Partitions of $\mathbf{Z}_m$ with the same weighted representation functions

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Abstract

Let $\mathbf{k} = (k_1, k_2, \cdots, k_t)$ be a t-tuple of integers, and m be a positive integer. For a subset $A \subset \mathbf{Z}_m$ and any $n \in \mathbf{Z}_m$, let $r_A^{\mathbf{k}}(n)$ denote the number of solutions of the equation $k_1 a_1 + \cdots + k_t a_t = n$ with $a_1, \cdots, a_t \in A$. In this paper, we give a necessary and sufficient condition on $(\mathbf{k}, m)$ such that there exists a subset $A \subset \mathbf{Z}_m$ satisifying $r_A^{\mathbf{k}} = r_{\mathbf{Z}_m \setminus A}^{\mathbf{k}}$. This settles a problem of Yang and Chen.

Keywords: Representation function, Partition, Sárközy problem.

1 Introduction

We use $\mathbf{N}$ to denote the set of nonnegative integers. For any subset $A \subset \mathbf{N}$ and $n \in \mathbf{N}$, define the representation functions $R_1(A, n)$, $R_2(A, n)$ and $R_3(A, n)$ to be the number of solutions of the equations

$$n = a + a', \quad a, a' \in A,$$
$$n = a + a', \quad a, a' \in A, \quad a < a',$$

and

$$n = a + a', \quad a, a' \in A, \quad a \leqslant a',$$

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respectively. Representation functions first appeared in the celebrated paper of Erdős and Turán [12], and were extensively studied by Erdős, Sárközy and Sós (see [7, 8, 11, 9, 10]).

Sárközy asked for each $i = 1, 2, 3$, whether there exist sets A and B with infinite symmetric difference such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers n. There have been quite some work around Sárközy's problem. Dombi [5] observed that the answer is negative for $i = 1$, and constructed a subset $A \subset \mathbf{N}$ such that $R_2(A, n) = R_2(\mathbf{N}\backslash A, n)$ for all $n \in \mathbf{N}$. An analogous example for $R_3(A, n)$ was constructed by Chen and Wang [3]. For $i = 2, 3$, Lev [6], Sándor [13] and Tang [14] determined all subsets $A \subset \mathbf{N}$ such that $R_i(A, n) = R_i(\mathbf{N}\backslash A, n)$ for all $n \geqslant 2N - 1$. The asymptotic behavior of the representation functions of these special sequences was studied by Chen and Tang (see [1, 2]).

Analogously, for any two positive integers k_1, k_2, any subset $A \subset \mathbf{N}$, one can define the weighted representation function $r_{k_1,k_2}(A, n)$ as the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. Cilleruelo and Rué [4] proved that $r_{k_1,k_2}(A, n)$ can not be eventually constant. Yang and Chen [15] proved that there exists a set $A \subset \mathbf{N}$ such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(\mathbf{N}\backslash A, n)$ for all sufficiently large n if and only if $k_1 \mid k_2$ and $k_1 < k_2$.

Let $\mathbf{k} = (k_1, k_2, \cdots, k_t)$ be a t-tuple of integers, and m be a positive integer. For any $A \subset \mathbf{Z}_m$ and $n \in \mathbf{Z}_m$, denote the number of solutions of the equation $k_1a_1 + \cdots + k_ta_t = n$ with $a_1, \cdots, a_t \in A$ by $r_A^{\mathbf{k}}(n)$. We call $r_A^{\mathbf{k}}$ the weighted representation function on $\mathbf{Z}_m$ with respect to A and weight $\mathbf{k}$. For $t = 2$, $\mathbf{k} = (k_1, k_2)$, Yang and Chen [16] characterized all subsets $A \subset \mathbf{Z}_m$ with the property that $r_A^{\mathbf{k}} = r_{\mathbf{Z}_m\backslash A}^{\mathbf{k}}$.

Note that if $A \subset \mathbf{Z}_m$ satisfying $r_A^{\mathbf{k}} = r_{\mathbf{Z}_m\backslash A}^{\mathbf{k}}$, then m is even and $|A| = \frac{m}{2}$. Indeed, this follows from the fact that

$$|A|^t = \sum_{n\in\mathbf{Z}_m} r_A^{\mathbf{k}}(n) = \sum_{n\in\mathbf{Z}_m} r_B^{\mathbf{k}}(n) = |B|^t.$$

For any nonzero integer k, we use $v_2(k)$ to denote the largest nonnegative integer l such that $2^l \mid k$. The following result is also proved in [16].

Theorem 1. *Let k_1, k_2 be nonzero integers, and $\mathbf{k} = (k_1, k_2)$. For a subset $A \subset \mathbf{Z}_m$ satisfying $r_A^{\mathbf{k}} = r_{\mathbf{Z}_m\backslash A}^{\mathbf{k}}$ to exist, it is necessary and sufficient that one of the following holds:*

(i) $k_1 + k_2$ is even;

(ii) $k_1 + k_2$ is odd and $v_2(k_1k_2) < v_2(m)$.

It is natural to consider the following problem suggested by Yang and Chen [16].

Problem 2. For $t \geqslant 3$, determine all $\mathbf{k} = (k_1, k_2, \cdots, k_t)$ and m such that there exists a subset $A \subset \mathbf{Z}_m$ with the property that $r_A^{\mathbf{k}} = r_{\mathbf{Z}_m\backslash A}^{\mathbf{k}}$.

In this paper, we give a complete answer to this problem. Since $k_1, \cdots, k_t$ are only considered modulo m, we may assume $k_1, \cdots, k_t$ are all positive integers, and write

$$|\mathbf{k}| = k_1 + k_2 + \cdots + k_t.$$

Let $\mathcal{A}(\mathbf{k},m)$ be the set of all subsets $A \subset \mathbf{Z}_m$ such that $r_A^{\mathbf{k}} = r_{\mathbf{Z}_m\setminus A}^{\mathbf{k}}$. We also identify an integer with its canonical image in $\mathbf{Z}_m$. Our main result is the following.

Theorem 3. *The following statements are equivalent:*

(i) $\mathcal{A}(\mathbf{k},m)$ *is nonempty;*

(ii) $\{0,1,\cdots,[\frac{m}{2}]-1\} \in \mathcal{A}(\mathbf{k},m)$*;*

(iii) m *is even, and either* $|\mathbf{k}|$ *is even, or* $0 < v_2(k_i) < v_2(m)$ *for some* $i \in [1,t]$*.*

Currently we have no answer for the following problem.

Problem 4. Determine the set $\mathcal{A}(\mathbf{k},m)$.

We give an example illustrating the complexity of Problem 4. For any even divisor $s \mid m$, a subset $A \subset \mathbf{Z}_m$ is said to be *balanced modulo* s if for any integer k, we have

$$|\{a \in A : a \equiv k \pmod{s}\}| = |\{a \in A : a \equiv k + \frac{s}{2} \pmod{s}\}|.$$

Example 5. Let $m = 2^l$, $l \geqslant 2$, $k_1 = 2$, $k_2 = \cdots = k_t = 1$.

(i) If t is even, then $A \in \mathcal{A}(\mathbf{k},m)$ if and only if $|A| = m/2$, and A is balanced modulo 2, in other words, A has same number of odd elements and even elements.

(ii) If t is odd, then $A \in \mathcal{A}(\mathbf{k},m)$ if and only if $|A| = m/2$, and for any integer $s \in [2,l]$, A is balanced modulo 2^{s-1} or 2^s, or both.

2 Proofs

For a subset $A \subset \mathbf{Z}_m$, we always use B to denote the complement $\mathbf{Z}_m \setminus A$. Let

$$f_A(x) = \sum_{a\in A} x^a,$$

and

$$T(x) = \prod_{i=1}^{t} f_A(x^{k_i}) - \prod_{i=1}^{t} f_B(x^{k_i}).$$

These polynomials are considered in the ring $\mathbf{Z}[x]/(x^m - 1)$.

Lemma 6. $A \in \mathcal{A}(\mathbf{k},m)$ *if and only if* $T(x) = 0$*.*

Proof. Since

$$\prod_{i=1}^{t} f_A(x^{k_i}) = \sum_{a_1,\cdots,a_t\in A} x^{k_1a_1+\cdots+k_ta_t} = \sum_{n\in\mathbf{Z}_m} r_A^{\mathbf{k}}(n)x^n,$$

and similarly

$$\prod_{i=1}^{t} f_B(x^{k_i}) = \sum_{n\in\mathbf{Z}_m} r_B^{\mathbf{k}}(n)x^n,$$

we conclude that $A \in \mathcal{A}(\mathbf{k},m)$ if and only if $T(x) = 0$. ☐

Let $d_i = (k_i, m)$, $i \in [1, t]$. For any positive integer d, we write $\xi_d = e^{2\pi i/d}$. For $d \mid m$, it makes sense to write $f(\xi_d)$ for $f(x) \in \mathbf{Z}[x]/(x^m - 1)$, and we use $I(d)$ to denote the set of indices $i \in [1, t]$ such that $d \nmid d_i$.

Lemma 7. *For $A \subset \mathbf{Z}_m$ with $|A| = m/2$, $A \in \mathcal{A}(\mathbf{k}, m)$ if and only if for any positive divisor $d \mid m$ with $|I(d)|$ odd, there exists $i \in I(d)$ such that $f_A(\xi_{d/(d,d_i)}) = 0$.*

Proof. By Lemma 6, $A \in \mathcal{A}(\mathbf{k}, m)$ if and only if $T(x) = 0$. This is true if and only if for every positive divisor $d \mid m$, we have $T(\xi_d) = 0$. For any $d \mid m$, if $d \mid k_i$, then

$$f_A(\xi_d^{k_i}) = f_B(\xi_d^{k_i}) = m/2. \tag{1}$$

If $d \nmid k_i$, then

$$f_A(\xi_d^{k_i}) + f_B(\xi_d^{k_i}) = \sum_{n=0}^{m-1} \xi_d^{nk_i} = 0,$$

thus

$$f_A(\xi_d^{k_i}) = -f_B(\xi_d^{k_i}). \tag{2}$$

Combining (1) and (2), we have

$$T(\xi_d) = \left(\frac{m}{2}\right)^{t-|I(d)|} \left(1 - (-1)^{|I(d)|}\right) \prod_{i \in I(d)} f_A(\xi_d^{k_i}).$$

If $|I(d)|$ is even, it is always true that $T(\xi_d) = 0$. If $|I(d)|$ is odd, then $T(\xi_d) = 0$ if and only if $f_A(\xi_d^{k_i}) = 0$ for some $i \in I(d)$. Since $\xi_d^{k_i}$ is a $d/(d, d_i)$-th primitive root of unity and f has rational coefficients, any primitive $d/(d, d_i)$-th root of unity is a root of f_A. In particular $f_A(\xi_{d/(d,d_i)}) = 0$ and vice versa. This completes the proof of Lemma 7. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. (ii)⇒(i) is trivial.

We now show that (i)⇒(iii). Assuming $\mathcal{A}(\mathbf{k}, m)$ is nonempty, m must be even. Suppose on the contrary that (iii) fails, then $|\mathbf{k}|$ is odd and either $v_2(k_i) = 0$ or $v_2(k_i) \geqslant v_2(m)$ for every $i \in [1, t]$, and it is clear that the number of $i \in [1, t]$ with $v_2(k_i) = 0$ is odd. For any positive number $s \leqslant v_2(m) =: l$, consider $d = 2^s \mid m$. Since $I(d) = \{i \in [1, t] : v_2(k_i) = 0\}$, $|I(d)|$ is odd. By Lemma 7, we have $f_A(\xi_d) = 0$. Since this is true for all $s \leqslant l$, we conclude that the product of all 2^s-th cyclotomic polynomials for $s \in [1, l]$ divides $f_A(x)$, i.e.

$$1 + x + \cdots + x^{2^l - 1} \mid f_A(x).$$

For $i \in [0, 2^l - 1]$, let n_i denote the number of elements $a \in A$ such that $a \equiv i \pmod{2^l}$. Then

$$f_A(x) = \sum_{a \in A} x^a \equiv \sum_{i=0}^{2^l - 1} n_i x^i \pmod{1 + x + \cdots + x^{2^l - 1}},$$

hence

$$1+x+\cdots+x^{2^l-1} \mid \sum_{i=0}^{2^l-1} n_i x^i.$$

It follows that $n_0 = n_1 = \cdots = n_{2^l-1} =: n$, $|A| = 2^l n$. However $|A| = m/2$, $v_2(|A|) = v_2(m) - 1 = l - 1$, and this contradicts $|A| = 2^l n$, therefore (iii) is true.

Finally we show that (iii)⇒(ii). So m is even, and we put

$$A = \{0, 1, \cdots, \frac{m}{2} - 1\}.$$

Then

$$f_B(x) = x^{m/2} f_A(x),$$

and

$$T(x) = \left(1 - x^{|\mathbf{k}|m/2}\right) \prod_{i=1}^{t} f_A(x^{k_i}).$$

If $|\mathbf{k}|$ is even, then $x^m - 1$ divides $1 - x^{|\mathbf{k}|m/2}$, thus $T(x) = 0$. By Lemma 6, we have $A \in \mathcal{A}(\mathbf{k}, m)$. Now suppose $|\mathbf{k}|$ is odd, and there exists $j \in [1, t]$ such that $0 < v_2(k_j) < v_2(m)$. Let d be any positive divisor of m such that $|I(d)|$ is odd. If $d \mid m/2$, then for any $i \in I(d)$, letting $d' = d/(d, d_i)$, we have

$$f_A(\xi_{d'}) = \sum_{i=0}^{m/2-1} \xi_{d'}^i = \frac{\xi_{d'}^{m/2} - 1}{\xi_{d'} - 1} = 0.$$

If $d \nmid m/2$, then $v_2(d) = v_2(m)$, and we have $j \in I(d)$. Since $2 \mid (d_j, d)$, therefore $d/(d, d_j) \mid m/2$. Let $d' = d/(d_j, d)$, then again,

$$f_A(\xi_{d'}) = \sum_{i=0}^{m/2-1} \xi_{d'}^i = \frac{\xi_{d'}^{m/2} - 1}{\xi_{d'} - 1} = 0.$$

By Lemma 7, we conclude that $A \in \mathcal{A}(\mathbf{k}, m)$. This completes the proof of Theorem 3. □

We now explain Example 5. Assume therefore that $m = 2^l$, $l \geqslant 2$, $k_1 = 2$, $k_2 = \cdots = k_t = 1$, and $A \subset \mathbf{Z}_m$ with $|A| = m/2$.

Lemma 8. *For any integer $s \in [1, l]$, $f_A(\xi_{2^s}) = 0$ if and only if A is blanced modulo 2^s.*

Proof. For $k \in [0, 2^s - 1]$, let n_k denote the number of elements $a \in A$ such that $a \equiv k \pmod{2^s}$. $f_A(\xi_{2^s}) = 0$ if and only if $(1 + x^{2^{s-1}}) \mid f_A(x)$. We have

$$f_A(x) = \sum_{a \in A} x^a \equiv \sum_{k=0}^{2^{s-1}-1} (n_k - n_{k+2^{s-1}}) x^k \pmod{1 + x^{2^{s-1}}}.$$

It follows that $(1 + x^{2^{s-1}}) \mid f_A(x)$ if and only if $n_k = n_{k+2^{s-1}}$ for any $k \in [0, 2^{s-1} - 1]$, i.e. A is balanced modulo 2^s. □

Explanation of Example 5. If t is even, consider $d \mid m$ with $|I(d)|$ odd, it is easy to see that $d = 2$. By Lemma 7, $A \in \mathcal{A}(\mathbf{k}, m)$ if and only if $f_A(\xi_2) = 0$. By Lemma 8, this is equivalent to A being balanced modulo 2.

If t is odd, then $d \mid m$ with $|I(d)|$ odd if and only if $d = 2^s$ such that $2 \leqslant s \leqslant l$. By Lemma 7, $A \in \mathcal{A}(\mathbf{k}, m)$ if and only if for any $s \in [2, l]$, we have either $f_A(\xi_{2^s}) = 0$ or $f_A(\xi_{2^{s-1}}) = 0$. By Lemma 8, this is equivalent to A being balanced modulo 2^{s-1} or 2^s, or both. □

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