New Multiple Harmonic Sum Identities

Helmut Prodinger

Department of Mathematics
University of Stellenbosch
7602 Stellenbosch, South Africa

hproding@sun.ac.za

Roberto Tauraso

Dipartimento di Matematica
Università di Roma "Tor Vergata"
00133 Roma, Italy

tauraso@mat.uniroma2.it

Abstract

We consider a special class of binomial sums involving harmonic numbers and we prove three identities by using the elementary method of the partial fraction decomposition. Some applications to infinite series and congruences are given.

Keywords: multiple harmonic sums, partial fraction decomposition, Bernoulli numbers, congruences, zeta values

1 Introduction

The *binomial transform* of a sequence, $\{a_n\}_{n \geqslant 0}$, is the sequence defined by

$$n \mapsto \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k.$$

This kind of transform has been widely studied, see for example the pioneering paper [2]. In it, the following integral representation (Nörlund-Rice) is used:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \frac{n!}{z(z-1)\cdots(z-n)} f(z) \, dz,$$

where $f(z)$ is an analytic extension of the sequence $f(k)$, and the curve $\mathcal{C}$ includes the poles $0, 1, \dots, n$ and no others. Enlarging the contour of integration leads to an asymptotic expansion, but often also to identities, in particular when $f(z)$ is a rational function. In this case the method is equivalent to considering the partial fraction expansion of $\frac{n!}{z(z-1)\cdots(z-n)} f(z)$. This is also the point of view that we adopt in the sequel. Our terms

$f(k)$ are not exactly rational functions in k, but not too far away. We consider for example *harmonic numbers* $H_n(s)$ with $s \in \mathbb{N}^+$, which can be written as

$$H_n(s) = \sum_{j=1}^n \frac{1}{j^s} = \sum_{j=1}^\infty \left(\frac{1}{j^s} - \frac{1}{(n+j)^s} \right),$$

and for fixed j, this *is* a rational function, leading to an identity, and these identities will be summed over all j.

In this paper we are interested in a variation of the binomial transform, that is

$$n \mapsto \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{n+k}{k}^{\pm 1} a_k,$$

in the special case where the generic term a_n involves harmonic numbers. As we will see in a moment, these types of sums can be evaluated in terms of *multiple harmonic sums* which are defined by

$$H_n(s_1, s_2, \ldots, s_l) = \sum_{1 \leqslant k_1 < k_2 < \cdots < k_l \leqslant n} \prod_{i=1}^l \frac{(\mathrm{sgn}(s_i))^{k_i}}{k_i^{|s_i|}}$$

for $l \in \mathbb{N}^+$, $\mathbf{s} = (s_1, s_2, \ldots, s_l) \in (\mathbb{Z}^*)^l$. The integers $l(\mathbf{s}) := l$ and $|\mathbf{s}| := \sum_{i=1}^l |s_i|$ are called the length (or depth) and the weight of a multiple harmonic sum. By $\{s_1, s_2, \ldots, s_j\}^m$ we denote the set formed by repeating m times $(s_1, s_2, \ldots, s_j)$.

The following three identities appear in [6, Th. 1 and Th 2]: for any integers $n, r > 0$ and $d > 2$,

$$\sum_{k=1}^n (-1)^k \binom{n}{k} \binom{n+k}{k} H_k(1) = 2(-1)^n H_n(1),$$

$$\sum_{k=1}^n (-1)^k \binom{n}{k} \binom{n+k}{k} H_k(d) = (-1)^{n-1} \sum_{\mathbf{s} \preceq (\{1\}^{d-2}, 2)} 2^{l(\mathbf{s})} H_n(s_1, \ldots, s_{l(\mathbf{s})-1}, -s_{l(\mathbf{s})}),$$

$$\sum_{k=1}^n \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k} = - \sum_{\mathbf{s} \preceq (\{1\}^r)} 2^{l(\mathbf{s})} H_n(\mathbf{s}),$$

where $\mathbf{s} \preceq \mathbf{t} = (t_1, t_2, \ldots, t_m) \in (\mathbb{N}^+)^m$ means that the sums are taken over all the *compositions* of $\mathbf{t}$, i. e. any $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ such that $s_i = \sum_{j_{i-1} < k \leqslant j_i} t_k$ for some $0 = j_0 < j_1 < j_2 < \cdots < j_l = m$. So, for example $\mathbf{s} \preceq (1, 1, 1, 2)$ if and only if

$$\mathbf{s} \in \{(1, 1, 1, 2), (2, 1, 2), (1, 2, 2), (1, 1, 3), (2, 3), (1, 4), (3, 2), (5)\}.$$

Moreover, let

$$S_{m,n}(s_1, \ldots, s_l) = \sum_{m \leqslant k_1 \leqslant \cdots \leqslant k_l \leqslant n} \frac{1}{k_1^{s_1} \cdots k_r^{s_l}},$$

for $\mathbf{s} = (s_1, s_2, \ldots, s_l) \in (\mathbb{N}^*)^l$, with the convention that $S_n(\mathbf{s}) = S_{1,n}(\mathbf{s})$. The following four identities appear in [4, Cor. 2.1 and Th. 2.3]: for any positive integers n, a, b,

$$2\sum_{k=1}^{n} \frac{(-1)^k}{k^{2b}} \binom{n}{k} \binom{n+k}{k}^{-1} = -S_n(\{2^b\}), \tag{1}$$

$$2\sum_{k=1}^{n} \frac{(-1)^k}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} = S_n(1, \{2^{b-1}\}),$$

$$2\sum_{k=1}^{n} \frac{(-1)^k}{k^{2b}} \binom{n}{k} \binom{n+k}{k}^{-1} (H_k(2a+1) + H_{k-1}(2a+1)) = -S_n(\{2^a\}, 3, \{2^{b-1}\}), \tag{2}$$

$$2\sum_{k=1}^{n} \frac{1}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} (H_k(-2a) + H_{k-1}(-2a)) = -S_n(\{2^a\}, 1, \{2^{b-1}\}).$$

Note that also in this case the right-hand side is a combination of multiple harmonic sums because

$$S_n(\mathbf{t}) = \sum_{\mathbf{s} \preceq \mathbf{t}} H_n(\mathbf{s}).$$

In the next two sections we provide three new identities which extend the pattern. The proofs are based on the elementary method of the partial fraction decomposition (see [5] and [6] for more applications of this technique). In the final section we give some applications to infinite series and to congruences. In particular we show the following *exotic sums*:

$$R_\infty(1,2) = \sum_{1 \leqslant j_1 < k_1 \leqslant k_2} \frac{1}{j_1^2(2k_1-1)k_2^2} = \frac{31}{8}\zeta(5) - \frac{7}{4}\zeta(3)\zeta(2),$$

$$R_\infty(2,2) = \sum_{1 \leqslant j_1 \leqslant j_2 < k_1 \leqslant k_2} \frac{1}{j_1^2 j_2^2(2k_1-1)k_2^2} = -\frac{635}{64}\zeta(7) + \frac{49}{16}\zeta(3)\zeta(4) + \frac{31}{8}\zeta(5)\zeta(2),$$

where

$$R_n(a,b) = \sum_{k=1}^{n} \frac{S_{1,k-1}(\{2\}^a)S_{k,n}(\{2\}^{b-1})}{(2k-1)} = \sum_{1 \leqslant j_1 \leqslant \cdots \leqslant j_a < k_1 \leqslant \cdots \leqslant k_b \leqslant n} \frac{1}{j_1^2 \cdots j_a^2(2k_1-1)k_2^2 \cdots k_b^2}$$

for positive integers a, b.

2 Results: first part

Theorem 1. *For $n, r \geqslant 1$ and $d \geqslant 2$*

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k} H_k(1) = \sum_{\mathbf{s} \preceq (2, \{1\}^{r-1})} 2^{l(\mathbf{s})} H_n(-s_1, s_2, \ldots, s_{l(\mathbf{s})}),$$

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k} H_k(d) = -\sum_{\mathbf{s} \preceq (\{1\}^{d-2}, 3, \{1\}^{r-1})} 2^{l(\mathbf{s})} H_n(\mathbf{s}).$$

Proof. For $n, j \geqslant 1$, let (factorials are defined via Gamma functions)

$$f(z) := \frac{(z+n)!(z-n-1)!}{z!(z-1)!} \cdot \frac{1}{z^{r+1}} \cdot \left[\frac{1}{j^d} - \frac{1}{(z+j)^d} \right],$$

and consider its partial fraction decomposition

$$f(z) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{1}{k^r} \left[\frac{1}{j^d} - \frac{1}{(k+j)^d} \right] \frac{1}{z-k}$$
$$+ \frac{C_r(n,j)}{z^r} + \cdots + \frac{C_1(n,j)}{z} + \frac{D_d(n,j)}{(z+j)^d} + \cdots + \frac{D_1(n,j)}{z+j}.$$

Multiply by z and let $z \to \infty$.

$$0 = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{1}{k^r} \left[\frac{1}{j^d} - \frac{1}{(k+j)^d} \right] + C_1(n,j) + D_1(n,j)$$

where

$$C_1(n,j) = [z^{-1}] \frac{(z+n)!(z-n-1)!}{z!(z-1)!} \frac{1}{z^{r+1}} \left[\frac{1}{j^d} - \frac{1}{(z+j)^d} \right]$$
$$= [z^r] \frac{(z+n)!(z-n-1)!}{z!(z-1)!} \left[\frac{1}{j^d} - \frac{1}{(z+j)^d} \right]$$
$$= -[z^r][w^{d-1}] \frac{(z+n)!(z-n-1)!}{z!(z-1)!} \left[\frac{1}{w-j} - \frac{1}{w-z-j} \right],$$

$$D_1(n,j) = [(z+j)^{-1}] \frac{(z+n)!(z-n-1)!}{z!(z-1)!} \frac{1}{z^{r+1}} \left[\frac{1}{j^d} - \frac{1}{(z+j)^d} \right]$$
$$= -[(z+j)^{d-1}] \frac{(z+n)!(z-n-1)!}{z!(z-1)!} \frac{1}{z^{r+1}}$$
$$= -[w^{d-1}] \frac{(w-j+n)!(w-j-n-1)!}{(w-j)!(w-j-1)!} \frac{1}{(w-j)^{r+1}}$$
$$= [z^r][w^{d-1}] \frac{(w-j+n)!(w-j-n-1)!}{(w-j)!(w-j-1)!} \frac{1}{z-(w-j)}$$
$$= [z^r][w^{d-1}] \frac{(w-j+n)!(w-j-n-1)!}{(w-j)!(w-j-1)!} \left[\frac{1}{w-j} - \frac{1}{w-z-j} \right].$$

Define $S(n)$ via

$$[z^r][w^{d-1}] S(n) = \sum_{j \geqslant 1} (C_1(n,j) + D_1(n,j))$$

then by summing over $j \geqslant 1$, we obtain

$$[z^r][w^{d-1}](-1)^{n-1} S(n) = \sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k} H_k(d).$$

Now set

$$F(n, j)$$
$$= \left(-\frac{(z+n)!(z-n-1)!}{z!(z-1)!} + \frac{(w-j+n)!(w-j-n-1)!}{(w-j)!(w-j-1)!} \right) \left[\frac{1}{w-j} - \frac{1}{w-z-j} \right],$$

and

$$G(n, j) = -\frac{(w-j+n+1)!(w-j-n-1)!}{(w-j)!(w-j)!} \frac{2z}{n+1}.$$

Then, as is easy to check,

$$(n+1-z)F(n+1, j) + (n+1+z)F(n, j) = G(n, j+1) - G(n, j).$$

Summing over $j \geqslant 1$, we get

$$(n+1-z)S(n+1) + (n+1+z)S(n) = \lim_{j\to\infty} G(n, j+1) - G(n, 1) = -\frac{2z}{n+1} - G(n, 1).$$

Let

$$T(n) = (-1)^n \frac{(n-z)!}{(n+z)!} S(n)$$

then, since

$$G(n, 1) = -\frac{(w+n)!(w-n-2)!}{(w-1)!(w-1)!} \frac{2z}{n+1},$$

we have

$$T(n+1) - T(n) = (-1)^n \frac{(n-z)!}{(n+z)!} \frac{1}{(n+1+z)} \left(\frac{2z}{n+1} + G(n, 1) \right)$$
$$= \frac{(-1)^n (n-z)!}{(n+1+z)!} \left(1 - \frac{(w+n)!(w-n-2)!}{(w-1)!(w-1)!} \right) \frac{2z}{n+1}.$$

Therefore

$$T(n) = \sum_{k=1}^{n} (T(k) - T(k-1))$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1-z)!}{(k+z)!} \left(1 - \frac{(w+k-1)!(w-k-1)!}{(w-1)!(w-1)!} \right) \frac{2z}{k},$$

and

$$(-1)^{n-1} S(n) = \frac{(n+z)!}{(n-z)!} \sum_{k=1}^{n} \frac{(-1)^k (k-1-z)!}{(k+z)!} \left(1 - \frac{(w+k-1)!(w-k-1)!}{(w-1)!(w-1)!} \right) \frac{2z}{k}$$
$$= 2 \sum_{k=1}^{n} \frac{z}{(1-z/k)} \prod_{j=k+1}^{n} \frac{1+z/j}{1-z/j} \cdot \left(\frac{(-1)^k}{k^2} - \frac{1}{k^3} \frac{w}{(1-w/k)} \prod_{j=1}^{k-1} \frac{1+w/j}{1-w/j} \right).$$

If $d = 1$ then

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k} H_k(1)$$

is given by

$$[z^r][w^0](-1)^{n-1} S(n) = 2[z^r] \sum_{k=1}^{n} \frac{z}{(1-z/k)} \prod_{j=k+1}^{n} \frac{1+z/j}{1-z/j} \cdot \left(\frac{(-1)^k}{k^2} \right)$$

$$= 2[z^{r-1}] \sum_{k=1}^{n} \frac{(-1)^k}{k^2} \left(1 + \sum_{s=1}^{\infty} \frac{z^s}{k^s} \right) \prod_{j=k+1}^{n} \left(1 + 2\sum_{s=1}^{\infty} \frac{z^s}{j^s} \right)$$

$$= \sum_{\mathbf{s} \preceq (2,\{1\}^{r-1})} 2^{l(\mathbf{s})} H_n(-s_1, s_2, \ldots, s_{l(\mathbf{s})}).$$

If $d \geqslant 2$ then

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k} H_k(d)$$

is given by

$$[z^r][w^{d-1}](-1)^{n-1} S(n) = -2[z^r][w^{d-1}] \sum_{k=1}^{n} \frac{z}{(1-z/k)} \prod_{j=k+1}^{n} \frac{1+z/j}{1-z/j}$$

$$\cdot \left(\frac{1}{k^3} \frac{w}{(1-w/k)} \prod_{j=1}^{k-1} \frac{1+w/j}{1-w/j} \right)$$

$$= -2[z^{r-1}][w^{d-2}] \sum_{k=1}^{n} \left(1 + \sum_{s=1}^{\infty} \frac{w^s}{k^s} \right) \prod_{j=1}^{k-1} \left(1 + 2\sum_{s=1}^{\infty} \frac{w^s}{j^s} \right)$$

$$\cdot \left(\frac{1}{k^3} \right) \cdot \left(1 + \sum_{s=1}^{\infty} \frac{z^s}{k^s} \right) \prod_{j=k+1}^{n} \left(1 + 2\sum_{s=1}^{\infty} \frac{z^s}{j^s} \right)$$

$$= - \sum_{\mathbf{s} \preceq (\{1\}^{d-2}, 3, \{1\}^{r-1})} 2^{l(\mathbf{s})} H_n(\mathbf{s}).$$

$\square$

3 Results: second part

Theorem 2. *For $n, a, b \geqslant 1$*

$$2\sum_{k=1}^{n} \frac{1}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} \left((-1)^k H_{k-1}(2a) - H_{k-1}(-2a) \right) = R_n(a, b). \tag{3}$$

Proof. For $n, j \geqslant 1$, let

$$f(z) := \frac{z!(z-n-1)!}{(z+n)!(z-1)!} \cdot \frac{1}{z^{r+1}} \cdot \left[\frac{1}{j^d} - \frac{1}{(z+j)^d}\right].$$

Then its partial fraction decomposition is

$$f(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{1}{k^r} \left[\frac{1}{j^d} - \frac{1}{(k+j)^d}\right] \frac{1}{z-k}$$

$$+ (-1)^r \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{1}{k^r} \left[\frac{1}{j^d}\right] \frac{1}{z+k}$$

$$+ (-1)^r \sum_{k=1, k \neq j}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{1}{k^r} \left[-\frac{1}{(-k+j)^d}\right] \frac{1}{z+k}$$

$$+ \frac{C_r(n,j)}{z^r} + \cdots + \frac{C_1(n,j)}{z} + \frac{D_{d+1}(n,j)}{(z+j)^{d+1}} + \cdots + \frac{D_1(n,j)}{z+j}.$$

Multiply by z and let $z \to \infty$.

$$0 = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{1}{k^r} \left[\frac{1}{j^d} - \frac{1}{(k+j)^d}\right]$$

$$+ (-1)^r \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{1}{k^r} \left[\frac{1}{j^d}\right]$$

$$+ (-1)^r \sum_{k=1, k \neq j}^{n} \frac{(-1)^{n-k}}{(n+k)!(n-k)!} \frac{1}{k^r} \left[-\frac{1}{(-k+j)^d}\right]$$

$$+ C_1(n,j) + D_1(n,j).$$

Moreover

$$C_1(n,j) = -[z^r][w^{d-1}] \frac{z!(z-n-1)!}{(z+n)!(z-1)!} \cdot \left[\frac{1}{w-j} - \frac{1}{w-(z+j)}\right]$$

and

$$D_1(n,j) = [z^r][w^{d-1}] \frac{(w-j)!(w-j-n-1)!}{(w-j+n)!(w-j-1)!} \cdot \left[\frac{1}{w-j} - \frac{1}{w-(z+j)}\right].$$

Define $S(n)$ via

$$[z^r][w^{d-1}]S(n) = (n!)^2 \sum_{j \geqslant 1} (C_1(n,j) + D_1(n,j))$$

then by summing over $j \geqslant 1$, we obtain

$$
\begin{aligned}
[z^r][w^{d-1}](-1)^{n-1}S(n) = {} & \sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} H_k(d) \\
& + (-1)^r \sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} \zeta(d) \\
& - (-1)^r \sum_{j=1}^{n} \sum_{k=1}^{j-1} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} \frac{1}{(j-k)^d} \\
& - (-1)^r \sum_{j=1}^{n} \sum_{k=j+1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} \frac{1}{(j-k)^d} \\
& - (-1)^r \sum_{j=n+1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} \frac{1}{(j-k)^d} \\
= {} & \sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} \left(H_k(d) - (-1)^{r+d} H_{k-1}(d) \right).
\end{aligned}
$$

Let

$$
F(n,j) = (n!)^2 \left(-\frac{z!(z-n-1)!}{(z+n)!(z-1)!} + \frac{(w-j)!(w-j-n-1)!}{(w-j+n)!(w-j-1)!} \right) \left[\frac{1}{w-j} - \frac{1}{w-(z+j)} \right]
$$

and

$$
G(n,j) = (n!)^2 \frac{(w-j-n-1)!}{(w-j+n+1)!} \left(\frac{z+n+1}{2(n+1)} - (w-j+z+1) \right) \frac{z}{2n+1}.
$$

Then $[z^r][w^{d-1}]F(n,j) = (n!)^2(C_1(n,j) + D_1(n,j))$,

$$
\left(1 - \frac{z^2}{(n+1)^2} \right) F(n+1,j) + F(n,j) = G(n,j+1) - G(n,j)
$$

and by summing over $j \geqslant 1$, we get

$$
\left(1 - \frac{z^2}{(n+1)^2} \right) S(n+1) + S(n) = \lim_{j \to \infty} G(n,j+1) - G(n,1) = -G(n,1).
$$

Let

$$
T(n) = (-1)^n \prod_{j=1}^{n} \left(1 - \frac{z^2}{j^2} \right) \cdot S(n)
$$

then,

$$
T(n+1) - T(n) = \prod_{j=1}^{n} \left(1 - \frac{z^2}{j^2} \right) \cdot (-1)^n G(n,1).
$$

Since

$$G(n,1) = (n!)^2 \frac{(w-n-2)!}{(w+n)!} \left(\frac{z+n+1}{2(n+1)} - (w+z) \right) \frac{z}{2n+1}$$

$$= (-1)^n \prod_{j=1}^{n} \left(1 - \frac{w^2}{j^2} \right)^{-1} \cdot \left(\frac{1}{2} \left(1 + \frac{z}{n+1} \right) - (w+z) \right) \frac{z}{(2n+1)w(w-n-1)},$$

we have that

$$(-1)^{n-1} S(n) = - \prod_{j=1}^{n} \left(1 - \frac{z^2}{j^2} \right)^{-1} \cdot T(n)$$

$$= - \prod_{j=1}^{n} \left(1 - \frac{z^2}{j^2} \right)^{-1} \cdot \sum_{k=1}^{n} (T(k) - T(k-1))$$

$$= - \prod_{j=1}^{n} \left(1 - \frac{z^2}{j^2} \right)^{-1} \cdot \sum_{k=1}^{n} \prod_{j=1}^{k-1} \left(1 - \frac{z^2}{j^2} \right) (-1)^{k-1} G(k-1,1)$$

$$= \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2} \right)^{-1} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{w^2}{j^2} \right)^{-1}$$

$$\cdot \left(\frac{1}{2} \left(1 + \frac{z}{k} \right) - (w+z) \right) \frac{z(1+w/k)}{k(2k-1)w(1-w^2/k^2)}$$

$$= \frac{z}{w} \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2} \right)^{-1} \cdot \prod_{j=1}^{k} \left(1 - \frac{w^2}{j^2} \right)^{-1}$$

$$\cdot \left(\frac{1}{2(2k-1)k} - \frac{w}{2k^2} - \frac{z}{2k^2} - \frac{w^2}{(2k-1)k^2} - \frac{zw}{2k^3} \right).$$

Hence

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^r} \binom{n}{k} \binom{n+k}{k}^{-1} (H_k(d) - (-1)^{r+d} H_{k-1}(d))$$

$$= [z^{r-1}][w^d] \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2} \right)^{-1} \cdot \prod_{j=1}^{k} \left(1 - \frac{w^2}{j^2} \right)^{-1}$$

$$\cdot \left(\frac{1}{2(2k-1)k} - \frac{w}{2k^2} - \frac{z}{2k^2} - \frac{w^2}{(2k-1)k^2} - \frac{zw}{2k^3} \right). \tag{4}$$

If $r = 2b - 1$ and $d = 2a$ then we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} (H_k(2a) + H_{k-1}(2a))$$

$$= [z^{2(b-1)}][w^{2a}] \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2}\right)^{-1} \cdot \prod_{j=1}^{k} \left(1 - \frac{w^2}{j^2}\right)^{-1}$$

$$\cdot \left(-\frac{1}{2k} + \frac{1}{2k-1}\left(1 - \frac{w^2}{k^2}\right)\right).$$

Since

$$\sum_{k=1}^{n} \frac{1}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} (H_k(-2a) + H_{k-1}(-2a))$$

$$= [z^{2(b-1)}][w^{2a}] \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2}\right)^{-1} \cdot \prod_{j=1}^{k} \left(1 - \frac{w^2}{j^2}\right)^{-1} \cdot \left(-\frac{1}{2k}\right)$$

and

$$(-1)^k(H_k(2a) + H_{k-1}(2a)) - (H_k(-2a) + H_{k-1}(-2a)) = 2((-1)^k H_{k-1}(2a) - H_{k-1}(-2a))$$

it follows that

$$2\sum_{k=1}^{n} \frac{1}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} ((-1)^k H_{k-1}(2a) - H_{k-1}(-2a))$$

$$= [z^{2(b-1)}][w^{2a}] \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2}\right)^{-1} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{w^2}{j^2}\right)^{-1} \cdot \left(\frac{1}{2k-1}\right)$$

$$= \sum_{k=1}^{n} \frac{S_{1,k-1}(\{2\}^a) S_{k,n}(\{2\}^{b-1})}{(2k-1)} = R_n(a,b).$$

$\square$

Note that if $r = 2b - 1$ and $d = 1$ then (4) yields (1),

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^{2b-1}} \binom{n}{k} \binom{n+k}{k}^{-1} \left(\frac{1}{k}\right)$$

$$= [z^{2(b-1)}][w^1] \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2}\right)^{-1} \cdot \prod_{j=1}^{k} \left(1 - \frac{w^2}{j^2}\right)^{-1} \cdot \left(-\frac{w}{2k^2}\right)$$

$$= -\frac{1}{2} S_n(\{2\}^b).$$

Moreover, if $r = 2b$ and $d = 2a + 1$ then (4) yields (2),

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^{2b}} \binom{n}{k} \binom{n+k}{k}^{-1} (H_k(2a+1) + H_{k-1}(2a+1))$$

$$= [z^{2b-1}][w^{2a+1}] \sum_{k=1}^{n} \prod_{j=k}^{n} \left(1 - \frac{z^2}{j^2}\right)^{-1} \cdot \prod_{j=1}^{k} \left(1 - \frac{w^2}{j^2}\right)^{-1} \cdot \left(-\frac{zw}{2k^3}\right)$$

$$= -\frac{1}{2} S_n(\{2\}^a, 3, \{2\}^{b-1}).$$

4   Two applications of identity (3)

Our first application involves infinite series. Evaluations for Euler sums of length two and of odd weight $m + n$, $H_\infty(m, -n)$ and $H_\infty(-m, n)$ in terms of zeta values are well known (see, for example, [1, Th. 7.2]):

$$2H_\infty(m, -n) = \overline{\zeta}(m+n) - (1 - (-1)^m)\zeta(m)\overline{\zeta}(n)$$

$$+ 2(-1)^m \sum_{r=1}^{(m+n-1)/2} \binom{2r}{n-1} \overline{\zeta}(2r+1)\overline{\zeta}(m+n-2r-1)$$

$$- 2(-1)^m \sum_{r=1}^{(m+n-1)/2} \binom{2r}{m-1} \zeta(2r+1)\overline{\zeta}(m+n-2r-1), \qquad (5)$$

$$2H_\infty(-n, m) = -2H_\infty(m, -n) - 2\zeta(m)\overline{\zeta}(n) + 2\overline{\zeta}(m+n), \qquad (6)$$

where $\overline{\zeta}(0) = 1/2$, $\overline{\zeta}(1) = \ln(2)$, $\overline{\zeta}(n) = (1 - 2^{1-n})\zeta(n)$, and $\zeta(n) = \sum_{k=1}^{\infty} 1/k^n$, for $n > 1$.

Theorem 3. *Let a, b positive integers with $b > 1$. Then*

$$R_\infty(a, b) = 4 \sum_{r=1}^{a+b-1} \left(\binom{2r}{2b-2} - \binom{2r}{2a-1} \right) \left(1 - \frac{1}{2^{2r+1}}\right) \zeta(2r+1)\overline{\zeta}(2(a+b-1-r))$$

Proof. By taking the limit $n \to \infty$ in (3),

$$R_\infty(a, b) = 2H_\infty(2a, -(2b-1)) - 2H_\infty(-2a, 2b-1).$$

Then use (5) and (6). $\qquad \square$

For the second application of (3), we first recall some congruences. Let m, n be positive integers and let p be any prime $p > w + 1$ where $w = m + n$,
i) if w is even then ([7, Th. 3.2])

$$H_{p-1}(m, n) \equiv \left((-1)^m n \binom{w+1}{m} - (-1)^m m \binom{w+1}{n} - w \right) \frac{pB_{p-w-1}}{2(w+1)} \pmod{p^2},$$

ii) if m is even and n is even then ([4, Lemma 3.1])

$$H_{p-1}(-m, -n) \equiv \left(\frac{(m-n)(1-2^{-w})}{2(m+1)(n+1)} \binom{w}{m} - \frac{w}{2(w+1)} \right) p B_{p-w-1} \pmod{p^2}.$$

iii) if w is odd then ([3, Lemma 1])

$$H_{\frac{p-1}{2}}(m, n) \equiv \left((-1)^n \binom{w}{m} + 2^w - 2 \right) \frac{B_{p-w}}{2w} \pmod{p},$$

iv) if m is even and n is odd then ([4, Lemma 3.2])

$$H_{\frac{p-1}{2}}(-m, -n) \equiv \left(\frac{1}{2^w} \binom{w}{m} + 1 \right) \frac{(2^{w-1}-1)B_{p-w}}{w} \pmod{p}.$$

Lemma 4. *Let m, n, r be positive integers. If $n+r$ is odd and m is even then for any prime $p > \max(m, n) + 1$,*

$$\sum_{k=1}^{p-1} \frac{(H_{k-1}(m) - (-1)^k H_{k-1}(-m))(H_k(n) + H_{k-1}(n))}{k^r} \equiv 0 \pmod{p}.$$

Proof. Let a be a positive integer or a negative even integer then, by Wolstenholme's theorem, for $p > a + 1$,

$$H_{p-1}(a) \equiv 0 \pmod{p}.$$

Thus

$$H_{p-k}(a) = \sum_{j=1}^{p-k} \frac{(\operatorname{sgn}(a))^j}{j^a} = \sum_{j=k}^{p-1} \frac{(\operatorname{sgn}(a))^{p-j}}{(p-j)^a}$$

$$\equiv (-1)^a \operatorname{sgn}(a) \sum_{j=k}^{p-1} \frac{(\operatorname{sgn}(a))^j}{j^a}$$

$$\equiv (-1)^a \operatorname{sgn}(a)(H_{p-1}(a) - H_{k-1}(a))$$

$$\equiv -(-1)^a \operatorname{sgn}(a) H_{k-1}(a) \pmod{p}.$$

Hence

$$L := \sum_{k=1}^{p-1} \frac{(H_{k-1}(m) - (-1)^k H_{k-1}(-m))(H_k(n) + H_{k-1}(n))}{k^r}$$

$$= \sum_{k=1}^{p-1} \frac{(H_{p-k-1}(m) - (-1)^{p-k} H_{p-k-1}(-m))(H_{p-k}(n) + H_{p-k-1}(n))}{(p-k)^r}$$

$$\equiv \sum_{k=1}^{p-1} \frac{(-H_k(m) + (-1)^k H_k(-m))(-1)^n(-H_{k-1}(n) - H_k(n))}{(-1)^r k^r}$$

$$\equiv (-1)^{n+r} L + (-1)^{n+r} \sum_{k=1}^{p-1} \frac{(1 - (-1)^{2k})(H_{k-1}(n) - H_k(n))}{k^{r+m}}$$

$$\equiv -L \pmod{p},$$

which implies that $L \equiv 0 \pmod{p}$. $\square$

Theorem 5. *For any prime $p > 2a + 2b - 1$,*

$$R_{(p-1)/2}(a,b) \equiv -\frac{2(1 - 2^{-(2a+2b-1)})}{2a+2b-1}\binom{2a+2b-1}{2a}B_{p-(2a+2b-1)} \pmod{p},$$

$$R_{p-1}(a,b) \equiv \frac{2(a-b+1)R_{(p-1)/2}(a,b)}{(2a+1)} \pmod{p}.$$

Proof. Setting $n = \frac{p-1}{2}$, we note that

$$\binom{n}{k}\binom{n+k}{k}^{-1} = (-1)^k\frac{\left(\frac{1}{2}-\frac{p}{2}\right)\left(\frac{3}{2}-\frac{p}{2}\right)\cdots\left(\frac{2k-1}{2}-\frac{p}{2}\right)}{\left(\frac{1}{2}+\frac{p}{2}\right)\left(\frac{3}{2}+\frac{p}{2}\right)\cdots\left(\frac{2k-1}{2}+\frac{p}{2}\right)} \equiv (-1)^k \pmod{p}.$$

Therefore, by i) and ii),

$$R_{(p-1)/2}(a,b) \equiv 2\sum_{k=1}^{n}\frac{1}{k^{2b-1}}(H_{k-1}(2a) - (-1)^k H_{k-1}(-2a))$$

$$\equiv 2H_{(p-1)/2}(2a, 2b-1) - 2H_{(p-1)/2}(-2a, -(2b-1))$$

$$\equiv -\frac{2(1 - 2^{-(2a+2b-1)})}{2a+2b-1}\binom{2a+2b-1}{2a}B_{p-(2a+2b-1)} \pmod{p}.$$

Now, setting $n = p - 1$ we have that

$$\binom{n}{k}\binom{n+k}{k}^{-1} = \frac{(-1)^k k}{p}\prod_{j=1}^{k}\left(1 - \frac{p}{j}\right)\cdot\prod_{j=1}^{k-1}\left(1 + \frac{p}{j}\right)^{-1}$$

$$\equiv \frac{(-1)^k k}{p}(1 - pH_k(1))(1 - pH_{k-1}(1))$$

$$\equiv (-1)^k\left(\frac{k}{p} - (H_k(1) + H_{k-1}(1))\right) \pmod{p},$$

and by the previous lemma, by iii) and iv), it follows that

$$R_{(p-1)/2}(a,b) \equiv 2\sum_{k=1}^{n}\frac{1}{k^{2b-1}}\left(\frac{k}{p} - (H_k(1) + H_{k-1}(1))\right)(H_{k-1}(2a) - (-1)^k H_{k-1}(-2a))$$

$$\equiv \frac{2}{p}\sum_{k=1}^{n}\frac{H_{k-1}(2a) - (-1)^k H_{k-1}(-2a)}{k^{2b-2}}$$

$$\equiv 2\left(\frac{H_{p-1}(2a, 2b-2)}{p} - \frac{H_{p-1}(-2a, -(2b-2))}{p}\right)$$

$$\equiv \frac{2(a-b+1)R_{(p-1)/2}(a,b)}{(2a+1)} \pmod{p}.$$

$\square$

References

[1] P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Exp. Math.*, 7(1):15–35, 1998.

[2] P. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice's integrals. *Theor. Comput. Sci.*, 144(1-2):101–124, 1995.

[3] Kh. Hessami Pilehrood, T. Hessami Pilehrood, and R. Tauraso. Congruences concerning Jacobi polynomials and Apéry-like formulae. *Int. J. Number Theory*, 8(7):1789–1811, 2012.

[4] Kh. Hessami Pilehrood, T. Hessami Pilehrood, and R. Tauraso. New properties of multiple harmonic sums modulo p and p-analogues of Leshchiner's series. *Trans. Amer. Math. Soc.*, 366(6):3131–3159, 2014.

[5] H. Prodinger. Human proofs of identities by Osburn and Schneider. *Integers*, 8:Art. A10, 8 pp. (electronic), 2008.

[6] H. Prodinger. Identities involving harmonic numbers that are of interest for physicists. *Util. Math.*, 83:291–299, 2010.

[7] J. Zhao. Wolstenholme type theorem for multiple harmonic sums. *Int. J. Number Theory*, 4(1): 73–106, 2008.