General Restriction of (s,t)-Wythoff's Game

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Abstract

A.S. Fraenkel introduces a new (s,t)-Wythoff's game which is the generalization of both Wythoff's game and a-Wythoff's game. Four new models of the restricted version of (s,t)-Wythoff's game, Odd-Odd (s,t)-Wythoff's Game, Even-Even (s,t)-Wythoff's Game, Odd-Even (s,t)-Wythoff's Game and Even-Odd (s,t)-Wythoff's Game, are investigated. Under normal or misère play convention, all its P-positions of these four models are given for arbitrary integers $s,t \geq 1$. For Even-Even (s,t)-Wythoff's Game, the structure of P-positions is given by recursive characterizations in term of mex function, which are just as Wythoff's sequence. For other models, the structures of P-positions are of algebraic form, which permits to decide in polynomial time whether or not a given game position (a,b) is a P-position.

Keywords: impartial combinatorial game; normal play convention; misère play convention; P-position; (s,t)-Wythoff's game

1 Introduction

By game we mean a combinatorial game; we restrict our attention to classical impartial games. There are two conventions: in *normal play convention*, the player first unable to

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move is the loser (his opponent the winner); in *misère play convention*, the player first unable to move is the winner (his opponent the loser). The positions from which the previous player can win regardless of the opponent's moves are called *P-positions* and those from which the next player can win regardless of the opponent's moves are called *N-positions*. The theory of such games can be found in [1, 2, 4, 8].

Throughout the paper, we use the following notations:

(1) By $Z^{\geq m}$ we denote the set of all integers not less than m, i.e., $Z^{\geq m} = \{x \geq m | x \text{ is an integer } \}$. Let $Z^{even} = \{2n | n \in Z^{\geq 0}\}$, $Z^{odd} = \{2n+1 | n \in Z^{\geq 0}\}$.

(2) For any set $U \subseteq Z^{\geq 0}$, by $\text{mex}(U)$ we denote the *Minimum EXcluded value* of U, i.e., the smallest nonnegative integer not in U. In particular, $\text{mex}(\emptyset) = 0$.

(3) By $\lfloor x \rfloor$ we denote the largest integer $\leq x$.

(4) We use the notation $(x_1, y_1) \rightarrow (x_2, y_2)$ if there is a legal move from (x_1, y_1) to (x_2, y_2).

1.1 Wythoff's game

Wythoff's game is played with two heaps of tokens. Each player can either remove any number of tokens from a single heap (*Nim rule*) or remove the same number of tokens from both heaps (*Wythoff's rule*). All P-positions of Wythoff's game under normal play convention are given in [19]. All P-positions of Wythoff's game under misère play convention are determined in [13].

1.2 Extension of Wythoff's game

In many papers devoted to variations of Wythoff's game, new rules are adjoined to the original ones. Such variations are called *extensions*.

As an example, *a-Wythoff's game* is investigated in [9]: Given an integer $a \geq 1$ and two heaps of finitely many tokens. Two rules of moves are allowed:

(*Nim Rule*) Take any positive number of tokens from a single heap, possibly the entire heap.

(*General Wythoff's Rule*) Take tokens from both heaps, $k > 0$ tokens from one heap, and $\ell > 0$ tokens from the other, and $| k - \ell | < a$, where $a > 0$ is a fixed integer parameter.

A.S. Fraenkel introduces a new (s,t)-*Wythoff's game* in [11]. Given two parameters $s, t \in Z^{\geq 1}$ and two heaps of finitely many tokens. There are two types of moves:

(*Nim Rule*) Take any positive number of tokens from a single heap, possibly the entire heap.

(*More General Wythoff's Rule*) Take tokens from both heaps, $k > 0$ from one heap and $\ell > 0$ from the other and

$$0 < \ k \leq \ell < sk + t. \tag{1}$$

In [11], the author gives the following results: By $\mathscr{P}_{s,t}$ we denote the set of all P-positions of the (s,t)-Wythoff's game under normal play convention. Then $\mathscr{P}_{s,t} =$

$\bigcup_{n=0}^{\infty}\{(A_n, B_n)\}$, where for $n \geq 0$,

$$\begin{cases} A_n = \text{mex}\{A_i, B_i | 0 \leq i < n\}, \\ B_n = sA_n + tn. \end{cases} \tag{2}$$

It is worth to mention that Wythoff's game is the special case $s = t = 1$ in (s, t)-Wythoff's game, and a-Wythoff's game is the special case $s = 1$ and $t = a$ in (s, t)-Wythoff's game. Thus (s, t)-Wythoff's game is the generalization of both Wythoff's game and a-Wythoff's game.

Under normal play convention, the set $\mathscr{P}_{1,a}$ of all P-positions of a-Wythoff's game and the set $\mathscr{P}_{1,1}$ of all P-positions of Wythoff's game are given by letting ($s = 1$ and $t = a$) and $s = t = 1$ in Eq. (2), respectively (see [9, 19]).

Under misère play convention, all P-positions of a-Wythoff's game are given in [13] and all P-positions of (s, t)-Wythoff's game have been determined in [17] for arbitrary integers $s, t \geq 1$.

Other examples of extensions of Wythoff's game are given in [3, 12, 14, 15, 18].

1.3 Restriction of (s, t)-Wythoff's game

There are a few papers where only subsets of Wythoff's moves are allowed (see [5, 6, 10]). Such variations are called *restrictions* of Wythoff's game.

We now introduce a new *General Restriction of (s, t)-Wythoff's Game*: Let S_h, S_v, D_1 and D_2 be subsets of $Z^{\geq 0}$. Given two parameters $s, t \in Z^{\geq 1}$ and two heaps of finitely many tokens. One of the heaps is designated as "first heap" and the other as "second heap" throughout the game. By (x, y) we denote a position of present game, where x and y denote the numbers of tokens in the first and the second heaps, respectively. There are three types of moves:

(*Horizontal Move*) A player chooses the first heap and takes $k \in (\{w | 0 < w \leq x\} \cap S_h)$ tokens, i.e.,

$$(x, y) \to (x - k, y) \text{ and } k \in (\{w | 0 < w \leq x\} \cap S_h). \tag{3}$$

In this case, we call that (x, y) is moved to $(x - k, y)$ in the *horizontal direction*, and k is the *horizontal distance*.

(*Vertical Move*) A player chooses the second heap and takes $\ell \in (\{z | 0 < z \leq y\} \cap S_v)$ tokens, i.e.,

$$(x, y) \to (x, y - \ell) \text{ and } \ell \in (\{z | 0 < z \leq y\} \cap S_v). \tag{4}$$

In this case, we call that (x, y) is moved to $(x, y - \ell)$ in the *vertical direction*, and ℓ is the *vertical distance*.

(*Extended Diagonal Move*) A player takes tokens from both heaps, $k \in (\{w | 0 < w \leq x\} \cap D_1)$ from the first heap and $\ell \in (\{z | 0 < z \leq y\} \cap D_2)$ from the second heap, and

$$0 \leq |\ell - k| < (s - 1)\lambda + t, \quad \lambda = \min\{k, \ell\} \in Z^{\geq 1}. \tag{5}$$

In this case, we call that (x, y) is moved to $(x - k, y - \ell)$ in the *extended diagonal direction*, and k, ℓ are the *extended diagonal distances*.

Remark 1. We note that Eq. (1) is equivalent to

$$0 \le \ell - k \ < (s-1)k + t, \quad k \in Z^{\ge 1}, \tag{6}$$

and Eq. (6) is equivalent to Eq. (5).

Remark 2. (s,t)-Wythoff's game introduced by A.S. Fraenkel in [11] is equivalent to $S_h = S_v = D_1 = D_2 = Z^{\ge 0}$ in General Restriction of (s,t)-Wythoff's Game. a-Wythoff's game investigated in [9] is equivalent to $S_h = S_v = D_1 = D_2 = Z^{\ge 0}$, $s = 1$ and $t = a$ in General Restriction of (s,t)-Wythoff's Game. Wythoff's game investigated in [19] is equivalent to $S_h = S_v = D_1 = D_2 = Z^{\ge 0}$, $s = 1$ and $t = 1$ in General Restriction of (s,t)-Wythoff's Game.

Remark 3. If $s = t = 1$ then Eq. (5) is equivalent to $0 < k = \ell$, the Extended Diagonal Move is reduced to a *Diagonal Move*, i.e., $(x,y) \to (x-m, y-m)$ with $0 < m \le min\{x,y\}$.

In [5], the authors investigate the case of $s = t = 1$ (Wythoff's game), where the horizontal, the vertical and the diagonal distances are bounded by a given positive integer R. This problem is equivalent to $S_h = S_v = D_1 = D_2 = \{n \le R | n \in Z^{\ge 0}\}$ and $s = t = 1$ in General Restriction of (s,t)-Wythoff's Game. The set of all P-positions of this game under normal play convention is determined in [5].

Remark 4. In [5], the authors present the following problems:

(1) One can investigate the case of Wythoff's game, where only the diagonal distance is bounded. This problem is equivalent to $S_h = S_v = Z^{\ge 0}$, $D_1 = D_2 = \{n \le R | n \in Z^{\ge 0}\}$ (R is a fixed positive integer) and $s = t = 1$ in General Restriction of (s,t)-Wythoff's Game.

(2) One can investigate the case of Wythoff's game, where the horizontal and the vertical distances are bounded, but the diagonal distance is infinite. This problem is equivalent to $S_h = S_v = \{n \le R | n \in Z^{\ge 0}\}$ (R is a fixed positive integer), $D_1 = D_2 = Z^{\ge 0}$ and $s = t = 1$ in General Restriction of (s,t)-Wythoff's Game. Under normal play convention, the set of all P-positions of this game is given in [16].

(3) One can investigate the bounded version of a-Wythoff's game($s = 1$ and $t = a$). If the horizontal and the vertical distances are bounded, but the extended diagonal distances are infinite. This problem is equivalent to $S_h = S_v = \{n \le R | n \in Z^{\ge 0}\}$ (R is a fixed positive integer), $D_1 = D_2 = Z^{\ge 0}$ and $s = 1$, $t = a$ in General Restriction of (s,t)-Wythoff's Game; If the horizontal, the vertical and the extended diagonal distances are bounded, then this problem is equivalent to $S_h = S_v = D_1 = D_2 = \{n \le R | n \in Z^{\ge 0}\}$ (R is a fixed positive integer), $s = 1$ and $t = a$ in General Restriction of (s,t)-Wythoff's Game.

1.4 Our results

For all extensions and restrictions of Wythoff's game, the main goal is to find characterizations of the sequence of P-positions, which almost always differs from the original Wythoff's sequence (see [7, 16]). In this paper, we investigate four models of General Restriction of (s,t)-Wythoff's Game. Let us now briefly present the content of this paper.

In Section 3, we define the first model, *Odd-Odd* (s,t)*-Wythoff's Game*, which is equivalent to $S_h = S_v = D_1 = D_2 = Z^{odd}$ in General Restriction of (s,t)-Wythoff's Game. Under normal play convention and for any $s,t \in Z^{\geq 1}$, the set of all P-positions is given by $\bigcup_{m=0}^{\infty}\bigcup_{n=0}^{\infty}\{(2m,2n)\}$; Under misère play convention and for any $s,t \in Z^{\geq 1}$, the set of all P-positions is given by

$$\{(0,2p+1),(2p+1,0)|p \in Z^{\geq 0}\} \cup \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\{(2m,2n)\}.$$

The structures of P-positions are of algebraic form (Theorems 7 and 9), which permits to decide in polynomial time whether or not a given game position (a,b) is a P-position.

In Section 4, we define the second model, *Even-Even* (s,t)*-Wythoff's Game*, which is equivalent to $S_h = S_v = D_1 = D_2 = Z^{even}$ in General Restriction of (s,t)-Wythoff's Game. Under normal or misère play convention, and for two integer parameters $s,t \geq 1$, the sets of all P-positions are given by recursive characterizations in term of mex function (Theorems 13 and 14), which are just as Eq. (2).

In Section 5, we define the third model, *Odd-Even* (s,t)*-Wythoff's Game*, which is equivalent to ($S_h = D_1 = Z^{odd}$ and $S_v = D_2 = Z^{even}$) in General Restriction of (s,t)-Wythoff's Game. Under normal or misère play convention, and for any $s,t \in Z^{\geq 1}$, the sets of all P-positions are given by algebraic characterizations (Theorems 17 and 20), which provide polynomial time procedures.

In Section 6, we define the fourth model, *Even-Odd* (s,t)*-Wythoff's Game*, which is equivalent to ($S_h = D_1 = Z^{even}$ and $S_v = D_2 = Z^{odd}$) in General Restriction of (s,t)-Wythoff's Game. Under normal or misère play convention, and for any $s,t \in Z^{\geq 1}$, the sets of all P-positions are given by explicit formulas (Corollaries 21 and 24), which provide polynomial time procedures.

2 Preliminaries

Given any game Γ, we say informally that a P-position is any position u of Γ from which the *Previous* player can force a win, that is, the opponent of the player moving from u. An N-position is any position v of Γ from which the *Next* player can force a win, that is, the player who moves from v. The set of all P-positions of Γ is denoted by $\mathscr{P}$, and the set of all N-positions of Γ is denoted by $\mathscr{N}$. Denote by *Option*(u) all the options of u, i.e., the set of all positions that can be reached in one move from the position u. It follows from Fraenkel [8] that

$$\begin{array}{l} u \in \mathscr{P} \Longleftrightarrow Option(u) \subseteq \mathscr{N}, \\ u \in \mathscr{N} \Longleftrightarrow Option(u) \cap \mathscr{P} \neq \emptyset. \end{array} \tag{7}$$

In order to better understand the legal moves of General Restriction of (s,t)-Wythoff's Game, we define the following notations:

$Option_h(x,y) = \{(x-k,y) | 0 < k \leq x\}$ to be the set of all positions that can be reached in one move in the horizontal direction from the position (x,y);

$Option_v(x,y) = \{(x,y-\ell) | 0 < \ell \leq y\}$ to be the set of all positions that can be reached in one move in the vertical direction from the position (x,y);

$Option_d(x,y) = \{(x-m,y-m) | 0 < m \leq \min\{x,y\}\}$ to be the set of all positions that can be reached in one move in the diagonal direction from the position (x,y);

$$Option_e(x,y) = \left\{ (x-k,y-\ell) \,\middle|\, \begin{array}{l} 0 < k \leq x, 0 < \ell \leq y, 0 \leq |\ell - k| < (s-1)\lambda + t, \\ \lambda = \min\{k,\ell\} \in Z^{\geq 1} \end{array} \right\}$$

to be the set of all positions that can be reached in one move in the extended diagonal direction from the position (x,y).

It is obvious that for any position (x,y),

(I) $Option(x,y) = Option_h(x,y) \cup Option_v(x,y) \cup Option_e(x,y)$;

(II) $Option_h(x,y)$, $Option_v(x,y)$ and $Option_e(x,y)$ are pairwise disjoint;

(III) $Option_d(x,y) \subseteq Option_e(x,y)$.

Example 5. Eq. (7) can be used to check whether or not a given game position (a,b) is a P-position. We consider General Restriction of (s,t)-Wythoff's Game under normal play convention and $s = t = 2$, where $S_h = S_v = D_1 = D_2 = Z^{\geq 0}$. Then $u = (1,4)$ is a P-position.

Proof. By $\mathscr{P}$ and $\mathscr{N}$ we denote the sets of all P-positions and all N-positions, respectively. It is obvious that $(0,0)$ is a P-position, i.e., $(0,0) \in \mathscr{P}$.

(1) The positions $(0,1),(0,2),(0,3),(0,4)$ are N-positions. In fact, fix $m \in \{1,2,3,4\}$ and let $w = (0,m)$, one can move $(0,m) \to (0,0)$ by taking m tokens in the vertical direction. Thus $(0,0) \in Option_v(w)$, i.e., $Option_v(w) \cap \mathscr{P} \neq \emptyset$. By Eq. (7), $(0,m)$ is a N-position.

(2) The position $(1,0)$ is an N-position. For $w = (1,0)$, one can move $(1,0) \to (0,0)$ by taking 1 tokens in the horizontal direction. Thus $(0,0) \in Option_h(w)$, i.e., $Option_h(w) \cap \mathscr{P} \neq \emptyset$. By Eq. (7), $(1,0)$ is an N-position.

(3) The positions $(1,1),(1,2),(1,3)$ are N-positions. In fact, fix $m \in \{1,2,3\}$ and let $w = (1,m)$. For $w = (1,m)$, one can move $(1,m) \to (0,0)$ by taking $k = 1$ token from the first heap and $\ell = m$ token from the second heap. Note that Eq. (5) is true:

$$|\ell - k| = m - 1 < 1 + 2 = (s-1)\lambda + t, \lambda = k = 1.$$

(4) $Option_e(1,4) = \{(0,1),(0,2),(0,3)\}$. For $w = (1,4)$, one can move $(1,4) \to (0,m)$ with $1 \leq m \leq 3$ by taking $k = 1$ token from the first heap and $\ell = m$ tokens from the second heap, and $|\ell - k| = m - 1 < 1 + 2 = (s-1)\lambda + t$.

(5) It is obvious that $Option_h(1,4) = \{(0,4)\}$, $Option_v(1,4) = \{(1,m) | 0 \leq m \leq 3\}$. Thus

$$\begin{aligned} Option(1,4) &= Option_h(1,4) \cup Option_v(1,4) \cup Option_e(1,4) \\ &= \{(0,4)\} \cup \{(1,m) | 0 \leq m \leq 3\} \cup \{(0,1),(0,2),(0,3)\}. \end{aligned}$$

It follows form (1), (2) and (3) that $Option(1,4) \subseteq \mathscr{N}$. By Eq. (7), the position $(1,4)$ is a P-position. □

Proposition 6. *(Characterization of the P-positions of an impartial acyclic game). The sets of P- and N-positions of any impartial acyclic game (like Wythoff's game) are uniquely determined by the following two properties:*

- *Any move from a P-position leads to an N-position (stability property of the P-positions).*
- *From any N-position, there exists a move leading to a P-position (absorbing property of the P-positions).*

Proof. See Proposition 1 in [7]. □

3 Odd-Odd (s,t)-Wythoff's Game

In this section, we introduce a new *Odd-Odd (s,t)-Wythoff's Game* (Denoted by OOW): Let S_h, S_v, D_1 and D_2 be subsets of $Z^{\geq 0}$. Given two parameters $s,t \in Z^{\geq 1}$ and two heaps of finitely many tokens. One of the heaps is designated as "first heap" and the other as "second heap" throughout the game. By (x,y) we denote a position of present game, where x and y denote the numbers of tokens in the first and the second heaps, respectively. Two rules of moves are allowed:

(*Odd-Odd Nim Rule*) A player chooses one heap and takes an arbitrary *odd* number k of tokens.

(*Odd-Odd More General Wythoff's Rule*) A player takes tokens from both heaps, *odd* $k > 0$ tokens from the first heap, *odd* $\ell > 0$ tokens from the second heap and

$$0 \leq |\ell - k| \ < (s-1)\lambda + t, \quad \lambda = \min\{k, \ell\} \in Z^{\geq 1}. \tag{8}$$

Obviously, OOW is equivalent to $S_h = S_v = D_1 = D_2 = Z^{odd}$ in General Restriction of (s,t)-Wythoff's Game.

By the definition of OOW, the positions (x,y) and (y,x) are equivalent, i.e., both (x,y) and (y,x) are P-positions, or are N-positions. Theorems 7 and and 9 will give the sets of all P-positions of OOW under normal or misère play convention, respectively. The corresponding winning strategies are also presented.

We define a function δ_n for $n \in Z^{\geq 0}$:

$$\delta_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 7. *By $\mathscr{P}_1$ we denote the set of all P-positions of OOW. Then for all $s,t \in Z^{\geq 1}$,*

$$\mathscr{P}_1 = \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} \{(2m, 2n)\}.$$

Proof. Let $\mathcal{M}_1 = \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} \{(2m, 2n)\}$. It suffices to show two things:

Fact A. No options of a position in $\mathcal{M}_1$ can be in $\mathcal{M}_1$.

Fact B. Any position not in $\mathcal{M}_1$ can land in a position in $\mathcal{M}_1$.

Proof of Fact A. Let $(x, y) \in \mathcal{M}_1$ be a position. Suppose that $(x, y) \to (x', y') \in \mathcal{M}_1$. By the definition of $\mathcal{M}_1$, x, y, x' and y' are even. Thus both $x - x'$ and $y - y'$ are even. This contradicts the rules of moves of OOW.

Proof of Fact B. Let $(x, y) \notin \mathcal{M}_1$ be a position. In this case, at least one of x and y is odd, i.e., $(\delta_x, \delta_y) = (0, 1)$ or $(1, 0)$ or $(1, 1)$. Thus we can move $(x, y) \to (x - \delta_x, y - \delta_y) \in \mathcal{M}_1$ by taking one token from an odd-size heap.

The proof is completed. □

Remark 8. Given a game Γ. Let $\mathcal{M}$ be the set of all P-positions of game Γ. The following facts are true:

Fact 1. No options of a position in $\mathcal{M}$ can be in $\mathcal{M}$.

Fact 2. Any position not in $\mathcal{M}$ can land in a position in $\mathcal{M}$ by a legal move.

We will determine the sets of all P-positions of the games investigated in this paper, respectively. In all these proofs, the validity of *Fact 1* and *Fact 2* will be proved. The method of the proofs is the same, though the proofs themselves vary greatly.

Theorem 9. *By $\mathscr{P}_2$ we denote the set of all P-positions of OOW under misère play convention. Then for all $s, t \in Z^{\geq 1}$,*

$$\mathscr{P}_2 = \{(0, 2p+1), (2p+1, 0) | p \in Z^{\geq 0}\} \cup \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{(2m, 2n)\}.$$

Proof. Let

$$\begin{aligned}
\mathcal{M}_2 &= \{(0, 2p+1), (2p+1, 0) | p \in Z^{\geq 0}\} \cup \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{(2m, 2n)\}, \\
\mathcal{M}_2' &= \{(0, 2p+1), (2p+1, 0) | p \in Z^{\geq 0}\}, \\
\mathcal{M}_2'' &= \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{(2m, 2n)\}.
\end{aligned}$$

Proof of Fact 1. Let (x, y) be a position in $\mathcal{M}_2$. For $(0, 2p+1) \in \mathcal{M}_2'$, $(0, 2p+1) \to (x', y') \in \mathcal{M}_2''$ is impossible since $x' > 0$; $(0, 2p+1) \to (0, 2q+1) (q < p)$ is also impossible, since $2(p - q)$ is even.

For $(2m, 2n) \in \mathcal{M}_2''$, $(2m, 2n) \to (0, 2p+1)$ (or $(2p+1, 0)$) is impossible, since $2m$ (or $2n$) is even; $(2m, 2n) \to (2m', 2n') \in \mathcal{M}_2''$ is also impossible, since $2(m - m')$ is even, which contradicts the rules of moves of OOW.

Proof of Fact 2. Let (x, y) with $x \leq y$ be a position not in $\mathcal{M}_2$.

If $(x, y) = (0, 2v)$ for some $v \in Z^{\geq 1}$, we move $(x, y) = (0, 2v) \to (0, 2v - 1)$ by taking one token from the heap of size $2v$. If $(x, y) = (0, 0)$ then next player wins without doing any thing.

If $(x, y) = (2m, 2n + 1)$ for some $m, n \in Z^{\geq 1}$ and $n \geq m$, we move $(2m, 2n + 1) \to (2m, 2n)$, by taking one token from the heap of size $2n + 1$.

If $(x, y) = (2u + 1, w)$ for some $u \in Z^{\geq 0}$ and $w \geq 2u + 1$. We need to consider two subcases:

(i) $u = 0$. In this case, $w \geq 1$. If w is odd, we move $(2u+1, w) = (1, w) \to (0, w) \in \mathcal{M}'_2$; If w is even, we move $(2u + 1, w) = (1, w) \to (0, w - 1) \in \mathcal{M}'_2$, by taking one token from each heap.

(ii) $u > 0$. In this case, $w \geq 2u + 1 \geq 3$. If w is odd, we move $(2u + 1, w) \to (2u, w - 1) \in \mathcal{M}''_2$. If w is even, thus we move $(2u + 1, w) \to (2u, w) \in \mathcal{M}''_2$.

The proof is completed. □

4 Even-Even (s, t)-Wythoff's Game

In this section, we introduce a new *Even-Even* (s, t)*-Wythoff's Game* (Denoted by EEW): Let S_h, S_v, D_1 and D_2 be subsets of $Z^{\geq 0}$. Given two parameters $s, t \in Z^{\geq 1}$ and two heaps of finitely many tokens. One of the heaps is designated as "first heap" and the other as "second heap" throughout the game. By (x, y) we denote a position of present game, where x and y denote the numbers of tokens in the first and the second heaps, respectively. Two rules of moves are allowed:

(*Even-Even Nim Rule*) A player chooses one heap and takes an arbitrary *even* number $k > 0$ of tokens.

(*Even-Even More General Wythoff's Rule*) A player takes tokens from both heaps, *even* $k > 0$ tokens from the first heap, *even* $\ell > 0$ tokens from the second heap and

$$0 \leq |\ell - k| \ < (s - 1)\lambda + t, \quad \lambda = \min\{k, \ell\} \in Z^{\geq 1}. \tag{9}$$

Obviously, EEW is equivalent to $S_h = S_v = D_1 = D_2 = Z^{even}$ in General Restriction of (s, t)-Wythoff's Game.

Note 10. The "symmetric" notation $\{x, y\}$ for unordered pairs of non-negative integers is used whenever the positions (x, y) and (y, x) are equivalent, i.e., both (x, y) and (y, x) are P-positions, or are N-positions.

Example 11. We consider EEW under normal play convention. Fix two integers $s = t = 1$. It is obvious that $(0, 1)$ is a P-position, $(1, 0)$ is also a P-position. Thus we use $\{0, 1\}$ to denote the two positions $(0, 1)$ and $(1, 0)$, i.e., $\{0, 1\} = \{(0, 1), (1, 0)\}$. Generally, by the definition of EEW, two positions (x, y) and (y, x) are equivalent, i.e., both (x, y) and (y, x) are P-positions, or are N-positions. Thus we use $\{x, y\}$ to denote the two positions (x, y) and (y, x), i.e., $\{x, y\} = \{(x, y), (y, x)\}$.

Theorems 13 and 14 will give the sets of all P-positions of EEW under normal or misère play convention, respectively. The corresponding winning strategies are also presented. Before the main results, we define two sequences and give some properties in Lemma 12.

We define two sequences A_n and B_n for $n \in Z^{\geq 0}$ and all $s, t \in Z^{\geq 1}$:

$$\begin{cases} A_n = \text{mex}\{A_i, A_i + 1, B_i, B_i + 1 | 0 \leq i < n\}, \\ B_n = sA_n + (t + \delta_t)n. \end{cases} \tag{10}$$

Tables 1 and 2 list the first few values of A_n and B_n for $s = t = 1$ and $s = t = 2$, respectively.

Table 1. The first few values of A_n and B_n for $s = t = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	2	6	8	12	16	18	22	24	28	32	34	38	42	44
B_n	0	4	10	14	20	26	30	36	40	46	52	56	62	68	72

Table 2. The first few values of A_n and B_n for $s = t = 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	2	4	8	10	14	16	18	20	24	26	30	32	34	36
B_n	0	6	12	22	28	38	44	50	56	66	72	82	88	94	100

Lemma 12. *Let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ be defined by Eq. (10). We have the following properties:*

(I) Both A_n and B_n are even for $n \geq 0$.

(II) Both A_n and B_n are strictly increasing sequences for $n \geq 0$.

(III) $B_n > A_n + 1 > A_n$ for $n \geq 1$.

(IV) Let

$$\begin{cases} A = \bigcup_{n=1}^{\infty} \{A_n\} \cup \bigcup_{n=1}^{\infty} \{A_n + 1\}, \\ B = \bigcup_{n=1}^{\infty} \{B_n\} \cup \bigcup_{n=1}^{\infty} \{B_n + 1\}. \end{cases} \tag{11}$$

Then A, B are complementary with respect to $Z^{\geq 2}$, i.e., $A \cup B = Z^{\geq 2}$ and $A \cap B = \emptyset$.

Proof. (I) Note that $t + \delta_t$ is even for $t \in Z^{\geq 1}$. We proceed by induction on n. Obviously, $A_0 = B_0 = 0$, $A_1 = 2$ and $B_1 = sA_1 + (t + \delta_t)$ are even. Suppose $m < n$, both A_m and B_m are even. We now show that A_n is even, and then $B_n = sA_n + (t + \delta_t)n$ is even.

Indeed, suppose that A_n is odd. Let $k = A_n$ and $S = \{A_i, A_i + 1, B_i, B_i + 1 | 0 \leq i < n\}$. By Eq. (10), $k = \text{mex}(S)$ implies that $k \notin S$ and $k - 1 \in S$.

We note that A_i and B_i are even for all $i \in \{0, 1, \cdots, n-1\}$. Since $k - 1 \in S$ and $k - 1$ is even, $k - 1 \neq A_i + 1$ or $B_i + 1$. If there exists an integer $i_0 \in \{0, 1, \cdots, n-1\}$ such that $k - 1 = A_{i_0}$ or B_{i_0}, then $k \in S$. This contradicts $k \notin S$.

(II) By the definition of A_n and *mex* property, A_n is strictly increasing sequence. B_n is also strictly increasing sequence. Indeed, for $m > n$,

$$B_m - B_n = s(A_m - A_n) + (t + \delta_t)(m - n) > 0.$$

(III) Note that $t+\delta_t \geq 2$ for $t \in Z^{\geq 1}$. By Eq. (10), we have $B_n = sA_n + (t+\delta_t)n \geq A_n + 2n > A_n + 1 > A_n$, for $n \in Z^{\geq 1}$.

(IV) In fact, $A \cup B = Z^{\geq 2}$ follows from the *mex* property and $A_0 = 0$, $B_0 = 0$, $A_1 = 2$. Suppose $A \cap B \neq \emptyset$. It follows (I) that $A_m + 1 \neq B_n$ and $A_m \neq B_n + 1$, thus the only possibility is $A_m = B_n$ for two integers $m, n \in Z^{\geq 1}$. If $m > n$, then A_m is *mex* of a set containing $B_n = A_m$, a contradiction. If $m \leq n$, then by (II) we have $B_n = sA_n + (t+\delta_t)n \geq sA_m + (t+\delta_t)m > A_m$, another contradiction.

The proof is completed. □

Theorem 13. *By $\mathscr{P}_3$ we denote the set of all P-positions of EEW under normal play convention. Then for all $s, t \in Z^{\geq 1}$,*

$$\mathscr{P}_3 = \bigcup_{i=0}^{\infty} \left\{ \begin{array}{l} \{A_i, B_i\}, \{A_i, B_i + 1\}, \\ \{A_i + 1, B_i\}, \{A_i + 1, B_i + 1\} \end{array} \right\},$$

where A_n and B_n are defined by Eq. (10).

Proof. Before we give the proof of Theorem 13, Tables 3 and 4 list the first few values of A_n and B_n, which show us how to determine the set $\mathscr{P}_3$ of all P-positions by using Theorem 13:

Table 3. The first few values of A_n and B_n for $s = t = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	2	6	8	12	16	18	22	24	28	32	34	38	42	44
A_n+1	1	3	7	9	13	17	19	23	25	29	33	35	39	43	45
B_n	0	4	10	14	20	26	30	36	40	46	52	56	62	68	72
B_n+1	1	5	11	15	21	27	31	37	41	47	53	57	63	69	73

For $s = t = 1$, it follows Table 3 that

$$\mathscr{P}_3 = \left\{ \begin{array}{l} (0,0), (0,1), (1,0), (1,1); \\ (2,4), (2,5), (3,4), (3,5); (4,2), (5,2), (4,3), (5,3); \\ (6,10), (6,11), (7,10), (7,11); (10,6), (11,6), (10,7), (11,7); \cdots \end{array} \right\}$$

Table 4. The first few values of A_n and B_n for $s = t = 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	2	4	8	10	14	16	18	20	24	26	30	32	34	36
A_n+1	1	3	5	9	11	15	17	19	21	25	27	31	33	35	37
B_n	0	6	12	22	28	38	44	50	56	66	72	82	88	94	100
B_n+1	1	7	13	23	29	39	45	51	57	67	73	83	89	95	101

For $s = t = 2$, it follows Table 4 that

$$\mathscr{P}_3 = \left\{ \begin{array}{l} (0,0), (0,1), (1,0), (1,1); \\ (2,6), (2,7), (3,6), (3,7); (6,2), (7,2), (6,3), (7,3); \\ (4,12), (4,13), (5,12), (5,13); (12,4), (13,4), (12,5), (13,5); \cdots \end{array} \right\}$$

We now give the proof of Theorem 13. Let

$$\mathcal{M}_3 = \bigcup_{i=0}^{\infty} \left\{ \begin{array}{l} \{A_i, B_i\}, \{A_i, B_i+1\}, \\ \{A_i+1, B_i\}, \{A_i+1, B_i+1\} \end{array} \right\}.$$

Proof of Fact 1. Let (x, y) with $x \le y$ be a position in $\mathcal{M}_3$. It follows (III) of Lemma 12 that there exists an integer n such that $x = A_n$ or $A_n + 1$, and $y = B_n$ or $B_n + 1$.

Suppose that $(x, y) \to (x', y) \in \mathcal{M}_3$ with $x' = A_m$ or $A_m + 1$, by Even-Even Nim Rule. Then $x' \le x - 2 < A_n$ implies that $m < n$. Thus $(x', y) \notin \mathcal{M}_3$, a contradiction. Suppose that $(x, y) \to (x, y') \in \mathcal{M}_3$ with $y' = B_m$ or $B_m + 1$, by Even-Even Nim Rule. Then $y' \le y - 2 < B_n$ implies that $m < n$. Thus $(x, y') \notin \mathcal{M}_3$, another contradiction.

Suppose that $(x, y) \to (x', y') \in \mathcal{M}_3$ by Even-Even More General Wythoff's Rule. Then $x - x' > 0$ is even and $y - y' > 0$ is even. It follows (I) and (III) of Lemma 12 that $k = x - x' = A_n - A_m$, $\ell = y - y' = B_n - B_m$ and $m < n$. Thus

$$0 < k \le \ell = s(A_n - A_m) + (t + \delta_t)(m - n) \ge sk + t,$$

which contradicts Eq. (9).

Proof of Fact 2. Let (x, y) be a position not in $\mathcal{M}_3$, without loss of generality, assume that $x \le y$.

If $x = 0$ or 1, we move $(x, y) \to (x, \delta_y) \in \mathcal{M}_3$. This move is legal, since $y - \delta_y$ is even and $(x, y) \notin \mathcal{M}_3$ imply that $y > 1 \ge \delta_y$.

If $x \ge 2$ then the integer x appears exactly once in exactly one of A and B, since A and B are complementary with respect to $Z^{\ge 2}$ (Lemma 12, (IV)). Therefore, we have one of the following two cases: (i) $x = B_n$ or $x = B_n + 1$, (ii) $x = A_n$ or $x = A_n + 1$, for some $n \ge 1$.

Cases (i): $x = B_n$ or $x = B_n + 1$, $n \ge 1$. We move

$$(x, y) \to (x, A_n + \delta_y) \in \mathcal{M}_3,$$

i.e., we take $y - A_n - \delta_y$ tokens from the heap of y tokens. It follows (I) and (III) of Lemma 12 that

$$y \ge x \ge B_n > A_n + 1 \ge A_n + \delta_y$$

and $y - A_n - \delta_y$ is even, thus the above move is legal.

Cases (ii): $x = A_n$ or $x = A_n + 1$, $n \ge 1$. In this cases, we have $y > B_n + 1$ or $x \le y < B_n$.

(1) $y > B_n + 1$. We move

$$(x, y) \to (x, B_n + \delta_y) \in \mathcal{M}_3,$$

i.e., we take $y - B_n - \delta_y$ tokens from the heap of y tokens. This ia a legal move, since $y > B_n + 1 \ge B_n + \delta_y$ and $y - B_n - \delta_y$ is even.

(2) $x \le y < B_n$. We distinguish the following two subcases: $x \le y < sA_n + t + \delta_t$ or $sA_n + t + \delta_t \le y < B_n$:

(2.1) $x \le y < sA_n + t + \delta_t$. We move

$$(x, y) \to (x - A_n, \delta_y) \in \mathcal{M}_3,$$

since $x - A_n = 0$ or 1, and $\delta_y = 0$ or 1. This move is legal:

1) $k = A_n$ is even;

2) $\ell = y - \delta_y$ is even;

3) $y \ge x$ implies that $\ell = y - \delta_y \ge A_n = k$. Note that $y < sA_n + t + \delta_t$ and $sA_n + t + \delta_t$ is even, so $y \le sA_n + t + \delta_t - 2 + \delta_y$. Hence,

$$\begin{array}{rcl} |\ell - k| & = & y - \delta_y - A_n \\ & \le & (s-1)A_n + t + \delta_t - 2 \\ & < & (s-1)A_n + t. \end{array}$$

(2.2) $sA_n + t + \delta_t \le y < B_n$. Put $m = \lfloor \frac{y - sA_n - \delta_y}{t + \delta_t} \rfloor$. We move

$$(x, y) \to (x - A_n + A_m, B_m + \delta_y) \in \mathcal{M}_3,$$

since $x - A_n = 0$ or 1, and $\delta_y = 0$ or 1. This move is legal:

(a) $k = A_n - A_m > 0$ and k is even. Firstly, we show that $0 \le m < n$. Note that $y - sA_n \ge t + \delta_t \ge 2 > \delta_y$, so $\frac{y - sA_n - \delta_y}{t + \delta_t} > 0$. Thus $m = \lfloor \frac{y - sA_n - \delta_y}{t + \delta_t} \rfloor \ge 0$; On the other hand, $y < B_n$ and B_n is even imply that $y - \delta_y < B_n$. Thus

$$y - sA_n - \delta_y < B_n - sA_n = (t + \delta_t)n,$$

and

$$m = \lfloor \frac{y - sA_n - \delta_y}{t + \delta_t} \rfloor \le \frac{y - sA_n - \delta_y}{t + \delta_t} < n.$$

It follows (I) and (II) of Lemma 12 that $k = A_n - A_m > 0$ and k is even.

(b) $\ell = y - B_m - \delta_y > 0$ and ℓ is even. By the definition of m, we have $m \le \frac{y - sA_n - \delta_y}{t + \delta_t}$, so

$$\begin{array}{rcl} y & \ge & (t + \delta_t)m + sA_n + \delta_y \\ & = & B_m + \delta_y + s(A_n - A_m) \\ & > & B_m + \delta_y. \end{array}$$

Thus $\ell = y - B_m - \delta_y > 0$ and ℓ is even.

(c) $|\ell - k| < (s-1)k + t$. By the definition of m, we have $m > \frac{y - sA_n - \delta_y}{t + \delta_t} - 1$, so $y < (t + \delta_t)(m + 1) + sA_n + \delta_y$. Thus

$$\begin{array}{rcl} \ell & = & y - B_m - \delta_y \\ & < & (t + \delta_t)(m + 1) + sA_n - sA_m - (t + \delta_t)m \\ & = & s(A_n - A_m) + t + \delta_t. \end{array}$$

We note that $y - B_m - \delta_y$ and $s(A_n - A_m) + t + \delta_t$ are even, so

$$\begin{array}{rcl} \ell & = & y - B_m - \delta_y \\ & \le & s(A_n - A_m) + t + \delta_t - 2 \\ & < & s(A_n - A_m) + t; \end{array}$$

On the other hand, $y - B_m - \delta_y \geq s(A_n - A_m) \geq A_n - A_m$ by virtue of (b). Therefore, $|\ell - k| < (s-1)k + t$.

The proof is completed. □

Theorem 14. *By $\mathscr{P}_4$ we denote the set of all P-positions of EEW under misère play convention. Then for all $s, t \in Z^{\geq 1}$,*

$$\mathscr{P}_4 = \bigcup_{i=0}^{\infty} \left\{ \begin{array}{l} \{E_i, H_i\}, \{E_i, H_i + 1\}, \\ \{E_i + 1, H_i\}, \{E_i + 1, H_i + 1\} \end{array} \right\},$$

where E_n and H_n are given by the following two cases:

(A) If $s \neq 1$ or $t > 2$ then for $n \geq 0$,

$$\begin{cases} E_n = mex\{E_i, E_i + 1, H_i, H_i + 1 | 0 \leq i < n\}, \\ H_n = sE_n + (t + \delta_t)n + 2. \end{cases} \tag{12}$$

(B) If $s = 1$ and $t \in \{1, 2\}$ then $E_0 = H_0 = 4$ and for $n \geq 1$,

$$\begin{cases} E_n = mex\{E_i, E_i + 1, H_i, H_i + 1 | 0 \leq i < n\}, \\ H_n = E_n + 2n. \end{cases} \tag{13}$$

Proof. Before we give the proof of Theorem 14, Tables 5 and 6 list the first few values of E_n and H_n, which show us how to determine the set $\mathscr{P}_4$ of all P-positions by using Theorem 14:

Table 5. The first few values of E_n and H_n for $s = t = 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_n	0	4	6	8	10	14	16	20	22	26	28	32	34	36	38
H_n	2	12	18	24	30	40	46	56	62	72	78	88	94	100	106

For $s = t = 2$, it follows Table 5 that

$$\mathscr{P}_4 = \left\{ \begin{array}{l} (0,2), (0,3), (1,2), (1,3); (2,0), (3,0), (2,1), (3,1); \\ (4,12), (4,13), (5,12), (5,13); (12,4), (13,4), (12,5), (13,5); \\ (6,18), (6,19), (7,18), (7,19); (18,6), (19,6), (18,7), (19,7); \cdots \end{array} \right\}$$

Table 6. The first few values of E_n and H_n for $s = t = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_n	4	0	6	8	12	16	18	22	24	28	32	34	38	42	44
H_n	4	2	10	14	20	26	30	36	40	46	52	56	62	68	72

For $s = t = 1$, it follows Table 6 that

$$\mathscr{P}_4 = \left\{ \begin{array}{l} (4,4), (4,5), (5,4), (5,5); \\ (0,2), (0,3), (1,2), (1,3); (2,0), (3,0), (2,1), (3,1); \\ (6,10), (6,11), (7,10), (7,11); (10,6), (11,6), (10,7), (11,7); \cdots \end{array} \right\}$$

We now give the proof of Theorem 14. Let

$$\mathcal{M}_4 = \bigcup_{i=0}^{\infty} \left\{ \begin{array}{l} \{E_i, H_i\}, \{E_i, H_i+1\}, \\ \{E_i+1, H_i\}, \{E_i+1, H_i+1\} \end{array} \right\},$$

and $E = \bigcup_{n=0}^{\infty}\{E_n\} \cup \bigcup_{n=0}^{\infty}\{E_n+1\}$, $H = \bigcup_{n=0}^{\infty}\{H_n\} \cup \bigcup_{n=0}^{\infty}\{H_n+1\}$. Then we claim that:

Fact A.1 If $s \neq 1$ or $t > 2$ then both E_n and H_n are even for $n \in Z^{\geq 0}$, and both E_n and H_n are strictly increasing sequences for $n \geq 0$. The proofs are similar to (I) and (II) of Lemma 12.

Fact A.2 If $s \neq 1$ or $t > 2$ then $E \cup H = Z^{\geq 0}$ and $E \cap H = \emptyset$. In fact, $E \cup H = Z^{\geq 0}$ follows from the definition of *mex*. Suppose $E \cap H \neq \emptyset$. It follows Fact A.1 that $E_m + 1 = H_n$ and $E_m = H_n + 1$ are impossible, thus there exist two integers $m, n \in Z^{\geq 0}$ such that $E_m = H_n$. If $m > n$ then $E_m = \text{mex}\{E_i, E_i+1, H_i, H_i+1 | 0 \leq i < m\}$, which contradicts $E_m = H_n$; If $m \leq n$ then

$$H_n = sE_n + (t+\delta_t)n + 2 \geq sE_m + (t+\delta_t)m + 2 > E_m,$$

which also contradicts $E_m = H_n$.

Fact B.1 If $s = 1$ and $t \in \{1, 2\}$ then both E_n and H_n are even for $n \in Z^{\geq 0}$, both E_n and H_n are strictly increasing sequences for $n \geq 1$. Indeed,

$$E_1 = 0 < 6 = E_2 < 8 = E_3 < 12 = E_4 < 16 = E_5 < \cdots$$

and

$$H_1 = 2 < 10 = H_2 < 14 = H_3 < 20 = E_4 < 26 = E_5 < \cdots.$$

Fact B.2 If $s = 1$ and $t \in \{1, 2\}$ then $E \cup H = Z^{\geq 0}$ and $E \cap H = \{4\}$. Its proof is similar to those of Fact A.2.

Proof of Fact 1. Let (x, y) with $x \leq y$ be a position in $\mathcal{M}_4$. By the definition of $\mathcal{M}_4$, there exists an integer $n \in Z^{\geq 0}$ such that $x = E_n$ or $E_n + 1$, and $y = H_n$ or $H_n + 1$.

Suppose that $(x, y) \to (x', y) \in \mathcal{M}_4$ with $x' = E_m$ or $E_m + 1$, by *Even-Even Nim Rule*. Then $x' \leq x - 2 < A_n$ implies that $m < n$. Thus $(x', y) \notin \mathcal{M}_4$, a contradiction; Similarly, $(x, y) \to (x, y') \in \mathcal{M}_4$ is also impossible.

Suppose that $(x, y) \to (x', y') \in \mathcal{M}_4$ by Even-Even More General Wythoff's Rule. Then $x - x' > 0$ is even and $y - y' > 0$ is even. It follows Fact A.1 and B.1 that $k = x - x' = E_n - E_m$, $\ell = y - y' = H_n - H_m$ and $m < n$. Thus

$$0 < k \leq \ell = s(E_n - E_m) + (t+\delta_t)(m-n) \geq sk + t,$$

which contradicts Eq. (9).

Proof of Fact 2. Let (x, y) be a position not in $\mathcal{M}_4$, without loss of generality, assume that $x \leq y$. By Fact A.2 and B.2, we have one of the following two cases: (i) $x = H_n$ or $x = H_n + 1$, (ii) $x = E_n$ or $x = E_n + 1$, for some $n \geq 0$.

Cases (i): $x = H_n$ or $x = H_n + 1$, $n \geq 0$. In this case, $(x, y) \notin \mathcal{M}_4$ implies that $y > E_n + 1$,

In fact, if $n = 0$, then $y \geq x \geq H_0 \geq E_0$, thus $y > E_0 + 1$; For $n \geq 1$, suppose that $y \leq E_n + 1$, it follows Eqs (12) and (13) that

$$x \geq H_n \geq E_n + 2 > E_n + 1 \geq y,$$

a contradiction.

We move

$$(x, y) \to (x, E_n + \delta_y) \in \mathcal{M}_4,$$

i.e., we take $y - E_n - \delta_y$ tokens from the heap of y tokens. Note that $y > E_n + 1 \geq E_n + \delta_y$ and $y - E_n - \delta_y$ is even, thus the above move is legal.

Cases (ii): $x = E_n$ or $x = E_n + 1$, $n \geq 0$. In this case, we have $y > H_n + 1$ or $x \leq y < H_n$.

(1) $y > H_n + 1$. In this case, we move

$$(x, y) \to (x, H_n + \delta_y) \in \mathcal{M}_4,$$

i.e., we take $y - H_n - \delta_y$ tokens from the heap of y tokens. This ia a legal move, since $y > H_n + \delta_y$ and $y - H_n - \delta_y$ is even.

(2) $x \leq y < H_n$. In this case, we need to consider the situations for (A) and for (B), respectively.

(2-A) $s \neq 1$ or $t > 2$. If $n = 0$ then $0 \leq x \leq y < 2$, we have $(x, y) = (0, 0)$ or $(0, 1)$ or $(1, 1)$. The next player wins without doing anything. It remains to consider the case of $n \geq 1$. We proceed by distinguishing the following two subcases: (2-A-1) $x \leq y < sE_n + t + \delta_t + 2$ and (2-A-2) $sE_n + t + \delta_t + 2 \leq y < H_n$.

(2-A-1) $x \leq y < sE_n + t + \delta_t + 2$. In this case, we move

$$(x, y) \to (x - E_n, 2 + \delta_y).$$

Note that $sE_n + t + \delta_t$ is even, so $y < sE_n + t + \delta_t + 1 + \delta_y$. This move is legal:

1) $k = E_n$ is even;

2) $\ell = y - 2 - \delta_y$ is even. Note that $y \geq x \geq E_n$ and E_n is even, thus $y - \delta_y \geq E_n$ and $\ell = y - 2 - \delta_y \geq E_n - 2 \geq E_1 - 2 > 0$;

3) It is easy to see that

$$\begin{aligned}
|\ell - k| &= |y - 2 - \delta_y - E_n| \\
&< (s - 1)E_n + t + \delta_t - 1 \\
&\leq (s - 1)E_n + t.
\end{aligned}$$

(2-A-2) $sE_n + t + \delta_t + 2 \leq y < H_n$. Put

$$m = \lfloor \frac{y - sE_n - 2 - \delta_y}{t + \delta_t} \rfloor. \tag{14}$$

We move

$$(x, y) \to (x - E_n + E_m, H_m + \delta_y) \in \mathcal{M}_4, \tag{15}$$

since $x - E_n = 0$ or 1, and $\delta_y = 0$ or 1. This move is legal:

(A-a) $k = E_n - E_m > 0$ and k is even. Firstly, we show that $0 \le m < n$. Note that

$$y - sE_n - 2 \ge t + \delta_t \ge 2 > \delta_y,$$

so $\frac{y - sE_n - 2 - \delta_y}{t + \delta_t} > 0$. Thus $m = \lfloor \frac{y - sE_n - 2 - \delta_y}{t + \delta_t} \rfloor \ge 0$; On the other hand, $y < H_n$ implies that $y - \delta_y < H_n$. Thus

$$y - sE_n - 2 - \delta_y < H_n - sE_n = (t + \delta_t)n,$$

and

$$m = \lfloor \frac{y - sE_n - 2 - \delta_y}{t + \delta_t} \rfloor \le \frac{y - sE_n - 2 - \delta_y}{t + \delta_t} < n.$$

By Facts A.1 and A.2, $k = E_n - E_m > 0$ and k is even.

(A-b) $\ell = y - H_m - \delta_y > 0$ and ℓ is even. By the definition of m, we have $m \le \frac{y - sE_n - 2 - \delta_y}{t + \delta_t}$, so

$$\begin{array}{rcl} y & \ge & (t + \delta_t)m + sE_n + 2 + \delta_y \\ & = & H_m + \delta_y + s(E_n - E_m) \\ & > & H_m + \delta_y. \end{array}$$

Thus $\ell = y - H_m - \delta_y > 0$ and ℓ is even.

(A-c) $0 \le \ell - k < (s - 1)k + t$. By the definition of m, we have

$$m > \frac{y - sE_n - 2 - \delta_y}{t + \delta_t} - 1,$$

i.e.,

$$y < (t + \delta_t)(m + 1) + sE_n + 2 + \delta_y.$$

By Eq. (12), we have $H_m = sE_m + (t + \delta_t)m + 2$. Thus

$$\begin{array}{rcl} \ell & = & y - H_m - \delta_y \\ & < & (t + \delta_t)(m + 1) + sE_n - sE_m - (t + \delta_t)m \\ & = & s(E_n - E_m) + t + \delta_t. \end{array}$$

We note that $y - H_m - \delta_y$ and $s(E_n - E_m) + t + \delta_t$ are even, so

$$y - H_m - \delta_y \le s(E_n - E_m) + t + \delta_t - 2 < s(E_n - E_m) + t;$$

On the other hand,

$$\ell \;\; = \;\; y - H_m - \delta_y \ge s(E_n - E_m) \ge E_n - E_m = k$$

by virtue of (A-b). Therefore, $0 \le \ell - k < (s - 1)k + t$.

(2-B) $s = 1$ and $t \in \{1, 2\}$. Note that $x \le y < H_n$. In this case $n = 0$ is impossible; if $n = 1$ then $0 = E_1 \le x \le y < H_1 = 2$, we have $(x, y) = (0, 0)$ or $(0, 1)$, or $(1, 1)$. The next player wins without doing anything. It remains to consider the case $n \ge 2$:

Put

$$m = \lfloor \frac{y - E_n - \delta_y}{2} \rfloor. \tag{16}$$

We move

$$(x, y) \to (x - E_n + E_m, H_m + \delta_y) \in \mathcal{M}_4, \tag{17}$$

since $x - E_n = 0$ or 1, and $\delta_y = 0$ or 1. This move is legal:

(B-a) $k = E_n - E_m > 0$ and k is even. Firstly, we show that $0 \leq m < n$. Note that if y is even, we have $y \geq x \geq E_n = E_n + \delta_y$; if y is odd, we have $y \geq E_n + 1 = E_n + \delta_y$, thus $y \geq E_n + \delta_y$, i.e., $m = \lfloor \frac{y-E_n-\delta_y}{2} \rfloor \geq 0$; On the other hand, $y < H_n$ implies that $y - \delta_y < H_n$. Thus

$$y - E_n - \delta_y < H_n - E_n = 2n,$$

and

$$m = \lfloor \frac{y - E_n - \delta_y}{2} \rfloor \leq \frac{y - E_n - \delta_y}{2} < n.$$

By Fact B.1, $k = E_n - E_m > 0$ and k is even.

(B-b) $\ell = y - H_m - \delta_y > 0$ and ℓ is even. By the definition of m, we have $m \leq \frac{y-E_n-\delta_y}{2}$. It follows Eq. (13) and $E_0 = H_0 = 4$ that

$$H_n = E_n + 2n \text{ for } n \geq 0. \tag{18}$$

Thus

$$\begin{aligned} y &\geq 2m + E_n + \delta_y \\ &= H_m + \delta_y + E_n - E_m \\ &> H_m + \delta_y. \end{aligned}$$

Thus $\ell = y - H_m - \delta_y > 0$ and ℓ is even.

(B-c) $0 \leq \ell - k < t$. By the definition of m, we have $m > \frac{y-E_n-\delta_y}{2} - 1$, i.e., $y < 2(m + 1) + E_n + \delta_y$. By Eq. (18), we have

$$\begin{aligned} \ell &= y - H_m - \delta_y \\ &< 2(m + 1) + E_n - E_m - 2m \\ &= E_n - E_m + 2. \end{aligned}$$

We note that $y - H_m - \delta_y$ and $E_n - E_m + 2$ are even, so $\ell = y - H_m - \delta_y \leq E_n - E_m < E_n - E_m + t$; On the other hand, $\ell = y - H_m - \delta_y \geq E_n - E_m = k$ by virtue of (B-b). Therefore, $0 \leq \ell - k < t$.

The proof is completed. □

5 Odd-Even (s, t)-Wythoff's Game

In present section, we introduce a new *Odd-Even (s, t)-Wythoff's Game* (denoted by OEW): Let S_h, S_v, D_1 and D_2 be subsets of $Z^{\geq 0}$. Given two parameters $s, t \in Z^{\geq 1}$ and two heaps of finitely many tokens. One of the heaps is designated as "first heap" and the other as "second heap" throughout the game. By (x, y) we denote a position of present game, where x and y denote the numbers of tokens in the first and the second heaps, respectively. Two rules of moves are allowed:

(*Odd-Even Nim Rule*) A player chooses the first heap and takes *odd* $k > 0$ tokens, or chooses the second heap and takes *even* $\ell > 0$ tokens.

(*Odd-Even More General Wythoff's Rule*) A player takes tokens from both heaps, *odd* $k > 0$ tokens from the first heap, *even* $\ell > 0$ tokens from the second heap, and

$$0 \leq |\ell - k| \ < (s-1)\lambda + t, \quad \lambda = \min\{k, \ell\} \in Z^{\geq 1}. \tag{19}$$

Obviously, OEW is equivalent to $S_h = D_1 = Z^{odd}$ and $S_v = D_2 = Z^{even}$ in General Restriction of (s,t)-Wythoff's Game. Therefore OEW is a restricted version of (s,t)-Wythoff's game.

Remark 15. OEW has no "symmetry", i.e., the positions (x, y) and (y, x) maybe not e-quivalent (see Notation 10). We consider OEW under normal play convention. Obviously, $(0, 0)$ is a P-position. The position $(0, 1)$ is a P-position, as the only possible move is taking 1 token from the second heap, but this is not a legal move. The position $(1, 0)$ is an N-position as one can move $(1, 0)$ to $(0, 0)$ by taking 1 tokens from the first heap. Thus the positions $(0, 1)$ and $(1, 0)$ are not equivalent. The position $(3, 8)$ is an option of position $(10, 8)$, as one can move $(10, 8)$ to $(3, 8)$ by taking 7 tokens from the first heap. But the position $(8, 3)$ is not an option of position $(8, 10)$, as the move of taking 7 token from the second heap is not legal.

Remark 16. In OEW, two parameters $s \geq 1$ and $t \geq 1$ are positive integers. If $s = t = 1$, then Eq. (19) can not hold:

$$|\ell - k| \geq 1 = (s-1)\lambda + t,$$

i.e., Odd-Even More General Wythoff's Rule is invalid. In other words, OEW is reduced to the *Odd-Even Nim Game*: A player chooses the first heap and takes *odd* $k > 0$ tokens, or chooses the second heap and takes *even* $\ell > 0$ tokens.

For the cases ($s = 1$ and $t > 1$) and ($s > 1$ and $t \geq 1$), Odd-Even More General Wythoff's Rule is valid. Therefore, we will give the results on the cases $s = t = 1$ and $s+t > 2$, respectively. All its P-positions of OEW under normal or misère play convention are given, and the corresponding winning strategies are also presented.

5.1 All P-positions of OEW: $s = t = 1$

Theorem 17. *Given two parameters $s = t = 1$. By $\mathscr{P}_5$ we denote the set of all P-positions of OEW under normal play convention. Then*

$$\mathscr{P}_5 = \bigcup_{n=0}^{\infty} \{(2n, 0), (2n, 1), (2n+1, 2), (2n+1, 3)\}.$$

Proof. Let
$$\mathscr{W} = \bigcup_{n=0}^{\infty}\{(2n,0),(2n,1),(2n+1,2),(2n+1,3)\}.$$

Proof of Fact 1. Let (a,b) and (a',b') are two distinct positions of $\mathscr{W}$. It is easy to see that there exists no legal move such that $(a,b) \to (a',b')$ or $(a',b') \to (a,b)$.

Proof of Fact 2. Let (a,b) be a position not in $\mathscr{W}$. It suffices to show that there exists a legal move such that $(a,b) \to (a',b') \in \mathscr{W}$.

(2.1) $a = 2n$ for some integer $n \in Z^{\geq 0}$. In this case, $(a,b) \notin \mathscr{W}$ implies that $b \geq 2$. We move
$$(a,b) \to (2n,\delta_b) \in \mathscr{W}.$$

(2.2) $a = 2n+1$ for some integer $n \in Z^{\geq 0}$. In this case, $(a,b) \notin \mathscr{W}$ implies that $b = 0$, $b = 1$ or $b \geq 4$:

- $b \in \{0,1\}$. We move $(a,b) = (2n+1,b) \to (2n,b) \in \mathscr{W}$.
- $b \geq 4$. We move $(a,b) = (2n+1,b) \to (2n+1,2+\delta_b) \in \mathscr{W}$.

The proof is completed. □

Theorem 18. *Given two parameters $s = t = 1$. By $\mathscr{P}_5'$ we denote the set of all P-positions of OEW under misère play convention. Then*
$$\mathscr{P}_5' = \bigcup_{n=0}^{\infty}\{(2n,2),(2n,3),(2n+1,0),(2n+1,1)\}.$$

Proof. Let
$$\mathscr{M} = \bigcup_{n=0}^{\infty}\{(2n,2),(2n,3),(2n+1,0),(2n+1,1)\}.$$

P*roof of Fact 1.* Let (a,b) and (a',b') are two distinct positions of $\mathscr{M}$. It is easy to see that there exists no legal move such that $(a,b) \to (a',b')$ or $(a',b') \to (a,b)$.

Proof of Fact 2. Let (a,b) be a position not in $\mathscr{M}$. It suffices to show that there exists a legal move such that $(a,b) \to (a',b') \in \mathscr{M}$.

(2.1) $a = 2n+1$ for some integer $n \in Z^{\geq 0}$. In this case, $(a,b) \notin \mathscr{M}$ implies that $b \geq 2$. We move
$$(a,b) \to (2n+1,\delta_b) \in \mathscr{M}.$$

(2.2) $a = 2n$ for some integer $n \in Z^{\geq 0}$. We distinguish the following two subcases: $n = 0$ or $n \geq 1$.

(2.2.1) $n = 0$. In this subcase, $(a,b) \notin \mathscr{M}$ implies that $b = 0$ or $b = 1$ or $b \geq 4$. It is obvious that $(0,0)$ and $(0,1)$ are N-positions. If $b \geq 4$, we move
$$(a,b) = (0,b) \to (0,2+\delta_b) \in \mathscr{M}.$$

(2.2.2) $n \geq 1$. In this subcase, $(a,b) \notin \mathscr{M}$ implies that $b = 0$ or $b = 1$ or $b \geq 4$.

- $b \in \{0,1\}$. We move $(a,b) = (2n,b) \to (2n-1,b) \in \mathscr{M}$.
- $b \geq 4$. We move $(a,b) = (2n,b) \to (2n, 2+\delta_b) \in \mathscr{M}$.

The proof is completed. $\square$

5.2 All P-positions of OEW: $s+t>2$

Theorem 19. *By $\mathscr{P}_6$ we denote the set of all P-positions of OEW under normal play convention. Then for all $s,t \in Z^{\geq 1}$ with $s+t>2$,*

$$\mathscr{P}_6 = \bigcup_{n=0}^{\infty} \{(A_n, B_n), (A_n, B_n')\},$$

where for $n \geq 0$,

$$\begin{cases} A_n = n, \\ B_n = \delta_n(sA_n + t + \delta_{s+t}), \\ B_n' = B_n + 1. \end{cases} \tag{20}$$

Proof. Before we give the proof of Theorem 19, Tables 7 and 8 list the first few values of A_n and B_n for $s=t=2$, $s=2$ and $t=3$, respectively, which show us how to determine the set $\mathscr{P}_6$ of all P-positions by using Theorem 19.

Table 7. The first few values of A_n and B_n for $s=t=2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	0	4	0	8	0	12	0	16	0	20	0	24	0	28	0
B_n'	1	5	1	9	1	13	1	17	1	21	1	25	1	29	1

For $s=t=2$, it follows Table 7 that

$$\mathscr{P}_6 = \left\{ \begin{array}{l} (0,0),(0,1),(1,4),(1,5),(2,0),(2,1),(3,8),(3,9),(4,0),(4,1), \\ (5,12),(5,13),(6,0),(6,1),(7,16),(7,17),(8,0),(8,1),\cdots \end{array} \right\}$$

Table 8. The first few values of A_n and B_n for $s=2$ and $t=3$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	0	6	0	10	0	14	0	18	0	22	0	26	0	30	0
B_n'	1	7	1	11	1	15	1	19	1	23	1	27	1	31	1

For $s=2$ and $t=3$, it follows Table 8 that

$$\mathscr{P}_6 = \left\{ \begin{array}{l} (0,0),(0,1),(1,6),(1,7),(2,0),(2,1),(3,10),(3,11),(4,0),(4,1), \\ (5,14),(5,15),(6,0),(6,1),(7,18),(7,19),(8,0),(8,1),\cdots \end{array} \right\}$$

We now give the proof of Theorem 19. Let $\mathcal{W} = \bigcup_{n=0}^{\infty} \{(A_n, B_n), (A_n, B_n')\}$.

Proof of Fact 1. By the definition of function δ_n, we have

$$B_n = \delta_n(sA_n + t + \delta_{s+t}) = \delta_n(sn + t + \delta_{s+t})$$

is even and $B'_n = B_n + 1$ is odd.

Given $(A_n, B_n) \in \mathcal{W}$. Note that $(A_n, B_n) \to (A_m, B'_m) \in \mathcal{W}$ is not a legal move, since $\ell = B_n - B'_m$ is odd. Similarly, $(A_n, B'_n) \to (A_m, B_m) \in \mathcal{W}$ is also impossible.

Given $(A_n, B_n) \in \mathcal{W}$ or $(A_n, B'_n) \in \mathcal{W}$. Suppose that $(A_n, B_n) \to (A_m, B_m) \in \mathcal{W}$ or $(A_n, B'_n) \to (A_m, B'_m) \in \mathcal{W}$. In both cases, we have $m < n$, $k = A_n - A_m = n - m$ is odd, and $\ell = B_n - B_m$.

If n is even and m is odd then

$$\ell = B_n - B_m = 0 - (sm + t + \delta_{s+t}) < 0,$$

which is impossible; If n is odd and m is even then

$$\ell = B_n - B_m = (sA_n + t + \delta_{s+t}) - 0 \geq sn \geq s(n - m) \geq k > 0,$$

and

$$\begin{aligned} \ell &= B_n - B_m = sA_n + t + \delta_{s+t} \\ &\geq sn + t \geq s(n - m) + t = sk + t, \end{aligned}$$

which contradicts Eq. (19).

Proof of Fact 2. Let (x, y) be a position not in $\mathcal{W}$. We will show that there exists a legal move such that $(x, y) \to (A_n, B_n) \in \mathcal{W}$ or $(x, y) \to (A_n, B'_n) \in \mathcal{W}$.

Put $x = n = A_n$ for some integer $n \in Z^{\geq 0}$. We distinguish two cases:

(2.1) $x = n = A_n$ is even. In this case, $B_n = 0$, $B'_n = 1$. The position $(x, y) \notin \mathcal{W}$ implies that $y \geq 2$. We move $(x, y) \to (A_n, B_n + \delta_y) \in \mathcal{W}$ by virtue of $\delta_y = 0$ or 1.

(2.2) $x = n = A_n$ is odd. In this case,

$$B_n = sx + t + \delta_{s+t} \text{ is even,} \tag{21}$$

and

$$B'_n = B_n + 1 \text{ is odd.} \tag{22}$$

The position $(x, y) \notin \mathcal{W}$ implies that $y \geq B_n + 2$ or $0 \leq y \leq B_n - 1$.

(2.2.1) $y \geq B_n + 2$. Now $y \geq B_n + 2 > B_n + \delta_y$ and $y - B_n - \delta_y$ is even. We move

$$(x, y) \to (A_n, B_n + \delta_y) \in \mathcal{W},$$

by taking $y - B_n - \delta_y$ tokens from the second heap.

(2.2.2) $0 \leq y \leq B_n - 1$. We distinguish the following three subcases: $y \in \{0, 1\}$ or $2 \leq y \leq x$ or $x + 1 \leq y \leq B_n - 1$.

- $y \in \{0, 1\}$. We move

$$(x, y) \to (x - 1, \delta_y) = (A_{n-1}, B_{n-1} + \delta_y) \in \mathcal{W},$$

since $n-1$ is odd, $B_{n-1} = 0$, and $y = B_{n-1} + \delta_y$.

• $2 \le y \le x$. We move

$$(x, y) \to (x - y + 1 + \delta_y, \delta_y),$$

by taking $y - 1 - \delta_y$ tokens from the first heap and $y - \delta_y$ tokens from the second heap. Let $x - y + 1 + \delta_y = m$. We note that $m = A_m$ is even and $B_m = 0$. Note that $\delta_y = B_m$ if y is even, $\delta_y = B_m + 1 = B'_m$ if y is odd. Thus

$$(x - y + 1 + \delta_y, \delta_y) = (A_m, B_m + \delta_y) \in \mathcal{W}.$$

This move is legal. Indeed,

1) $k = y - 1 - \delta_y > 0$ is odd;

2) $\ell = y - \delta_y > 0$ is even;

3) $0 \le |\ell - k| = 1 < (s-1)k + t$.

• $x + 1 \le y \le B_n - 1$. We move $(x, y) \to (0, \delta_y) \in \mathcal{W}$, by taking x tokens from the first heap and $y - \delta_y$ tokens from the second heap. This move is legal. Indeed,

1) $k = x$ is odd;

2) $\ell = y - \delta_y$ is even;

3) We note that $y \ge x + 1$ implies that $\ell = y - \delta_y \ge x = k$. By Eq. (21), $B_n - 1$ is odd, so $y \le B_n - 1$ implies that $y \le B_n - 2 + \delta_y$. Hence,

$$\begin{array}{rcl} 0 \le |\ell - k| & = & y - \delta_y - x \\ & \le & sx + t + \delta_{s+t} - 2 - x \\ & = & (s-1)k + t + \delta_{s+t} - 2 \\ & < & (s-1)k + t, \end{array}$$

by virtue of $k \ge 1$.

The proof is completed. □

Theorem 20. *By $\mathscr{P}'_6$ we denote the set of all P-positions of OEW under misère play convention. Then for all $s, t \in Z^{\ge 1}$ with $s + t > 2$,*

$$\mathscr{P}'_6 = \bigcup_{n=0}^{\infty} \{(E_n, H_n), (E_n, H'_n)\},$$

where $E_0 = 0$, $H_0 = 2$, $H'_0 = 3$ and for $n \ge 1$

$$\begin{cases} E_n = n, \\ H_n = (1 - \delta_n)(sE_n - s + t + \delta_{s+t}), \\ H'_n = H_n + 1. \end{cases} \tag{23}$$

Proof. Before we give the proof of Theorem 20, Tables 9 and 10 list the first few values of E_n, H_n and H'_n for $s = t = 2$, $s = 1$ and $t = 2$, respectively, which show us how to determine the set $\mathscr{P}'_6$ of all P-positions by using Theorem 20:

Table 9. The first few values of E_n, H_n and H'_n for $s = t = 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
H_n	2	0	4	0	8	0	12	0	16	0	20	0	24	0	28
H'_n	3	1	5	1	9	1	13	1	17	1	21	1	25	1	29

Table 10. The first few values of E_n, H_n and H'_n for $s = 1, t = 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
H_n	2	0	5	0	9	0	13	0	17	0	21	0	25	0	29
H'_n	3	1	6	1	10	1	14	1	18	1	22	1	26	1	30

We now give the proof of Theorem 20. Let

$$\mathcal{G} = \{(0,2),(0,3)\} \cup \bigcup_{n=1}^{\infty} \{(E_n, H_n), (E_n, H'_n)\}.$$

Proof of Fact 1. By the definition of function δ_n, if n is odd then $H_n = 0$, if $n \geq 1$ is even then $H_n = s(n-1) + t + \delta_{s+t}$ is even. Thus for $n \in Z^{\geq 1}$, H_n is even and $H'_n = H_n + 1$ is odd.

Note that $(0,2)$ or $(0,3)$ can not move to (E_n, H_n) or (E_n, H'_n) as $E_n > 0$. If n is odd, $H_n = 0$ implies that (E_n, H_n) or (E_n, H'_n) can not move to $(0,2)$ or $(0,3)$; If n is even, $E_n = n$ is even implies that (E_n, H_n) or (E_n, H'_n) can not move to $(0,2)$ or $(0,3)$.

Given $(E_n, H_n) \in \mathcal{G}$. Note that $(E_n, H_n) \to (E_m, H'_m) \in \mathcal{G}$ is not a legal move, since $\ell = H_n - H'_m$ is odd. Similarly, $(E_n, H'_n) \to (E_m, H_m) \in \mathcal{G}$ is also impossible.

Given $(E_n, H_n) \in \mathcal{G}$ or $(E_n, H'_n) \in \mathcal{G}$. Suppose that $(E_n, H_n) \to (E_m, H_m) \in \mathcal{G}$ or $(E_n, H'_n) \to (E_m, H'_m) \in \mathcal{G}$. In both cases, we have $1 \leq m < n$, $k = E_n - E_m = n - m$ is odd, and $\ell = H_n - H_m$.

If n is odd and m is even then

$$\ell = H_n - H_m = 0 - (s(m-1) + t + \delta_{s+t}) < 0,$$

which is impossible; If n is even and m is odd, then

$$\ell = H_n - H_m = s(n-1) + t + \delta_{s+t} - 0 \geq s(n-m) + t > sk \geq k > 0,$$

and

$$\begin{aligned}
\ell &= H_n - H_m \\
&= sE_n - s + t + \delta_{s+t} \\
&\geq s(n-1) + t \\
&= s(n-m) + s(m-1) + t \\
&\geq s(n-m) + t = sk + t,
\end{aligned}$$

by virtue of $m \geq 1$, which contradicts Eq. (19).

Proof of Fact 2. Let (x, y) be a position not in $\mathcal{G}$. It suffices to show that there exists a legal move such that $(x, y) \to (E_n, H_n) \in \mathcal{G}$ or $(x, y) \to (E_n, H'_n) \in \mathcal{G}$.

Put $x = n = E_n$ for some integer $n \in Z^{\geq 0}$.

(2.1) $x = n = 0$. In this case, we have $y = 0$ or $y = 1$ or $y \geq 4$. It is obvious that $(0, 0)$ and $(0, 1)$ are N-positions. If $y \geq 4$, then we move $(0, y) \to (0, 2 + \delta_y) \in \mathcal{G}$ by virtue of $\delta_y = 0$ or 1.

(2.2) $x = n > 0$ is odd. In this case, $H_n = 0$, $H'_n = 1$. The position $(x, y) \notin \mathcal{G}$ implies that $y \geq 2$. We move $(x, y) \to (E_n, H_n + \delta_y) \in \mathcal{G}$.

(2.3) $x = n > 0$ is even. In this case,

$$H_n = sx - s + t + \delta_{s+t} \text{ is even,} \tag{24}$$

and

$$H'_n = H_n + 1 \text{ is odd.} \tag{25}$$

The position $(x, y) \notin \mathcal{G}$ implies that $y \geq H_n + 2$ or $0 \leq y \leq H_n - 1$.

(2.3.1) $y \geq H_n + 2$, $n \in Z^{\geq 1}$. We move

$$(x, y) \to (E_n, H_n + \delta_y) \in \mathcal{G},$$

which is legal, since $y - H_n - \delta_y > 0$ is even.

(2.3.2) $0 \leq y \leq H_n - 1$, $n \in Z^{\geq 1}$. We distinguish the following three subcases: $y \in \{0, 1\}$ or $2 \leq y \leq x - 1$ or $x \leq y \leq H_n - 1$.

- $y \in \{0, 1\}$. We move

$$(x, y) \to (x - 1, \delta_y) = (E_{n-1}, H_{n-1} + \delta_y) \in \mathcal{G},$$

since $n - 1$ is odd, $H_{n-1} = 0$, and $\delta_y = 0$ or 1.

- $2 \leq y \leq x - 1$. We move

$$(x, y) \to (x - y - 1 + \delta_y, \delta_y),$$

by taking $y + 1 - \delta_y$ tokens from the first heap and $y - \delta_y$ tokens from the second heap. Let $x - y - 1 + \delta_y = m$. We note that m is odd, $H_m = 0$, and $\delta_y = H_m + \delta_y$. Obviously, $\delta_y = H_m$ if y is even, $\delta_y = H_m + 1 = H'_m$ if y is odd. Thus

$$(x - y - 1 + \delta_y, \delta_y) = (E_m, H_m + \delta_y) \in \mathcal{G}.$$

This move is legal. Indeed,

1) $k = y - \delta_y + 1 > 0$ is odd;

2) $\ell = y - \delta_y > 0$ is even and $\ell \geq k$;

3) $0 < |\ell - k| = 1 < (s - 1)k + t$.

- $x \leq y \leq H_n - 1$, $n \in Z^{\geq 1}$. We move

$$(x, y) \to (1, \delta_y) \in \mathcal{G},$$

by taking $x-1$ tokens from the first heap and $y-\delta_y$ tokens from the second heap. If y is even, $\delta_y = 0 = H_1$; If y is odd, $\delta_y = 1 = H_1 + 1 = H_1'$. Thus $(1, \delta_y) = (E_1, H_1 + \delta_y) \in \mathcal{G}$. This move is legal. Indeed,

1) $k = x - 1$ is odd;

2) $\ell = y - \delta_y$ is even;

3) We note that $y \geq x$ implies that

$$\ell = y - \delta_y \geq x > x - 1 = k.$$

By Eq. (24), $H_n - 1$ is odd, so $y \leq H_n - 1$ implies that $y \leq H_n - 2 + \delta_y$. Hence,

$$\begin{aligned} 0 \leq |\ell - k| &= y - \delta_y - x + 1 \\ &\leq sx - s + t + \delta_{s+t} - 2 - x + 1 \\ &= (s-1)(k+1) - s + t + \delta_{s+t} - 1 \\ &= (s-1)k + t + \delta_{s+t} - 2 \\ &< (s-1)k + t, \end{aligned}$$

by virtue of $k \geq 1$.

The proof is completed. □

6 Conclusion

Three new models of the restricted version of (s,t)-Wythoff's game, Odd-Odd (s,t)-Wythoff's Game, Even-Even (s,t)-Wythoff's Game, Odd-Even (s,t)-Wythoff's Game, are investigated. Under normal or misère play convention, all its P-positions of these three models are given for two arbitrary integers $s, t \geq 1$.

Similar to Odd-Even (s,t)-Wythoff's Game, we can define the fourth model, *Even-Odd (s,t)-Wythoff's Game* (Denoted by EOW): Let S_h, S_v, D_1 and D_2 be subsets of $Z^{\geq 0}$. Given two parameters $s, t \in Z^{\geq 1}$ and two heaps of finitely many tokens. One of the heaps is designated as "first heap" and the other as "second heap" throughout the game. By (x, y) we denote a position of present game, where x and y denote the numbers of tokens in the first and the second heaps, respectively. Two rules of moves are allowed:

(*Even-Odd Nim Rule*) A player chooses the first heap and takes *even* $k > 0$ tokens, or chooses the second heap and takes *odd* $\ell > 0$ tokens.

(*Even-Odd More General Wythoff's Rule*) A player takes tokens from both heaps, *even* $k > 0$ tokens from the first heap, *odd* $\ell > 0$ tokens from the second heap, and

$$0 \leq |\ell - k| < (s-1)\lambda + t, \quad \lambda = \min\{k, \ell\} \in Z^{\geq 1}. \tag{26}$$

Obviously, EOW is equivalent to $S_h = D_1 = Z^{even}$ and $S_v = D_2 = Z^{odd}$ in General Restriction of (s,t)-Wythoff's Game.

If (x, y) is a P-position of OEW, then (y, x) is a P-position of EOW. Thus we have

Corollary 21. *Given two parameters $s = t = 1$. By $\mathscr{P}_7$ we denote the set of all P-positions of EOW under normal play convention. Then*

$$\mathscr{P}_7 = \bigcup_{n=0}^{\infty} \{(0, 2n), (1, 2n), (2, 2n+1), (3, 2n+1)\}.$$

Corollary 22. *Given two parameters $s = t = 1$. By $\mathscr{P}'_7$ we denote the set of all P-positions of EOW under misère play convention. Then*

$$\mathscr{P}'_7 = \bigcup_{n=0}^{\infty} \{(2, 2n), (3, 2n), (0, 2n+1), (1, 2n+1)\}.$$

Corollary 23. *By $\mathscr{P}_8$ we denote the set of all P-positions of EOW under normal play convention. Then for all $s, t \in Z^{\geq 1}$ with $s + t > 2$,*

$$\mathscr{P}_8 = \bigcup_{n=0}^{\infty} \{(B_n, A_n), (B'_n, A_n)\},$$

where for $n \geq 0$,

$$\begin{cases} A_n = n, \\ B_n = \delta_n(sA_n + t + \delta_{s+t}), \\ B'_n = B_n + 1. \end{cases} \tag{27}$$

Corollary 24. *By $\mathscr{P}'_8$ we denote the set of all P-positions of EOW under misère play convention. Then for all $s, t \in Z^{\geq 1}$ with $s + t > 2$,*

$$\mathscr{P}'_8 = \{(2, 0), (3, 0)\} \cup \bigcup_{n=1}^{\infty} \{(H_n, E_n), (H'_n, E_n)\},$$

where for $n \geq 1$,

$$\begin{cases} E_n = n, \\ H_n = (1 - \delta_n)(sE_n - s + t + \delta_{s+t}), \\ H'_n = H_n + 1. \end{cases} \tag{28}$$

Recall that Wythoff's game is the special case $s = t = 1$ in (s, t)-Wythoff's game, and a-Wythoff's game is the special case $s = 1$ and $t = a$ in (s, t)-Wythoff's game. Thus (s, t)-Wythoff's game is the generalization of both Wythoff's game and a-Wythoff's game. Under normal play convention, the set of all P-positions of a-Wythoff's game and the set of all P-positions of Wythoff's game can be obtained by letting ($s = 1$ and $t = a$) and $s = t = 1$ in Eq. (2), respectively (see [19],[9]).

Our results on OOW, EEW, OEW and EOW are given for all integers $s \geq 1$ and $t \geq 1$, thus the corresponding results of cases ($s = 1$ and $t = a$) and $s = t = 1$ have been obtained.

Given two integer $K \geq 1$ and $r \in \{0, 1, 2 \cdots, K-1\}$. We use notation $Z_K^{(r)} = \{Kn + r | n \in Z^{\geq 0}\}$.

Open Problem 25. We add restrictions $S_h = S_v = D_1 = D_2 = Z_K^{(0)}$ in General Restriction of (s,t)-Wythoff's Game, how to determine all its P-positions under normal or misère play convention?

Note that $Z^{even} = Z_2^{(0)}$, thus Theorems 13 and 14 have settled the special case $K = 2$ of this problem. Can we generalize Theorems 13 and 14 from $K = 2$ to an arbitrary integer $K \geq 3$?

Open Problem 26. We add restrictions $S_h = S_v = D_1 = D_2 = Z_K^{(r)}$ in General Restriction of (s,t)-Wythoff's Game, where $r \in \{0, 1, 2 \cdots, K-1\}$ be a fixed integer. How to determine all its P-positions under normal or misère play convention?

Note that $Z^{odd} = Z_2^{(1)}$, thus Theorems 7 and 9 have settled the special case $K = 2$ and $r = 1$ of this problem. Can we generalize Theorems 7 and 9 to an arbitrary integer $K \geq 3$?

Open Problem 27. We add restrictions $S_h = D_1 = Z_K^{(r_1)}$ and $S_v = D_2 = Z_K^{(r_2)}$ in General Restriction of (s,t)-Wythoff's Game, where $r_1, r_2 \in \{0, 1, 2 \cdots, K-1\}$ and $r_1 \neq r_2$ be fixed integers. How to determine all its P-positions under normal or misère play convention?

Note that $Z^{even} = Z_2^{(0)}$ and $Z^{odd} = Z_2^{(1)}$, thus Theorems 17, 18, 19 and 20 have settled the special case $K = 2$ of this problem. Can we generalize these results from ($K = 2$, $r_1 = 0$ and $r_2 = 1$) to arbitrary integers $K \geq 3$ and $r_1, r_2 \in \{0, 1, 2 \cdots, K-1\}$?

Open Problem 28. One can investigate the case of (s,t)-Wythoff's game which is only restricted on Extended Diagonal Moves. As an example, let $S_h = S_v = Z^{\geq 0}$ and $D_1 = D_2 = Z_K^{(r)}$ (K is a fixed positive integer, $r \in \{0, 1, 2 \cdots, K-1\}$) in General Restriction of (s,t)-Wythoff's Game. Can we obtain the corresponding results?

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