More on the Wilson $W_{tk}(v)$ matrices

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Abstract

For integers $0 \leqslant t \leqslant k \leqslant v-t$, let X be a v-set, and let $W_{tk}(v)$ be a $\binom{v}{t} \times \binom{v}{k}$ inclusion matrix where rows and columns are indexed by t-subsets and k-subsets of X, respectively, and for row T and column K, $W_{tk}(v)(T,K) = 1$ if $T \subseteq K$ and zero otherwise. Since $W_{tk}(v)$ is a full rank matrix, by reordering the columns of $W_{tk}(v)$ we can write $W_{tk}(v) = (S|N)$, where N denotes a set of independent columns of $W_{tk}(v)$. In this paper, first by classifying t-subsets and k-subsets, we present a new decomposition of $W_{tk}(v)$. Then by employing this decomposition, the Leibniz Triangle, and a known right inverse of $W_{tk}(v)$, we construct the inverse of N and consequently special basis for the null space (known as the standard basis) of $W_{tk}(v)$.

Keywords: Signed t-design; Leibniz Triangle; Standard basis; Right inverse; Root of a block; $\mathcal{R}$-ordering; B-changer

1 Introduction

Integers t, k, and v with $0 \leqslant t \leqslant k \leqslant v-t$ are considered. Let X be a linearly ordered v-set, and let

$$\binom{X}{i} := \{A \subseteq X : |A| = i\}, \qquad 0 \leqslant i \leqslant v.$$

For the sake of brevity, we will denote a set $\{a_1, \dots, a_i\}$ by the string "$a_1 \dots a_i$", and assuming that $a_1 < a_2 < \cdots < a_i$. The elements of $\binom{X}{k}$ and $\binom{X}{t}$ are called *blocks* and *t-subsets*, respectively.

The *inclusion matrix* $W_{tk}(v)$ (known as Wilson matrix) is defined to be a $\binom{v}{t}$ by $\binom{v}{k}$ $(0,1)$-matrix whose rows and columns are indexed by (and referred to) the members of

$\binom{X}{t}$ and $\binom{X}{k}$, respectively, and where

$$W^v_{tk}(T,K) := \begin{cases} 1 & \text{if } T \subseteq K \\ 0 & \text{otherwise} \end{cases}, \qquad T \in \binom{X}{t},\ K \in \binom{X}{k}.$$

For the sake of convenience, sometimes we use W_{tk} or just a bare W for $W_{tk}(v)$.

Let $S = x_1x_2\ldots x_n$ be a finite set, and let $\mathbb{F}$ be an arbitrary ring. An $\mathbb{F}$-*collection* of the elements of S is a function $f : S \to \mathbb{F}$, with the vector representation $\big(f(x_1), \cdots, f(x_n)\big)^T$, for i, $1 \leqslant i \leqslant n$, $f(x_i)$ is defined to be *the value of* x_i *in* f.

It is well known that W_{tk} is a full rank matrix over $\mathbb{Q}$ [8]. As a linear operator, W_{tk} acts on a $\mathbb{Z}$-collection of blocks, and algebraically counts the number of times that any member of $\binom{X}{t}$ appears in the blocks of the collection.

In the set of our notations, for any matrix M, the free $\mathbb{Z}$-module generated by rows and columns of matrix M will be denoted by $\mathrm{row}_{\mathbb{Z}}(M)$ and $\mathrm{col}_{\mathbb{Z}}(M)$, respectively, and $\mathrm{null}_{\mathbb{Z}}(M)$ will be the free $\mathbb{Z}$-module orthogonal to $\mathrm{row}_{\mathbb{Z}}(M)$.

Let $\mathbf{1}$ be the all 1 vector, and let λ be a nonnegative integer. We call the following equation the *fundamental equation* of design theory:

$$W_{tk}\, f = \lambda \mathbf{1}. \tag{1}$$

Every integral solution of equation (1) is called a *signed* t-(v, k, λ) *design*. For more on this, see [4, 8].

Since W is a full rank matrix, by reordering the columns of W we can write W as $W = (S|N)$, where N denotes a set of independent columns of W. Therefore, there is a matrix C such that $N^{-1}(S|N) = (C|I)$. Let $\mathbb{S}$ be a matrix defined by stacking an identity matrix above the matrix $-C$,i.e., $\mathbb{S} := \left(\frac{I}{-C}\right)$. Since $W\mathbb{S} = 0$ and W is full rank, the columns of $\mathbb{S}$ form a basis for $\mathrm{null}_{\mathbb{Z}}(W)$.

Now, we would like to give a rather comprehensive view on the problem addressed in this paper: We start with *the halving conjecture*. In 1987 A. Hartman [9] stated the following conjecture which is now known as the halving conjecture:

> *For* $0 \leqslant i \leqslant t$, *there is a* $(1, -1)$*-vector in* $null_{\mathbb{Z}}(W)$ *if and only if* $\binom{v-i}{k-i} \equiv 0 \,(mod\, 2)$.

Up to our knowledge, the conjecture has been settled for $t = 2$ utilizing a recursive construction [2], and some infinite classes have been constructed too [10].

Since every $(1, -1)$-vector in $null_{\mathbb{Z}}(W)$ is a linear combination of the columns of $\mathbb{S}$, therefore the null space of the W should be studied more carefully. For this, we have to know the components, row structure and column structure of $\mathbb{S}$.

- In what follows, an explicit formula for the entries of N^{-1} and consequently a closed formula for the entries of $\mathbb{S}$ are presented.
- For the row structure of $\mathbb{S}$, there are two conjectures on the table:
  - The elements of every row of $\mathbb{S}$ have the same sign.

– For $t > 1$, the matrix $\mathbb{S}$ contains a nowhere zero row.

In [1] these two conjectures have been settled for $t = 2$ and $k = 3$.

- In [1] the columns of $\mathbb{S}_{23}(v)$ have been classified into five classes and by utilizing these classes the correctness of the halving conjecture has been established.

2 Classification of blocks and t-subsets

Definition 1. For m, $1 \leqslant m \leqslant k$, let $A = \{a_{k-m+1}, \dots, a_k\} \in \binom{X}{m}$, and

$$L_A = \{i : a_i \leqslant 2i - k + t,\ k - m + 1 \leqslant i \leqslant k\}.$$

Let $\ell_A = \max(L_A)$. Note that $\max(\varnothing) := 0$, here. Now, $\mathcal{R}_{tk}(A) := \{a_{k-m+1}, \dots, a_{\ell_A}\}$ is called the root of A.

Example 2. For $A = 3458$, we obtain:

i	a_i	$2i-k+t$	ℓ_A
4	8	7	
3	5	5	✓
2	4	3	
1	3	1	

$\ell_A = 3 \Rightarrow \mathcal{R}_{34}(3458) = 345$

i	a_i	$2i-k+t$	ℓ_A
4	8	6	
3	5	4	
2	4	2	
1	3	0	

$L_A = \varnothing \Rightarrow \mathcal{R}_{24}(3458) = \varnothing$

For $B = 1478$, we obtain:

i	a_i	$2i-k+t$	ℓ_B
5	8	8	✓
4	7	6	
3	4	4	
2	1	2	

$\ell_B = 5 \Rightarrow \mathcal{R}_{35}(1478) = 1478$

i	a_i	$2i-k+t$	ℓ_B
5	8	7	
4	7	5	
3	4	3	
2	1	1	✓

$\ell_B = 2 \Rightarrow \mathcal{R}_{25}(1478) = 1$

In [6, 14], a decomposition of $W_{tk}(v)$ is presented:

$$W_{tk}(v) = \begin{array}{|c|c|} \hline W_{t-1,k-1}(v-1) & 0 \\ \hline W_{t,k-1}(v-1) & W_{tk}(v-1) \\ \hline \end{array} \tag{2}$$

Now, we propose a new ordering of blocks and t-subsets and consequently a new decomposition.

Definition 3. For given t, k, and X, let

$$\begin{cases} \mathcal{B}_i = \{B : |\mathcal{R}_{tk}(B)| = i, B \in \binom{X}{k}\}, \\ \mathcal{T}_j = \{T : |\mathcal{R}_{tk}(T)| = j, T \in \binom{X}{t}\}, \end{cases} \qquad 0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant t.$$

If we order every B_i and T_j in reverse lexicographic ordering, then $B_0, B_1, \ldots, B_k$ and $T_0, T_1, \ldots, T_t$ are orders on $\binom{X}{k}$ and $\binom{X}{t}$, respectively. This ordering is called $\mathcal{R}$-*ordering.*

$$W_{tk}(v) = \begin{array}{|c|c|} \hline E & 0 \\ \hline \mathcal{A} & W_{tk}(t+k) \\ \hline \end{array} \tag{3}$$

Here the rows and the columns of $W_{tk}(t+k)$ are indexed by $\mathcal{T}_t$ and $\mathcal{B}_k$ elements, respectively. In passing we note that the matrix E contains $(v-k-t)$ copies of intersecting submatrices $W_{t-1,k-1}(v-1)$.

Example 4. The above decomposition of $W_{23}(7)$ is:

		$\mathcal{B}_0$															$\mathcal{B}_1$					$\mathcal{B}_2$						$\mathcal{B}_3$									
		567	467	457	456	367	357	356	347	346	267	257	256	247	246	167	157	156	147	146	237	236	137	136	127	126	345	245	235	234	145	135	134	125	124	123
$\mathcal{T}_0$	67	1	1			1					1					1																					
	57	1		1			1					1					1																				
	56	1			1			1					1					1																			
	47		1	1					1					1					1																		
	46		1		1					1					1					1																	
$\mathcal{T}_1$	37					1	1		1														1		1												
	36					1		1		1													1		1											
	27										1	1		1							1				1												
	26										1		1		1							1				1											
	17															1	1		1				1		1												
	16															1		1		1				1		1											
$\mathcal{T}_2$	45			1	1																						1	1			1					
	35						1	1																			1		1			1				
	34								1	1																	1			1			1			
	25											1	1																1	1					1		
	24													1	1													1		1					1	
	23																					1	1							1	1						1
	15																1	1														1	1		1		
	14																		1	1												1		1		1	
	13																						1	1								1	1			1
	12																								1	1								1	1	1

$\mathbf{W}_{23}(5)$

Table 1. The decompostion of $W_{23}(7)$.

(In tables throughout this paper, unless otherwise indicated, blanks are zeros.)

Remark 5. To obtain the inverse of N, first we construct the inverse of $W_{tk}(t+k)$. In the next section, we introduce a right inverse of W.

3 Right inverse of W and Leibniz Triangle

Around 1980, Graham, Li, and Li [7] presented a right inverse for W with a closed formula. Later on, Bapat [3] constructed a right inverse for W in a recursive form. The elements of these right inverses are multiples of the entries of Leibniz Triangle (Table 2).

						1						
					$\frac{1}{2}$		$\frac{1}{2}$					
				$\frac{1}{3}$		$\frac{1}{6}$		$\frac{1}{3}$				
			$\frac{1}{4}$		$\frac{1}{12}$		$\frac{1}{12}$		$\frac{1}{4}$			
		$\frac{1}{5}$		$\frac{1}{20}$		$\frac{1}{30}$		$\frac{1}{20}$		$\frac{1}{5}$		
	$\frac{1}{6}$		$\frac{1}{30}$		$\frac{1}{60}$		$\frac{1}{60}$		$\frac{1}{30}$		$\frac{1}{6}$	
$\frac{1}{7}$		$\frac{1}{42}$		$\frac{1}{105}$		$\frac{1}{140}$		$\frac{1}{105}$		$\frac{1}{42}$		$\frac{1}{7}$

Table 2. Leibniz Triangle.

For given $0 \leqslant r \leqslant n$, the (n,r)-th position of Leibniz Triangle was introduced as the (n,r)-th *harmonic coefficient* which is defined to be

$$\mathcal{H}_r^n = \frac{1}{(n+1)\binom{n}{r}} = \frac{1}{(r+1)\binom{n+1}{r+1}}. \tag{4}$$

Now we index the rows and the columns of the right inverse of W by the elements of $\binom{X}{k}$ and $\binom{X}{t}$, respectively. According to [7] and (4) every entry of this matrix comes from the following relation:

$$\begin{aligned} \mathrm{GLL}(B,T) &= \frac{(-1)^{(k-t)}(k-t)}{(-1)^{|B-T|}|B-T|} \cdot \frac{1}{\binom{v-t}{|B-T|}} \\ &= (-1)^{k-t+|B-T|}(k-t)\mathcal{H}_{|B-T|}^{v-t-1}, \end{aligned} \tag{5}$$

where $B \in \binom{X}{k}$, $T \in \binom{X}{t}$.

⋆⋆⋆⋆⋆

Now back to the inverse of $W_{tk}(t+k)$. We replace $v-t$ by k in (5), and then every element of the inverse of $W_{tk}(t+k)$, denoted by $F(B,T)$, is defined as

$$F(B,T) := (-1)^{(t-\theta)}(k-t)\mathcal{H}_\theta^{k-1}, \tag{6}$$

where $\theta = |B \cap T|$.

Let B be an arbitrary block such that $|\mathcal{R}_{tk}(B)| = k$. Suppose that $\mathbf{b} = (b_1, \ldots, b_{\binom{v}{t}})$ is a vector where $b_i = F(B, T_i)$, $T_i \in \binom{X}{t}$. Now we compute the product of $\mathbf{b}$ in the column B' of W. The product is the sum of those $F(B,T_i)$ where $T_i \subseteq B'$.

$$\mathbf{b}.B' = (k-t)\sum_{\theta=0}^{t}(-1)^{t-\theta}\binom{s}{\theta}\binom{k-s}{t-\theta}\mathcal{H}_\theta^{k-1} = (-1)^t\binom{k-s-1}{t}, \tag{7}$$

where $s = |B \cap B'|$.

The above formula is easily verified by Maple [13] and exhibits a very interesting relation between Leibniz Triangle and binomial triangle.

4 The inverse of N

The construction of the inverse of N is based on (6), but first we should partition W into independent and dependent columns. The function which is defined on blocks in [5, 11], classifies the blocks into $t+2$ classes. Although through that classification independent and dependent columns are separated, the partitioning is not refined enough to be useful for the inverse construction. Here we introduce a new function to partition subsets of X, which is based on $\mathcal{R}$-ordering.

Definition 6. The block B is called *a starting block* if $0 \leqslant |\mathcal{R}_{tk}(B)| < k-t$, and *a non-starting block* if $k-t \leqslant |\mathcal{R}_{tk}(B)| \leqslant k$.

Notation. $k_B{:=}|\mathcal{R}_{tk}(B)|$.

Now, we omit the columns indexed by the starting blocks from W and we denote the remaining matrix by N_{tk}. If we $\mathcal{R}$-order the t-subsets and non-starting blocks, then:

$N_{tk} =$

$|\mathcal{R}_{tk}(T)|$ \ $k_B-(k-t)$	0	1	2	$\cdots$	t
0		0	0	0	0
1			0	0	0
2				0	0
$\vdots$					0
t					

Note. The entries of shaded boxes could be zero or one.

Let B be a non-starting block and $T \in \binom{X}{t}$. If $k_B-(k-t)=i$ and $|\mathcal{R}_{tk}(T)|<i$, then $k-k_B < t-|\mathcal{R}_{tk}(T)|$. That is to say that there exists an element in T which is not in B. Therefore, $N_{tk}(T,B)=0$.

Lemma 7. *For given t, k, and X, the number of non-starting blocks is $\binom{v}{t}$.*

Proof. For $0 \leqslant m \leqslant t$, let $A=\{a_{k-t+m},\dots,a_k\}$ and $\mathcal{R}_{tk}(A)=\varnothing$. If $A\subseteq T$ and $|\mathcal{R}_{tk}(T)|=m-1$, then $T\setminus\mathcal{R}_{tk}(T)=A$. Let $\mathcal{R}_{tk}(T)=\{a_{k-t+1},\dots,a_{k-t+m-1}\}$, by Definition 2.1, $a_{k-t+m-1}\leqslant k-t+2m-2$. Therefore, the number of T such that $A\subseteq T$ is equal to $\binom{k-t+2m-2}{m-1}$.

Similarly the number of non-starting blocks B such that $A\subseteq B$ and $|\mathcal{R}_{tk}(B)| = k-t+m-1$ is equal to $\binom{k-t+2m-2}{k-t+m-1}$.

Now, we have to show that different A's with the same size, produce different t-subsets and different blocks.

Let A_1, A_2, D_1, and D_2 be subsets of X. Suppose $R_{tk}(A_1) = R_{tk}(A_2) = \varnothing$, $A_1 \neq A_2$, and $|A_1| = |A_2|$. We show that, if

$$A_1 \subseteq D_1,\ A_2 \subseteq D_2,\ \text{and } |\mathcal{R}_{tk}(D_1)| = |\mathcal{R}_{tk}(D_2)| = |D_1| - |A_1|,$$

then $D_1 \neq D_2$.

Suppose $D_1 = D_2$. Since $A_1 \neq A_2$, there is an $e \in A_1$, such that $e \in \mathcal{R}_{tk}(D_2)$. Hence, by Definition 2.1 $|\mathcal{R}_{tk}(D_2)| = |D_2| - |A_2| - 1$, and this is a contradiction. Therefore, there exists a bijection from the set of non-starting blocks to all the t-subsets. □

Corollary 8. *The main diagonal boxes of N_{tk} are square matrices.*

Example 9. Table 3 demonstrates the boxing structure of $N_{23}(6)$.

$N_{23}(6) =$

		$k_B = 1$		$k_B = 2$			$k_B = 3$									
		156	**146**	**236**	**136**	**126**	**345**	**245**	**235**	**234**	**145**	**135**	**134**	**125**	**124**	**123**
$\lvert\mathcal{R}_{23}(T)\rvert = 0$	**56**	1														
	46		1													
$\lvert\mathcal{R}_{23}(T)\rvert = 1$	**36**			1	1											
	26			1		1										
	16	1	1		1	1										
$\lvert\mathcal{R}_{23}(T)\rvert = 2$	**45**						1	1			1					
	35						1		1			1				
	34						1			1			1			
	25							1	1					1		
	24							1		1					1	
	23			1					1	1						1
	15	1									1	1		1		
	14		1								1		1		1	
	13				1							1	1			1
	12					1								1	1	1

Definition 10. For given t, k, and X, let B be a non-starting block. For any $A \subseteq X$ such that $|A| \leqslant |B|$, $A \setminus \big(B \setminus \mathcal{R}_{tk}(B)\big)$ denoted by $\mathcal{R}_{tk}(A, B)$ is called *the root of* A with respect to B.

Example 11. $\mathcal{R}_{24}(58, 3458) = 5$, $\mathcal{R}_{34}(5678, 1234) = 5678$, and $\mathcal{R}_{34}(5678, 1478) = 56$.

Now, to show that the columns of N_{tk} are linearly independent, first we define a matrix $\mathcal{F}_{tk}(v)$, whose rows and columns are indexed by non-starting blocks and t-subsets, respectively. We note that the non-starting blocks and t-subsets are $\mathcal{R}$-ordered. Then $\mathcal{F}_{tk}(v)$ is defined as:

$$\mathcal{F}_{tk}(v)(B,T) := \begin{cases} F(\mathcal{R}_{tk}(B), \mathcal{R}_{tk}(B,T)) & k - k_B = t - |\mathcal{R}_{tk}(B,T)|, \\ 0 & k - k_B \neq t - |\mathcal{R}_{tk}(B,T)|, \end{cases} \tag{8}$$

where $F(B,T) = (-1)^{(t-\theta)}(k-t)\mathcal{H}_{\theta}^{k-1}$ as in (6). Now let $M := \mathcal{F}_{tk}(v)N_{tk}$. Naturally the rows and the columns of M are indexed by non-starting blocks.

Example 12.

$\mathcal{F}_{23}(6) =$

	56	**46**	**36**	**26**	**16**	**45**	**35**	**34**	**25**	**24**	**23**	**15**	**14**	**13**	**12**
156	1														
146		1													
236	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$										
136	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$										
126	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$										
345	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
245	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
235	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$
234	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
145	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$
135	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
134	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$
125	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
124	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$
123	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

Notation. $t_B := t - (k - k_B)$.

Lemma 13. *If for two non-starting blocks B and B', $k_B = k_{B'}$, then $|\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(B')| \geqslant k_B - t_B$.*

Proof. Since $k_B = k_{B'}$, every element of $\mathcal{R}_{tk}(B)$ and $\mathcal{R}_{tk}(B')$ is at most $2k_B - k + t = k_B + t_B$ by Definition 1. Hence $|\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(B')| \geqslant k_B - t_B$. □

Now, we have

$$M(B,B') = \begin{cases} (-1)^{t_B} \binom{k_B - |\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(B',B)| - 1}{t_B} & k_B = |\mathcal{R}_{tk}(B',B)|, \\ 0 & k_B \neq |\mathcal{R}_{tk}(B',B)|. \end{cases} \tag{9}$$

For clarity we add the following statements:

- $k_B - |\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(B',B)| - 1 = -1$ if and only if $B = B'$, and $\binom{-1}{t_B} = (-1)^{t_B}$;
- $k_B = |\mathcal{R}_{tk}(B',B)|$ and $B \neq B'$,then $\mathcal{R}_{tk}(B',B)| = \mathcal{R}_{tk}(B')$, and $0 \leqslant k_B - |\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(B',B)| - 1 < t_B$, implying that the binomial coefficient is 0.

By Corollary 8, (9), and the above statements, the main diagonal boxes of matrix M are identity matrices. Therefore, M is a lower triangular matrix.

Example 14.

$M_{23}(6) =$

	156	**146**	**236**	**136**	**126**	**345**	**245**	**235**	**234**	**145**	**135**	**134**	**125**	**124**	**123**
156	1														
146		1													
236	−1	−1	1												
136				1											
126					1										
345					1	1									
245				1			1								
235		1						1							
234	1								1						
145			1							1					
135											1				
134												1			
125													1		
124														1	
123															1

Theorem 15. *The columns indexed by non-starting blocks in W are linearly independent.*

Definition 16. For a given non-starting block B, a block A is called a B-changer, if the following conditions hold:

(*i*) $k_B > k_A$,

(*ii*) $|\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(A,B)| < k_B - t_B$,

(*iii*) $B \setminus \mathcal{R}_{tk}(B) \subseteq A \setminus \mathcal{R}_{tk}(A,B)$.

Lemma 17. *Let B and A be two non-starting blocks such that $A \neq B$. Then A is a B-changer if and only if $M(B,A) \neq 0$.*

Proof. For a given block B, let A be a B-changer. By Definition 16 we have $|\mathcal{R}_{tk}(A) \cap \mathcal{R}_{tk}(B)| \leqslant k_B - t_B - 1$. Therefore, by (9) it follows that $M(B, A) \neq 0$.

Now assume that $M(B, A) \neq 0$, again by (9) we have $k_B \geqslant k_A$ and $|\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(A, B)| < k_B - t_B - 1$. If $k_B = k_A$, then based on Lemma 13 and (9) we have $M(B, A) = 0$, which is a contradiction. Therefore, $k_B > k_A$ and A is a B-changer. □

Theorem 18. *Suppose that the rows and the columns of matrix N^{-1} are indexed by non-starting blocks and t-subsets, respectively. For a block B and a t-subset T, we have:*

$$N_{tk}^{-1}(B, T) = \begin{cases} F(\mathcal{R}_{tk}(B), \mathcal{R}_{tk}(T, B)) - \sum\limits_{A} M(B, A) N_{tk}^{-1}(A, T) & k - k_B = t - |\mathcal{R}_{tk}(T, B)|, \\ 0 & k - k_B \neq t - |\mathcal{R}_{tk}(T, B)|. \end{cases} \tag{10}$$

Proof. The correctness of the statement of the theorem can be easily established by the elementary row operations. □

Example 19.

$N_{23}^{-1}(6) =$

	56	**46**	**36**	**26**	**16**	**45**	**35**	**34**	**25**	**24**	**23**	**15**	**14**	**13**	**12**
156	1														
146		1													
236	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$										
136	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$										
126	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$										
345	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
245	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
235	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$
234	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
145	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$
135	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
134	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$
125	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
124	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$
123	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

5 Standard basis and the unique signed design

Let $\mathcal{F}_{tk}(v)$ and S be defined as before and let B and B^s be a non-starting and a starting block, respectively. Suppose $M^s := \mathcal{F}_{tk}(v)S$. Clearly, the rows and the columns of M^s are indexed by non-starting blocks and starting blocks, respectively. By Definition 16 we have $k_B > k_{B^s}$. Every entry of matrix M^s, based on proof (9) is obtained as:

$$M^s(B, B^s) = \begin{cases} (-1)^{t_B} \binom{k_B - |\mathcal{R}_{tk}(B) \cap \mathcal{R}_{tk}(B^s, B)| - 1}{t_B} & k_B = |\mathcal{R}_{tk}(B^s, B)|, \\ 0 & k_B \neq |\mathcal{R}_{tk}(B^s, B)|. \end{cases}$$

Example 20.

$M^s_{23}(6) =$

	456	**356**	**346**	**256**	**246**
156	1	1		1	
146	1		1		1
236	−1				
136	−1			−1	−1
126	−1	−1	−1		
345					
245					
235					
234					
145					
135					1
134				1	
125			1		
124		1			
123	1				

Note 21. We recall that $C = N^{-1}S$. From this, it follows that the rows and the columns of C are indexed by non-starting and starting blocks, respectively.

Theorem 22. *Let B and B^s be a non-starting and a starting block, respectively. Every entry of C is given by*

$$C(B, B^s) = \begin{cases} M^s(B, B^s) - \sum_A M(B, A)C(A, B^s) & k_B = |\mathcal{R}_{tk}(B^s, B)|, \\ 0 & k_B \neq |\mathcal{R}_{tk}(B^s, B)|, \end{cases}$$

where A is a B-changer.

Example 23.

$C_{23}(6) =$

	456	**356**	**346**	**256**	**246**
156	1	1		1	
146	1		1		1
236	1	1	1	1	1
136	−1			−1	−1
126	−1	−1	−1		
345	1	1	1		
245	1			1	1
235	−1		−1		−1
234	−1	−1		−1	
145	−1	−1	−1	−1	−1
135					1
134				1	
125			1		
124		1			
123	1				

In [12] Khosrovshahi and Tayfeh-Rezaie showed that by subtracting $\mathbf{1}$ from the sum of the columns of the standard basis of W, one obtains a unique signed t-design D. For more on this subject see [15]. Here we show that D is also obtained by the sum of the columns of the inverse of N.

Let $\left(s_{i_1},\ldots,s_{i_{\binom{v}{k}-\binom{v}{t}}}\right)$ be the i-th row of $\mathbb{S}_{tk}$ and $D=\left(d_1,\ldots,d_{\binom{v}{k}}\right)^T$. Therefore,

$$d_i=\sum_{j=1}^{\binom{v}{k}-\binom{v}{t}} s_{i_j}-\mathbf{1}.$$

Let $\left(\gamma_{i_1},\ldots\gamma_{i_{\binom{v}{t}}}\right)$ be the i-th row of N_{tk}^{-1}. We have the following identities:

$$\binom{v-t}{k-t}\sum_{m=1}^{\binom{v}{t}}\gamma_{i_m}=\sum_{m=1}^{\binom{v}{t}}\sum_{j=1}^{\binom{v}{k}}\gamma_{i_m}W_{mj}=\mathbf{1}-\sum_{j=1}^{\binom{v}{k}-\binom{v}{t}} s_{i_j}=d_i.$$

Theorem 24. *Let* $\eta=\sum_{i=1}^{\binom{v}{t}}\Gamma_i$, *where* Γ_i*'s are the columns of* N_{tk}^{-1}, *then* $\binom{v-t}{k-t}\eta=D$.

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