Proof of two divisibility properties of binomial coefficients conjectured by Z.-W. Sun

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Abstract

For all positive integers n, we prove the following divisibility properties:

$$(2n+3)\binom{2n}{n}\,\Big|\,3\binom{6n}{3n}\binom{3n}{n} \quad \text{and} \quad (10n+3)\binom{3n}{n}\,\Big|\,21\binom{15n}{5n}\binom{5n}{n}.$$

This confirms two recent conjectures of Z.-W. Sun. Some similar divisibility properties are given. Moreover, we show that, for all positive integers m and n, the product $am\binom{am+bm-1}{am}\binom{an+bn}{an}$ is divisible by $m+n$. In fact, the latter result can be further generalized to the q-binomial coefficients and q-integers case, which generalizes the positivity of q-Catalan numbers. We also propose several related conjectures.

Keywords: congruences, binomial coefficients, p-adic order, q-Catalan numbers, reciprocal and unimodal polynomials

1 Introduction

In [18, 19], Z.-W. Sun proved some divisibility properties of binomial coefficients, such as

$$2(2n+1)\binom{2n}{n}\,\Big|\,\binom{6n}{3n}\binom{3n}{n}, \tag{1.1}$$

$$(10n+1)\binom{3n}{n}\,\Big|\,\binom{15n}{5n}\binom{5n-1}{n-1}. \tag{1.2}$$

Some similar divisibility results were later obtained by Guo [10] and Guo and Krattenthaler [11]. A generalization of (1.1) was recently given by Sepanski [15]. It is worth mentioning that Bober [6] has completely described when ratios of factorial products of the form

$$\frac{(a_1 n)! \cdots (a_k n)!}{(b_1 n)! \cdots (b_{k+1} n)!}$$

with $a_1 + \cdots + a_k = b_1 + \cdots + b_{k+1}$ are always integers.

Let

$$S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}}, \quad \text{and} \quad t_n = \frac{\binom{15n}{5n}\binom{5n-1}{n-1}}{(10n+1)\binom{3n}{n}}.$$

In this paper we first prove the following two results conjectured by Z.-W. Sun [18, 19].

Theorem 1.1 (see [18, Conjecture 3(i)]) *Let n be a positive integer. Then*

$$3S_n \equiv 0 \pmod{2n+3}. \tag{1.3}$$

Theorem 1.2 [19, Conjecture 1.3] *Let n be a positive integer. Then*

$$21t_n \equiv 0 \pmod{10n+3}.$$

We shall also give more congruences for S_n and t_n as follows.

Theorem 1.3 *Let n be a positive integer. Then*

$$105S_n \equiv 0 \pmod{2n+5}, \tag{1.4}$$
$$315S_n \equiv 0 \pmod{2n+7}, \tag{1.5}$$
$$6435S_n \equiv 0 \pmod{2n+9}, \tag{1.6}$$
$$3003t_n \equiv 0 \pmod{2n+1}, \tag{1.7}$$
$$88179t_n \equiv 0 \pmod{10n+7}, \tag{1.8}$$
$$43263t_n \equiv 0 \pmod{10n+9}. \tag{1.9}$$

Let $\mathbb{Z}$ denote the set of integers. Another result in this paper is the following.

Theorem 1.4 *Let a, b, m, n be positive integers. Then*

$$\frac{abm}{(a+b)(m+n)}\binom{am+bm}{am}\binom{an+bn}{an} = \frac{am}{m+n}\binom{am+bm-1}{am}\binom{an+bn}{an} \in \mathbb{Z}. \tag{1.10}$$

Letting $a = b = 1$ in (1.10), we get the following result, of which a combinatorial interpretation was given by Gessel [9, Section 7].

Corollary 1.5 *Let m, n be positive integers. Then*

$$\frac{m}{2(m+n)}\binom{2m}{m}\binom{2n}{n} \in \mathbb{Z}. \tag{1.11}$$

In the next section, we give some lemmas. The proofs of Theorems 1.1–1.3 will be given in Sections 3–5 respectively. A proof of the q-analogue of Theorem 1.4 will be given in Section 6. We close our paper with some further remarks and open problems in Section 7.

2 Some lemmas

For the p-adic order of $n!$, there is a known formula

$$\mathrm{ord}_p n! = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor, \tag{2.1}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. In this section, we give some results on the floor function $\lfloor x \rfloor$.

Lemma 2.1 *For any real number x, we have*

$$\lfloor 6x \rfloor + \lfloor x \rfloor \geqslant \lfloor 3x \rfloor + 2\lfloor 2x \rfloor, \tag{2.2}$$

$$\lfloor 15x \rfloor + \lfloor 2x \rfloor \geqslant \lfloor 10x \rfloor + \lfloor 4x \rfloor + \lfloor 3x \rfloor. \tag{2.3}$$

Proof. See [6, Theorem 1.1] and one of the 52 sporadic step functions given in [6, Table 2, line# 32]. □

Lemma 2.2 *Let m and n be two positive integers such that $m|2n+3$ and $m \geqslant 5$. Then*

$$\left\lfloor \frac{6n}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{3n}{m} \right\rfloor + 2\left\lfloor \frac{2n}{m} \right\rfloor + 1. \tag{2.4}$$

Proof. Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Then (2.4) is equivalent to

$$\left\{ \frac{6n}{m} \right\} + \left\{ \frac{n}{m} \right\} = \left\{ \frac{3n}{m} \right\} + 2\left\{ \frac{2n}{m} \right\} - 1. \tag{2.5}$$

Now suppose that $m|2n+3$ and $m \geqslant 5$. We have

$$\left\{ \frac{2n}{m} \right\} = \frac{m-3}{m} > \frac{1}{3}, \quad \text{and} \quad \left\lfloor \frac{2n}{m} \right\rfloor = \frac{2n+3}{m} - 1 \equiv 0 \pmod 2.$$

It follows that

$$\left\{ \frac{6n}{m} \right\} = \begin{cases} \dfrac{2m-9}{m}, & \text{if } m = 5, 7, \\[2ex] \dfrac{m-9}{m}, & \text{if } m \geqslant 9, \end{cases}$$

$$\left\{ \frac{n}{m} \right\} = \frac{m-3}{2m},$$

$$\left\{ \frac{3n}{m} \right\} = \begin{cases} \dfrac{3m-9}{2m}, & \text{if } m = 5, 7, \\[2ex] \dfrac{m-9}{2m}, & \text{if } m \geqslant 9. \end{cases}$$

Therefore, the identity (2.5) is true for any positive integer $m \geqslant 5$. □

Lemma 2.3 *Let m and n be two positive integers such that $m|10n+3$ and $m \geqslant 9$. Then*

$$\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + 1. \tag{2.6}$$

Proof. It is easy to see that (2.6) is equivalent to

$$\left\{ \frac{15n}{m} \right\} + \left\{ \frac{2n}{m} \right\} = \left\{ \frac{10n}{m} \right\} + \left\{ \frac{4n}{m} \right\} + \left\{ \frac{3n}{m} \right\} - 1. \tag{2.7}$$

Now suppose that $m|10n+3$ and $m \geqslant 9$. We have

$$\left\{ \frac{10n}{m} \right\} = \frac{m-3}{m} \geqslant \frac{2}{3}, \quad \text{and} \quad A := \left\lfloor \frac{10n}{m} \right\rfloor = \frac{10n+3}{m} - 1 \equiv 0, 2, 6, 8 \pmod{10}.$$

It is easy to check that

$$\left\{ \frac{15n}{m} \right\} = \frac{m-9}{2m},$$

$$\left(\left\{ \frac{2n}{m} \right\}, \left\{ \frac{4n}{m} \right\}, \left\{ \frac{3n}{m} \right\} \right) = \begin{cases} \left(\frac{2m-6}{10m}, \frac{4m-12}{10m}, \frac{3m-9}{10m} \right), & \text{if } A \equiv 0 \pmod{10}, \\ \left(\frac{6m-6}{10m}, \frac{2m-12}{10m}, \frac{9m-9}{10m} \right), & \text{if } A \equiv 2 \pmod{10}, \\ \left(\frac{4m-6}{10m}, \frac{8m-12}{10m}, \frac{m-9}{10m} \right), & \text{if } A \equiv 6 \pmod{10}, \\ \left(\frac{8m-6}{10m}, \frac{6m-12}{10m}, \frac{7m-9}{10m} \right), & \text{if } A \equiv 8 \pmod{10}, \end{cases}$$

and so the identity (2.7) holds. □

3 Proofs of Theorem 1.1

First Proof. Let $\gcd(a, b)$ denote the greatest common divisor of two integers a and b. For any positive integer n, since $\gcd(2n+3, 4n+2) = 1$, to prove Theorem 1.1, it is enough to show that

$$(2n+3) \left| \frac{3\binom{6n}{3n}\binom{3n}{n}}{\binom{2n}{n}} \right. . \tag{3.1}$$

By (2.1), for any odd prime p, the p-adic order of

$$\frac{\binom{6n}{3n}\binom{3n}{n}}{(2n+3)\binom{2n}{n}} = \frac{(2n+2)!(6n)!(n)!}{(2n+3)!(3n)!(2n)!^2}$$

is given by

$$\sum_{i=1}^{\infty}\left(\left\lfloor\frac{2n+2}{p^i}\right\rfloor+\left\lfloor\frac{6n}{p^i}\right\rfloor+\left\lfloor\frac{n}{p^i}\right\rfloor-\left\lfloor\frac{2n+3}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor-2\left\lfloor\frac{2n}{p^i}\right\rfloor\right). \tag{3.2}$$

Note that

$$\left\lfloor\frac{2n+2}{p^i}\right\rfloor-\left\lfloor\frac{2n+3}{p^i}\right\rfloor=\begin{cases}-1, & \text{if } p^i|2n+3,\\ 0, & \text{otherwise.}\end{cases}$$

By Lemmas 2.1 and 2.2, for $p \geqslant 5$, the summation (3.2) is clearly greater than or equal to 0. For $p = 3$, we have (3.2) $\geqslant -1$ because if the positive integer i satisfies $3^i|2n+3$ and $3^i < 5$ then we must have $i = 1$. This proves that

$$\frac{3\binom{6n}{3n}\binom{3n}{n}}{(2n+3)\binom{2n}{n}}$$

is always an integer. Hence (3.1) holds. □

Second Proof (*provided by T. Amdeberhan and V.H. Moll*). Replacing n by $n+1$ in (1.1), we see that (after some rearrangement)

$$\frac{\binom{6n+6}{3n+3}\binom{3n+3}{n+1}}{2(2n+3)\binom{2n+2}{n+1}}=\frac{6(6n+5)(6n+1)S_n}{(n+1)(2n+3)}\in\mathbb{Z}.$$

Hence, $(2n+3)|6(6n+5)(6n+1)S_n$. Since $\gcd(2n+3,2) = \gcd(2n+3,6n+5) = \gcd(2n+3,6n+1) = 1$, we must have $(2n+3)|3S_n$. □

Remark. Z.-W. Sun [18, Conjecture 3(i)] also conjectured that S_n is odd if and only if n is a power of 2. After reading a previous version of this paper, Quan-Hui Yang told me that it is easy to show that $\mathrm{ord}_2((6n)!n!/(3n)!(2n)!^2)$ equals the number of 1's in the binary expansion of n by noticing

$$\mathrm{ord}_2(6n)! = 3n + \mathrm{ord}_2(3n)!,\quad \mathrm{ord}_2(2n)! = n + \mathrm{ord}_2 n!,$$

and using Legendre's theorem. T. Amdeberhan and V.H. Moll also pointed out this.

4 Proof of Theorem 1.2

For any positive integer n, since $\gcd(10n+3,10n+1) = 1$, to prove Theorem 1.2, it is enough to show that

$$(10n+3)\left|\frac{21\binom{15n}{5n}\binom{5n-1}{n-1}}{\binom{3n}{n}}\right.. \tag{4.1}$$

Furthermore, since $\gcd(10n+3,5)=1$ and $\binom{5n}{n}=5\binom{5n-1}{n-1}$, one sees that (4.1) is equivalent to

$$(10n+3)\,\left|\,\frac{21\binom{15n}{5n}\binom{5n}{n}}{\binom{3n}{n}}\right.. \tag{4.2}$$

By (2.1), for any odd prime p, the p-adic order of

$$\frac{\binom{15n}{5n}\binom{5n}{n}}{(10n+3)\binom{3n}{n}}=\frac{(10n+2)!(15n)!(2n)!}{(10n+3)!(10n)!(4n)!(3n)!}$$

is given by

$$\sum_{i=1}^{\infty}\left(\left\lfloor\frac{10n+2}{p^i}\right\rfloor+\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n+3}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor\right). \tag{4.3}$$

Note that

$$\left\lfloor\frac{10n+2}{p^i}\right\rfloor-\left\lfloor\frac{10n+3}{p^i}\right\rfloor=\begin{cases}-1, & \text{if } p^i|10n+3,\\ 0, & \text{otherwise.}\end{cases}$$

By Lemmas 2.1 and 2.3, for $p\geqslant 11$, or $p=5$, the summation (4.3) is clearly greater than or equal to 0. For $p=3,7$, we have (4.3) $\geqslant -1$ because there is at most one index $i\geqslant 1$ satisfying $p^i|10n+3$ and $p^i<9$ in this case. This proves that

$$\frac{21\binom{15n}{5n}\binom{5n}{n}}{(10n+3)\binom{3n}{n}}$$

is always an integer. Namely, (4.2) is true.

5 Proof of Theorem 1.3

Lemma 5.1 *Let m and n be two positive integers. Then*

$$\left\lfloor\frac{6n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor=\left\lfloor\frac{3n}{m}\right\rfloor+2\left\lfloor\frac{2n}{m}\right\rfloor+1, \tag{5.1}$$

if $m|2n+5$ and $m\geqslant 9$, or $m|2n+7$ and $m\geqslant 11$, or $m|2n+9$ and $m\geqslant 15$.

Proof. The proof is similar to that of Lemma 2.2. We only consider the case when $m|2n+5$ and $m\geqslant 9$. In this case, we have

$$\left\{\frac{2n}{m}\right\}=\frac{m-5}{m}>\frac{1}{3},\quad\text{and}\quad\left\lfloor\frac{2n}{m}\right\rfloor=\frac{2n+5}{m}-1\equiv 0\pmod 2.$$

It follows that

$$\left\{\frac{6n}{m}\right\} = \begin{cases} \dfrac{2m-15}{m}, & \text{if } m = 9, 11, 13, \\ \dfrac{m-15}{m}, & \text{if } m \geqslant 15, \end{cases}$$

$$\left\{\frac{n}{m}\right\} = \frac{m-5}{2m},$$

$$\left\{\frac{3n}{m}\right\} = \begin{cases} \dfrac{3m-15}{2m}, & \text{if } m = 9, 11, 13, \\ \dfrac{m-15}{2m}, & \text{if } m \geqslant 15, \end{cases}$$

and so

$$\left\{\frac{6n}{m}\right\} + \left\{\frac{n}{m}\right\} = \left\{\frac{3n}{m}\right\} + 2\left\{\frac{2n}{m}\right\} - 1.$$

This proves (5.1). □

Lemma 5.2 *Let m and n be two positive integers. Then*

$$\left\lfloor\frac{15n}{m}\right\rfloor + \left\lfloor\frac{2n}{m}\right\rfloor = \left\lfloor\frac{10n}{m}\right\rfloor + \left\lfloor\frac{4n}{m}\right\rfloor + \left\lfloor\frac{3n}{m}\right\rfloor + 1, \tag{5.2}$$

if $m|2n+1$ and $m \geqslant 15$, or $m|10n+7$ and $m \geqslant 21$, or $m|10n+9$ and $m \geqslant 27$.

Proof. The proof is similar to that of Lemma 2.3. We only consider the case when $m|10n+9$ and $m \geqslant 27$. In this case, we have

$$\left\{\frac{10n}{m}\right\} = \frac{m-9}{m} \geqslant \frac{2}{3}, \quad \text{and} \quad A := \left\lfloor\frac{10n}{m}\right\rfloor = \frac{10n+9}{m} - 1 \equiv 0, 2, 6, 8 \pmod{10}.$$

It follows that

$$\left\{\frac{15n}{m}\right\} = \frac{m-27}{2m},$$

$$\left(\left\{\frac{2n}{m}\right\}, \left\{\frac{4n}{m}\right\}, \left\{\frac{3n}{m}\right\}\right) = \begin{cases} \left(\dfrac{2m-18}{10m}, \dfrac{4m-36}{10m}, \dfrac{3m-27}{10m}\right), & \text{if } A \equiv 0 \pmod{10}, \\ \left(\dfrac{6m-18}{10m}, \dfrac{2m-36}{10m}, \dfrac{9m-27}{10m}\right), & \text{if } A \equiv 2 \pmod{10}, \\ \left(\dfrac{4m-18}{10m}, \dfrac{8m-36}{10m}, \dfrac{m-27}{10m}\right), & \text{if } A \equiv 6 \pmod{10}, \\ \left(\dfrac{8m-18}{10m}, \dfrac{6m-36}{10m}, \dfrac{7m-27}{10m}\right), & \text{if } A \equiv 8 \pmod{10}. \end{cases}$$

Hence,

$$\left\{\frac{15n}{m}\right\}+\left\{\frac{2n}{m}\right\}=\left\{\frac{10n}{m}\right\}+\left\{\frac{4n}{m}\right\}+\left\{\frac{3n}{m}\right\}-1,$$

which means that (5.2) holds. $\square$

Proof of Theorem 1.3. Since the proofs of the congruences (1.4)–(1.9) are similar in view of Lemmas 5.1 and 5.2, we only give proofs of (1.5) and (1.9). Noticing that $\gcd(2n+1, 2n+7)=1$ or 3, to prove (1.5), it suffices to show that

$$(2n+7)\,\Bigg|\,\frac{105\binom{6n}{3n}\binom{3n}{n}}{\binom{2n}{n}}. \tag{5.3}$$

Let

$$X_n := \frac{\binom{6n}{3n}\binom{3n}{n}}{(2n+7)\binom{2n}{n}} = \frac{(2n+6)!(6n)!(n)!}{(2n+7)!(3n)!(2n)!^2}.$$

By (2.1), for any odd prime p, we have

$$\mathrm{ord}_p X_n = \sum_{i=1}^{\infty}\left(\left\lfloor\frac{2n+6}{p^i}\right\rfloor+\left\lfloor\frac{6n}{p^i}\right\rfloor+\left\lfloor\frac{n}{p^i}\right\rfloor-\left\lfloor\frac{2n+7}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor-2\left\lfloor\frac{2n}{p^i}\right\rfloor\right).$$

Note that (5.1) is also true for $m=3$ and $n\equiv 1 \pmod 3$, and

$$\left\lfloor\frac{2n+6}{p^i}\right\rfloor-\left\lfloor\frac{2n+7}{p^i}\right\rfloor=\begin{cases}-1, & \text{if } p^i \mid 2n+7,\\ 0, & \text{otherwise.}\end{cases}$$

By Lemmas 2.1 and 5.1, we obtain

$$\begin{cases}\mathrm{ord}_p X_n \geqslant 0, & \text{if } p\geqslant 11,\\ \mathrm{ord}_p X_n \geqslant -1, & \text{if } p=3,5,7.\end{cases}$$

This proves (5.3).

Similarly, since $\gcd(10n+9, 10n+1)=\gcd(10n+9,5)=1$, the congruence (1.9) is equivalent to

$$(10n+9)\,\Bigg|\,\frac{43263\binom{15n}{5n}\binom{5n}{n}}{\binom{3n}{n}}. \tag{5.4}$$

Let

$$Y_n := \frac{\binom{15n}{5n}\binom{5n}{n}}{(10n+9)\binom{3n}{n}} = \frac{(10n+8)!(15n)!(2n)!}{(10n+9)!(10n)!(4n)!(3n)!}.$$

Then, for any odd prime p, $\mathrm{ord}_p Y_n$ is given by

$$\sum_{i=1}^{\infty}\left(\left\lfloor\frac{10n+8}{p^i}\right\rfloor+\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n+9}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor\right).$$

Note that (5.2) also holds for $m = 7, 13, 17$ and any positive integer n such that $m|10n+9$. Similarly as before, we have

$$\begin{cases} \mathrm{ord}_p Y_n \geqslant 0, & \text{if } p = 5, 7, 13, 17, \text{ or } p \geqslant 29, \\ \mathrm{ord}_p Y_n \geqslant -1, & \text{if } p = 11, 19, 23, \\ \mathrm{ord}_p Y_n \geqslant -2, & \text{if } p = 3. \end{cases}$$

Observing that $43263 = 3^2 \cdot 11 \cdot 19 \cdot 23$, we complete the proof of (5.4).

6 A q-analogue of Theorem 1.4

Recall that the *q-binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

We begin with the announced strengthening of Theorem 1.4.

Theorem 6.1 *Let $a, b, m, n \geqslant 1$. Then*

$$\frac{1-q^{\gcd(am,m+n)}}{1-q^{m+n}}\begin{bmatrix} am+bm-1 \\ am \end{bmatrix}_q \begin{bmatrix} an+bn \\ an \end{bmatrix}_q \tag{6.1}$$

is a polynomial in q with non-negative integer coefficients.

Corollary 6.2 *Let $a, b, m, n \geqslant 1$. Then*

$$\frac{1-q^{am}}{1-q^{m+n}}\begin{bmatrix} am+bm-1 \\ am \end{bmatrix}_q \begin{bmatrix} an+bn \\ an \end{bmatrix}_q \tag{6.2}$$

is a polynomial in q with non-negative integer coefficients.

It is easily seen that Theorem 1.4 can be obtained upon letting $q \to 1$ in Corollary 6.2. Moreover, when $a = b = m = 1$, the numbers (6.2) reduce to the q-Catalan numbers

$$C_n(q) = \frac{1-q}{1-q^{2n+1}}\begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

It is well known that the q-Catalan numbers $C_n(q)$ are polynomials with non-negative integer coefficients (see [2, 3, 5, 7]). There are many different q-analogues of the Catalan numbers (see Fürlinger and Hofbauer [7]). For the so-called q,t-Catalan numbers, see [8, 13, 12].

Recall that a polynomial $P(q) = \sum_{i=0}^{d} p_i q^i$ in q of degree d is called *reciprocal* if $p_i = p_{d-i}$ for all i, and that it is called *unimodal* if there is an integer r with $0 \leqslant r \leqslant d$ and $0 \leqslant p_0 \leqslant \cdots \leqslant p_r \geqslant \cdots \geqslant p_d \geqslant 0$. An elementary but crucial property of reciprocal and unimodal polynomials is the following.

Lemma 6.3 *If $A(q)$ and $B(q)$ are reciprocal and unimodal polynomials, then so is their product $A(q)B(q)$.*

Lemma 6.3 is well known and its proof can be found, e.g., in [1] or [16, Proposition 1].

Similarly to the proof of [11, Theorem 3.1], the following lemma plays an important role in the proof of Theorem 6.1. It is a slight generalization of [14, Proposition 10.1.(iii)], which extracts the essentials out of Andrews [4, Proof of Theorem 2].

Lemma 6.4 *Let $P(q)$ be a reciprocal and unimodal polynomial and m and n positive integers with $m \leqslant n$. Furthermore, assume that $A(q) = \frac{1-q^m}{1-q^n} P(q)$ is a polynomial in q. Then $A(q)$ has non-negative coefficients.*

Proof. See [11, Lemma 5.1]. □

Proof of Theorem 6.1. It is well known that the q-binomial coefficients are reciprocal and unimodal polynomials in q (cf. [17, Ex. 7.75.d]), and by Lemma 6.3, so is the product of two q-binomial coefficients. In view of Lemma 6.4, for proving Theorem 6.1 it is enough to show that the expression (6.1) is a polynomial in q. We shall accomplish this by a count of cyclotomic polynomials.

Recall the well-known fact that

$$q^n - 1 = \prod_{d|n} \Phi_d(q),$$

where $\Phi_d(q)$ denotes the d-th cyclotomic polynomial in q. Consequently,

$$\frac{1 - q^{\gcd(am,m+n)}}{1 - q^{m+n}} \begin{bmatrix} am + bm - 1 \\ am \end{bmatrix}_q \begin{bmatrix} an + bn \\ an \end{bmatrix}_q = \prod_{d=2}^{\min\{am+bm-1,\, an+bn\}} \Phi_d(q)^{e_d},$$

with

$$\begin{aligned} e_d = \chi(d \mid \gcd(am, m+n)) - \chi(d \mid m+n) + \left\lfloor \frac{am+bm-1}{d} \right\rfloor + \left\lfloor \frac{an+bn}{d} \right\rfloor \\ - \left\lfloor \frac{am}{d} \right\rfloor - \left\lfloor \frac{bm-1}{d} \right\rfloor - \left\lfloor \frac{an}{d} \right\rfloor - \left\lfloor \frac{bn}{d} \right\rfloor, \end{aligned} \tag{6.3}$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. This is clearly non-negative, unless $d \mid m+n$ and $d \nmid \gcd(am, m+n)$.

So, let us assume that $d \mid m+n$ and $d \nmid \gcd(am, m+n)$, which means that $d \nmid am$ and therefore

$$\begin{aligned}\left\lfloor\frac{am+bm-1}{d}\right\rfloor + \left\lfloor\frac{an+bn}{d}\right\rfloor &= \frac{(a+b)(m+n)}{d} - 1,\\ \left\lfloor\frac{am}{d}\right\rfloor + \left\lfloor\frac{an}{d}\right\rfloor &= \frac{a(m+n)}{d} - 1,\\ \left\lfloor\frac{bm-1}{d}\right\rfloor + \left\lfloor\frac{bn}{d}\right\rfloor &= \frac{b(m+n)}{d} - 1,\end{aligned}$$

and so $e_d = 0$ is also non-negative in this case. This completes the proof of polynomiality of (6.1). $\square$

Proof of Corollary 6.2. This follows immediately from Theorem 6.1 and the fact that $\gcd(am, m+n) \mid am$. $\square$

7 Concluding remarks and open problems

On January 2, 2014 T. Amdeberhan and V.H. Moll (personal communication) found the following generalization of Theorem 1.1, which was soon proved by Q.-H. Yang [21] and C. Krattenthaler.

Conjecture 7.1 *Let a, b and n be positive integers with $a > b$. Then*

$$(2bn+1)(2bn+3)\binom{2bn}{bn} \,\Big|\, 3(a-b)(3a-b)\binom{2an}{an}\binom{an}{bn}.$$

Let $[m]! = (1-q)\cdots(1-q^m)$. By a result of Warnaar and Zudilin [20, Proposition 3], one sees that, for any positive integer n, the polynomial

$$\frac{[6n]![n]!}{[3n]![2n]!^2}$$

has non-negative integer coefficients. Similarly as before, we can prove the following generalization of congruences (1.3)–(1.5).

Theorem 7.2 *Let n be a positive integer. Then all of*

$$\frac{(1-q)[6n]![n]!}{(1-q^{2n+1})[3n]![2n]!^2}, \quad \frac{(1-q^3)[6n]![n]!}{(1-q^{2n+3})[3n]![2n]!^2}, \quad \frac{(1-q)(1-q^3)[6n]![n]!}{(1-q^{2n+1})(1-q^{2n+3})[3n]![2n]!^2},$$

$$\frac{(1-q^3)(1-q^5)(1-q^7)[6n]![n]!}{(1-q^{2n+3})(1-q^{2n+5})(1-q^{2n+7})[3n]![2n]!^2} \quad (n \geqslant 2),$$

$$\frac{(1-q^3)^2(1-q^5)(1-q^7)[6n]![n]!}{(1-q^{2n+1})(1-q^{2n+3})(1-q^{2n+5})(1-q^{2n+7})[3n]![2n]!^2} \quad (n \geqslant 2)$$

are polynomials in q.

We have the following two related conjectures.

Conjecture 7.3 *All the polynomials in Theorem* 7.2 *have non-negative integer coefficients.*

Conjecture 7.4 *Let* $n \geqslant 2$. *Then the polynomial* $\frac{[6n]![n]!}{[3n]![2n]!^2}$ *is unimodal.*

It is obvious that the polynomial $\frac{[6n]![n]!}{[3n]![2n]!^2}$ is reciprocal. If Conjecture 7.4 is true, then, applying Lemma 6.3, we conclude that the first two polynomials in Theorem 7.2 have non-negative integer coefficients.

It was conjectured by Warnaar and Zudilin (see [20, Conjecture 1]) that

$$\frac{[15n]![2n]!}{[10n]![4n]![3n]!}$$

has non-negative integer coefficients. Similarly, we have the following generalization of Theorem 1.2.

Theorem 7.5 *Let* n *be a positive integer. Then both*

$$\frac{(1-q)[15n]![2n]!}{(1-q^{10n+1})[10n]![4n]![3n]!}, \quad \textit{and} \quad \frac{(1-q^3)(1-q^7)[15n]![2n]!}{(1-q)(1-q^{10n+3})[10n]![4n]![3n]!}$$

are polynomials in q.

We end the paper with the following conjecture, strengthening the above theorem.

Conjecture 7.6 *The two polynomials in Theorem* 7.5 *have non-negative integer coefficients.*

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References

[1] G.E. Andrews, A theorem on reciprocal polynomials with applications to permutations and compositions, *Amer. Math. Monthly* 82 (1975), 830–833.

[2] G.E. Andrews, Catalan numbers, q-Catalan numbers and hypergeometric series, *J. Combin. Theory Ser. A* 44 (1987), 267–273.

[3] G.E. Andrews, On the difference of successive Gaussian polynomials, *J. Statist. Plann. Inference* 34 (1993), 19–22.

[4] G.E. Andrews, The Friedman–Joichi–Stanton monotonicity conjecture at primes, *Unusual Applications of Number Theory* (M. Nathanson, ed.), *DIMACS Ser. Discrete Math. Theor. Comp. Sci.*, vol. 64, *Amer. Math. Soc.*, Providence, R.I., 2004, pp. 9–15.

[5] G.E. Andrews, q-Catalan identities, in: *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, Springer, New York, 2010, pp. 183–190.

[6] J.W. Bober, Factorial ratios, hypergeometric series, and a family of step functions, *J. Lond. Math. Soc.* 79 (2009), 422–444.

[7] J. Fürlinger and J. Hofbauer, q-Catalan numbers, *J. Combin. Theory Ser. A* 40 (1985), 248–264.

[8] A.M. Garsia and J. Haglund, A proof of the q, t-Catalan positivity conjecture, *Discrete Math.* 256 (2002), 677–717.

[9] I.M. Gessel, Super ballot numbers, *J. Symbolic Comput.* 14 (1992), 179–194.

[10] V.J.W. Guo, Proof of Sun's conjecture on the divisibility of certain binomial sums, *Electron. J. Combin.* 20(4) (2013), #P20.

[11] V.J.W. Guo and C. Krattenthaler, Some divisibility properties of binomial and q-binomial coefficients, *J. Number Theory* 135 (2014), 167–184.

[12] J. Haglund, Conjectured statistics for the q, t-Catalan numbers, *Adv. Math.* 175 (2003), 319–334.

[13] M. Haiman, t, q-Catalan numbers and the Hilbert scheme, *Discrete Math.* 193 (1998), 201–224.

[14] V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* 108 (2004), 17–50.

[15] M.R. Sepanski, On divisibility of convolutions of central binomial coefficients, *Electron. J. Combin.* 21(1) (2014), #P1.32.

[16] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in: *Graph Theory and Its Applications: East and West (Jinan, 1986), Ann. New York Acad. Sci.*, 576, New York Acad. Sci., New York, 1989, pp. 500–535.

[17] R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.

[18] Z.-W. Sun, On divisibility of binomial coefficients, *J. Austral. Math. Soc.* 93 (2012), 189–201.

[19] Z.-W. Sun, Products and sums divisible by central binomial coefficients, *Electron. J. Combin.* 20(1) (2013), #P9.

[20] S. O. Warnaar and W. Zudilin, A q-rious positivity, *Aequat. Math.* 81 (2011), 177–183.

[21] Q.-H. Yang, Proof of a conjecture related to divisibility properties of binomial coefficients, preprint, 2014, arXiv:1401.1108.