{n-1}\sigma$, where

$$I_{n-1} = (n-2,n-1)\cdots(2,3)(1,2)s_0(1,2)(2,3)\cdots(n-2,n-1)\,.$$

This yields another sign-reversing involution on the summands in the final displayed line above, hence

$$\partial\Gamma = 0\,.$$

Since $\Gamma = \tilde{\rho}^{(n-1)}_{n-1}$, it follows that

$$\dim H^{(n-1)}_{n-2}(\Delta_G) = 1\,.$$

Since $\chi_G(\lambda) = (\lambda-1)\lambda^{n-1} = \lambda^n - \lambda^{n-1}$, our base case holds.

In the case where E consists of only an edge, we can without loss of generality consider the edge to be $\{n-1,n\}$. If we set $\gamma = (1|2|\cdots|\{n,n-1\}|-1\ -2\ \cdots -n)$, then the same analysis as given above holds, establishing this base case as well.

Induction: Let G be a signed graph with $n \geqslant 2$ edges, and assume by way of induction that Theorem 26 holds for any signed graph with fewer than n edges. Let e be an edge of G; without loss of generality, we may assume e is either a half-edge or a positive edge, since by Proposition 7 and Lemma 25 we may switch a negative e to obtain a new signed graph with the same chromatic polynomial and Hodge structure. Let E be the graph with vertex set $V(G)$ and edge set $\{e\}$. Let $C_r(E)$ denote the space spanned by the chains $P = (P_1|\cdots|P_{r+2})$ that contain e. Let D_r denote the space spanned by the chains $P = (P_1|\cdots|P_{r+2})$ in Δ_G that do not contain e. It follows that

$$C_r(G) = D_r \oplus C_r(E).$$

Notice that the action of B_{r+2} commutes with the isomorphism, and thus,

$$C_r^{(j)}(G) = D_r^{(j)} \oplus C_r^{(j)}(E)\,. \tag{8}$$

Considering D_r next, we claim that

$$D_r \simeq C_r(G\backslash e)/(C_r(G\backslash e) \cap C_r(E))\,.$$

To prove this, observe that $C_r(G\backslash e)$ has as a spanning set the set of chains $P = (P_1|\cdots|P_{r+2})$ where at least one of the P_i contains an edge of the graph $G\backslash e$. This spanning set consists of chains P that contain an edge of the graph $G\backslash e$ as well as the edge e, and it consists of chains P that contain an edge of the graph $G\backslash e$ but do not also contain the edge e. The set of all chains P that contain an edge of $G\backslash e$ but not the edge e form a spanning set for $C_r(G\backslash e)/(C_r(G\backslash e) \cap C_r(E))$. Notice that this set is also a spanning set for D_r, and the isomorphism then follows. Since the action B_{r+2} commutes with this isomorphism, we have

$$D_r^{(j)} \simeq C_r^{(j)}(G\backslash e)/(C_r^{(j)}(G\backslash e) \cap C_r^{(j)}(E)).\,. \tag{9}$$

We finally claim that

$$C_r^{(j)}(G\backslash e) \cap C_r^{(j)}(E) \simeq C_r^{(j)}(G/e)\,. \tag{10}$$

Suppose first that $e = \{a, b\}$ is a positive edge. Consider the map sending

$$P = (P_1|P_2|\cdots|P_{r+1}|P_{r+2}) \in C_r(G\backslash e) \cap C_r(E)$$

to the chain $Q = (Q_1|\cdots|Q_{r+2}) \in C_r(G/e)$ obtained by replacing the pair $\{a, b\}$ by the symbol $a = b$ representing the contracted vertex in G/e. It is straightforward that this map gives a bijection inducing our desired isomorphism between $C_r(G\backslash e) \cap C_r(E)$ and $C_r(G/e)$, because the pair of symbols $\{a, b\}$ in any basis chain $P \in C_r(G\backslash e) \cap C_r(E)$ is simply replaced by the contracted vertex symbol $a = b$. Note that surjectivity follows since any chain $Q \in C_r(G/e)$ will contain an edge of G/e in some block, which will by definition correspond to an edge in $G\backslash e$, and hence a preimage under our map may be found. It is clear that this map is invariant under the hyperoctahedral group action, as a and b always are moved as part of the same block in both settings.

Next, suppose that $e = \{j\}$ is a half-edge, and observe that $C_r(E)$ is spanned by chains with $j, -j$ in P_{r+2}. Consider the map which takes a chain $(P_1|\cdots|P_{r+2}) \in C_r(G\backslash e)\cap C_r(E)$

and deletes the $j, -j$ in P_{r+2} to obtain a chain $(P_1|\cdots|P_{r+2}\ \{j,-j\}) \in C_r(G/e)$. We claim that this is a bijection inducing our desired isomorphism. To prove this, note that for any chain $(Q_1|\cdots|Q_{r+2}) \in C_r(G/e)$ we can add $\{j,-j\}$ to Q_{r+2} and obtain a new chain. This is actually a chain in the spanning set for $C_r(G\backslash e) \cap C_r(E)$; that it is in $C_r(E)$ is clear. To see that it is in $C_r(G\backslash e)$, we consider two possible cases. First, if a block Q_k contains an edge in G not incident to j, in which case this is also an edge in $G\backslash e$, our chain is a spanning element of $C_r(G\backslash e)$ and we are done. Second, if no block Q_k contains an edge in G/e, then for some i where i, j is an edge of G, we must have that $i, -i$ is in Q_{r+2}. Then when we add in $j, -j$ to Q_{r+2}, we have all of $i, j, -i, -j$ in Q_{r+2}. This implies that $(Q_1|\cdots|Q_{r+2} \cup \{j,-j\})$ can be obtained by merging the last two blocks in $(Q_1|\cdots|Q_{r+1}|\{i,j\}|(Q_{r+2} \setminus \{i\}) \cup \{-j\})$, and this longer chain corresponds to a spanning element of $C_{r+1}(G\backslash e)$. Thus, $(Q_1|\cdots|Q_{r+2} \cup \{i,-i\})$ must also be in the spanning set $C_r(G\backslash e)$. It is immediate that these maps are invariant under the hyperoctahedral group action, as the final block containing $j, -j$ is always fixed by the group.

From (8), (9), and (10), it follows that

$$\dim(C_r^{(j)}(G)) = \dim(C_r^{(j)}(G\backslash e)) - \dim(C_r^{(j)}(G/e)) + \dim(C_r^{(j)}(E)) .$$

Using the fact that the reduced homology $H_*(\Delta_G)$ is only nonvanishing in top dimension, along with the Euler-Poincare identity and our inductive hypothesis, we conclude that

$$\begin{aligned}
& \sum_j \dim(H_{n-2}^{(j)}(G))\lambda^j \\
= & \sum_{j,r} (-1)^{(n-2)-r} \dim(H_r^{(j)}(G))\lambda^j \\
= & \sum_{j,r} (-1)^{n-2-r} \dim(C_r^{(j)}(G))\lambda^j \\
= & \sum_{j,r} (-1)^{n-2-r} (\dim(C_r^{(j)}(G\backslash e)) - \dim(C_r^{(j)}(G/e)) + \dim(C_r^{(j)}(E))\lambda^j \\
= & \sum_j \dim(H_{n-2}^{(j)}(G\backslash e))\lambda^j - (-1)\sum_j \dim(H_{n-1}^{(j)}(G/e))\lambda^j + \sum_j \dim(H_{n-2}^{(j)}(E))\lambda^j \\
= & (-1)^n \left[(\chi_{G\backslash e}(-\lambda) - (-\lambda)^n) - (\chi_{G/e}(-\lambda) - (-\lambda)^{n-1}) + (\chi_E(-\lambda) - (-\lambda)^n)\right] \\
= & (-1)^n \left[\chi_{G\backslash e}(-\lambda) - \chi_{G/e}(-\lambda) - (-\lambda)^n\right] \\
= & (-1)^n(\chi_G(-\lambda) - (-\lambda)^n).
\end{aligned}$$

□

Remark 27. At the end of his paper regarding type A Hodge decompositions and ordinary chromatic polynomials, Hanlon [8] states that it would be interesting to find ways to use properties of chromatic polynomials to prove algebraic results about Hodge decompositions of coloring complexes, and vice versa. It is reasonable to ask for similar uses of Theorem 26 in the context of signed graphs. The existence of these results in both type A and type B are evidence that such connections might be waiting to be uncovered, though at present they remain elusive.

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