An Erdős-Ko-Rado theorem for permutations with fixed number of cycles

Cheng Yeaw Ku
Department of Mathematics
National University of Singapore
Singapore 117543
matkcy@nus.edu.sg

Kok Bin Wong
Institute of Mathematical Sciences
University of Malaya
50603 Kuala Lumpur, Malaysia
kbwong@um.edu.my

Submitted: Feb 4, 2014; Accepted: Jul 18, 2014; Published: Jul 25, 2014
Mathematics Subject Classifications: 05D05

Abstract

Let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For a positive integer k, define $S_{n,k}$ to be the set of all permutations of $[n]$ with exactly k disjoint cycles, i.e.,

$$S_{n,k} = \{\pi \in S_n : \pi = c_1c_2 \cdots c_k\},$$

where $c_1, c_2, \ldots, c_k$ are disjoint cycles. The size of $S_{n,k}$ is $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k}s(n, k)$, where $s(n, k)$ is the Stirling number of the first kind. A family $\mathcal{A} \subseteq S_{n,k}$ is said to be *t-cycle-intersecting* if any two elements of $\mathcal{A}$ have at least t common cycles. In this paper we show that, given any positive integers k, t with $k \geqslant t + 1$, if $\mathcal{A} \subseteq S_{n,k}$ is t-cycle-intersecting and $n \geqslant n_0(k, t)$ where $n_0(k, t) = O(k^{t+2})$, then

$$|\mathcal{A}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix},$$

with equality if and only if $\mathcal{A}$ is the stabiliser of t fixed points.

Keywords: t-intersecting family; Erdős-Ko-Rado; permutations; Stirling number of the first kind

1 Introduction

Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \geqslant t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1 (Erdős, Ko, and Rado [12], Frankl [13], Wilson [36]). *Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \geqslant (k-t+1)(t+1)$, we have*

$$|\mathcal{A}| \leqslant \binom{n-t}{k-t}.$$

Moreover, if $n > (k-t+1)(t+1)$ then equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : T \subseteq A\}$ for some t-set T.

Later, Ahlswede and Khachatrian [1] extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n. There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 27, 28, 29, 30, 32, 33, 34, 35, 37]).

Let S_n denote the set of permutations of $[n]$. A family of S_n is said to be *t-intersecting* if any two permutations in the family agree in at least t points. In 1977, Deza and Frankl [9] proved that a 1-intersecting family has size at most $(n-1)!$. Much later, it was proved by several authors via different approaches that the only families achieving the maximum size are the cosets of stabilisers of a point (see [7, 15, 29, 35]). More recently, Ellis, Friedgut and Pilpel [11] have settled an old conjecture of Deza and Frankl regarding an analogue of the Erdős-Ko-Rado theorem for permutations. In particular, they showed that for sufficiently large n depending on t, a t-intersecting family $\mathcal{A}$ of S_n has size at most $(n-t)!$, with equality if and only if $\mathcal{A}$ is a coset of the stabilizer of t points. Their proof uses spectral methods and representations of the symmetric group.

The concept of t-cycle intersection for permutations was introduced by Ku and Renshaw [23]. A family $\mathcal{A}$ of permutations is *t-cycle-intersecting* if any two permutations in $\mathcal{A}$, when written in their cycle decomposition form, have at least t cycles in common. Obviously, if $\mathcal{A}$ is t-cycle-intersecting, it is t-intersecting but the converse is not true.

In this paper, we are interested in the t-cycle-intersection problem for permutations that have a prescribed number of cycles. For a positive integer k, define $S_{n,k}$ to be the set of all permutations of $[n]$ with exactly k disjoint cycles, i.e.,

$$S_{n,k} = \{\pi \in S_n : \pi = c_1 c_2 \cdots c_k\},$$

where $c_1, c_2, \ldots, c_k$ are disjoint cycles. It is well known that the size of $S_{n,k}$ is given by $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)$, where $s(n,k)$ is the *Stirling number of the first kind.*

We shall use the following notations:

(a) Let $N(c) = \{a_1, a_2, \ldots, a_l\}$ denote the set of points occurring in the cycle $c = (a_1, a_2, \ldots, a_l)$;

(b) Let $M(\pi) = \{c_1, c_2, \ldots, c_k\}$ denote the set of cycles in the cycle decomposition of the permutation $\pi = c_1 c_2 \ldots c_k \in S_{n,k}$.

Thus, a family $\mathcal{A} \subseteq S_{n,k}$ is t-cycle-intersecting if $|M(\pi_1) \cap M(\pi_2)| \geqslant t$ for any $\pi_1, \pi_2 \in \mathcal{A}$.

Theorem 2. *Suppose k, t are positive integers with $k \geqslant t+1$. There exists a function $n_0(k,t) = O(k^{t+2})$ such that if $\mathcal{A} \subseteq S_{n,k}$ is t-cycle-intersecting and $n \geqslant n_0(k,t)$, then*

$$|\mathcal{A}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix},$$

with equality if and only if $\mathcal{A}$ is the stabiliser of t fixed points, i.e. $\mathcal{A}$ consists of all permutations in $S_{n,k}$ with some t fixed cycles of length one.

2 Stirling number revisited

The unsigned Stirling number $\begin{bmatrix} n \\ k \end{bmatrix}$ satisfies the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}, \tag{1}$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} = 0$, $n > 0$. Note that $\begin{bmatrix} n \\ n \end{bmatrix} = 1$, and $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$.

It is well-known that the sequence $\begin{bmatrix} n \\ k \end{bmatrix}$, $1 \leqslant k \leqslant n$, is *strongly log-concave* (SLC):

$$\begin{bmatrix} n \\ k \end{bmatrix}^2 > \begin{bmatrix} n \\ k+1 \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix}, \quad \text{for } 2 \leqslant k \leqslant n-1. \tag{2}$$

Using Newton's inequality for symmetric functions, Lieb [31] obtained the following inequality which implies the SLC property.

Theorem 3 (Lieb [31])**.** *The following sequences are strictly decreasing for any $n = 3, 4, \ldots$:*

$$\frac{k-1}{n-k+1} \begin{bmatrix} n \\ k \end{bmatrix} \Big/ \begin{bmatrix} n \\ k-1 \end{bmatrix}, \quad \text{for } 2 \leqslant k \leqslant n. \tag{3}$$

Corollary 4. *The following inequalities are obtained by considering the ends of the sequence in (3):*

$$\frac{n-k+1}{(k-1)(n-1)} H_{n-1} \geqslant \begin{bmatrix} n \\ k \end{bmatrix} \Big/ \begin{bmatrix} n \\ k-1 \end{bmatrix} \geqslant \frac{2(n-k+1)}{(k-1)n}, \quad \text{for } 2 \leqslant k \leqslant n, \tag{4}$$

where $H_m = \sum_{i=1}^{m} \frac{1}{i}$ is the m-th Harmonic number.

3 Proof of Theorem 2

We may assume that $\mathcal{A}$ is maximally t-intersecting (with respect to inclusion).

Suppose $k = t+1$. Since $\mathcal{A}$ is t-intersecting, there are $\pi_1, \pi_2 \in \mathcal{A}$ such that

$$\pi_1 = c_1 c_2 \dots c_t d_1$$
$$\pi_2 = c_1 c_2 \dots c_t d_2$$

where $c_1, \dots, c_t, d_1$ are disjoint cycles, $d_2 \neq d_1$ and $N(d_2) = N(d_1)$. Suppose there is a $\pi \in \mathcal{A}$ with $c_{i_0} \notin M(\pi)$ for some i_0. Then $d_1, d_2 \in M(\pi)$. But this is impossible as $N(d_1) = N(d_2)$. Hence,

$$\mathcal{A} = \{\pi \in S_{n,k} \ : \ c_i \in M(\pi) \ \text{ for } \ i = 1, 2, \dots, t\},$$

since $\mathcal{A}$ is maximally t-intersecting. Let $P = \bigcup_{i=1}^{t} N(c_i)$. Then

$$|\mathcal{A}| = \begin{bmatrix} n-|P| \\ 1 \end{bmatrix} \leqslant \begin{bmatrix} n-t \\ 1 \end{bmatrix},$$

with equality if and only if $|N(c_i)| = 1$ for $i = 1, 2 \dots, t$, i.e., $\mathcal{A}$ is the stabilizer of at least t fixed points. Now, if $n \geqslant t+2$, then $\mathcal{A}$ is the stabilizer of t fixed points.

From now on, we may suppose that $k \geqslant t+2$ and that $\mathcal{A} \subseteq S_{n,k}$ is a t-intersecting family of maximum size. Assuming that $\mathcal{A}$ is not the stabilizer of t points, we shall prove that $|\mathcal{A}| < \begin{bmatrix} n-t \\ k-t \end{bmatrix}$.

Pick any $\pi \in \mathcal{A}$ and assume that $\pi = c_1 c_2 \cdots c_k$, where $k \geqslant t+1$. Let $T = \{c_1, \dots, c_t\}$. We set

$$\mathcal{A}(T) = \{\pi \in \mathcal{A} \ : \ T \subseteq M(\pi)\}.$$

If $\mathcal{A} \subseteq \mathcal{A}(T)$, then at least one of the c_i, $1 \leqslant i \leqslant t$, must be of size greater than 1 (otherwise $\mathcal{A}$ will be the stabilizer of t points). Thus, in this case, $|\mathcal{A}| \leqslant |\mathcal{A}(T)| \leqslant \begin{bmatrix} n-\sum_{i=1}^{t} |N(c_i)| \\ k-t \end{bmatrix} < \begin{bmatrix} n-t \\ k-t \end{bmatrix}$, where the last inequality follows from (1).

Therefore, there must be a permutation $\sigma \in \mathcal{A}$ that does not contain all the t-cycles in T. Every permutation in the subfamily $\mathcal{A}(T)$ must t-intersect σ, and so it must contain an additional cycle from σ that is different from those in T. There are at most $k-1$ ways to pick such a cycle from σ since at least one cycle from σ would contain elements from the cycles in T. Consequently, the maximum size of the subfamily $\mathcal{A}(T)$ is at most

$$(k-1)\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}.$$

Since every permutation in $\mathcal{A}$ must t-intersect π, we deduce that

$$|\mathcal{A}| = \left| \bigcup_{T \subseteq M(\pi)} A(T) \right| \leqslant \binom{k}{t}(k-1)\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}.$$

Since we can write

$$\begin{bmatrix} n-t \\ k-t \end{bmatrix} = \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (n-t-1) \begin{bmatrix} n-t-1 \\ k-t \end{bmatrix}$$

it remains to show that

$$\binom{k}{t}(k-1) \quad < \quad 1 + (n-t-1) \begin{bmatrix} n-t-1 \\ k-t \end{bmatrix} \Big/ \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}. \tag{5}$$

By Corollary 4, it is enough to show that

$$\binom{k}{t}(k-1) \quad < \quad 1 + (n-t-1) \frac{2(n-k)}{(k-t-1)(n-t-1)} \tag{6}$$

which implies that

$$n > k + \frac{(k-t-1)}{2} \left((k-1)\binom{k}{t} - 1 \right).$$

This concludes the proof of Theorem 2.

4 Dependence of n on k and t

For $0 \leqslant i \leqslant \lfloor (k-t)/2 \rfloor$, define the family

$$\mathcal{F}_i = \{\pi \in S_{n,k} : |M(\pi) \cap \{(1), (2), \ldots, (t+2i)\}| \geqslant t+i\}.$$

These families are analogous to those first defined by Frankl for set systems:

$$\left\{ F \in \binom{[n]}{k} : |F \cap [t+2i]| \geqslant t+i \right\}.$$

Clearly, $\mathcal{F}_i$ is t-cycle-intersecting for $0 \leqslant i \leqslant \lfloor (k-t)/2 \rfloor$. Note that $\mathcal{F}_0$ is the stabiliser of t fixed points. Therefore, Theorem 2 says that if $n \geqslant n_0(k,t)$, then the only largest t-cycle-intersecting families of $S_{n,k}$ are those isomorphic to $\mathcal{F}_0$. The following proposition shows that the condition $n \geqslant n_0(k,t)$ cannot be replaced by $n \geqslant n_0(t)$ or $k \geqslant k_0(t)$.

Proposition 5. *Let r be a fixed positive integer. If $k = n - r$ then $|\mathcal{F}_1| > |\mathcal{F}_0|$ for all sufficiently large n in terms of r, regardless of the value of t.*

Proof. Note that

$$|\mathcal{F}_1| = \binom{t+2}{t+1} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - (t+1) \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}, \quad |\mathcal{F}_0| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}.$$

It follows that

$$
\begin{aligned}
|\mathcal{F}_1| - |\mathcal{F}_0| &= \binom{t+2}{t+1} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - (t+1) \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
&\quad - \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (n-t-1) \begin{bmatrix} n-t-1 \\ k-t \end{bmatrix} \right) \\
&= (t+1) \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - (t+1) \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} - (n-t-1) \begin{bmatrix} n-t-1 \\ k-t \end{bmatrix} \\
&= (t+1)(n-t-2) \begin{bmatrix} n-t-2 \\ k-t-1 \end{bmatrix} - (n-t-1) \begin{bmatrix} n-t-2 \\ k-t-1 \end{bmatrix} \\
&\quad - (n-t-1)(n-t-2) \begin{bmatrix} n-t-2 \\ k-t \end{bmatrix} \\
&= (t(n-t-1) - (t+1)) \begin{bmatrix} n-t-2 \\ k-t-1 \end{bmatrix} - (n-t-1)(n-t-2) \begin{bmatrix} n-t-2 \\ k-t \end{bmatrix}.
\end{aligned} \tag{7}
$$

To show that the right-hand side of (7) is greater than 0, we shall prove the inequality

$$
\begin{bmatrix} n-t-2 \\ k-t \end{bmatrix} \Big/ \begin{bmatrix} n-t-2 \\ k-t-1 \end{bmatrix} < \frac{(t(n-t-1) - (t+1))}{(n-t-1)(n-t-2)}. \tag{8}
$$

By Corollary 4,

$$
\begin{aligned}
\begin{bmatrix} n-t-2 \\ k-t \end{bmatrix} \Big/ \begin{bmatrix} n-t-2 \\ k-t-1 \end{bmatrix} &\leqslant \frac{n-k-1}{(k-t-1)(n-t-3)} H_{n-t-3} \\
&< \frac{n-k-1}{(k-t-1)(n-t-3)} (1 + \ln(n-t-3)) \\
&= \frac{r-1}{(n-r-t-1)(n-t-3)} (1 + \ln(n-t-3)).
\end{aligned} \tag{9}
$$

Clearly, the right-hand side of (9) is now less than $\frac{(t(n-t-1)-(t+1))}{(n-t-1)(n-t-2)}$ for sufficiently large n in terms of r, regardless of the value of t. $\square$

5 Concluding remarks

The technique used in the proof of the main theorem of this paper is the *kernel method* introduced by Hajnal and Rothschild [16]. The limitation of this method is that it only works from some threshold. This method usually yields short and easy proofs but rarely gives the exact range of results. We were rather cavalier in our estimates of the function $n_0(k,t)$. A better bound is perhaps possible by using a more delicate approximation of the unsigned Stirling number of the first kind. We believe that the bound $n_0(k,t) = O(k^{t+2})$ is not optimal, and that a different technique similar to the shifting operation and compression for set systems may be needed to derive the exact optimal bound for n.

In the original EKR theorem (Theorem 1), the condition $n \geqslant 2k$ is optimal for $t = 1$ since taking all k-subsets of $[n]$ with $n = 2k-1$ yields a 1-intersecting family of k-subsets which has size greater than $\binom{n-1}{k-1} = \binom{2k-2}{k-1}$. A similar construction for permutations can be given as follows. Suppose $n = 2k-3$, where $k \geqslant 4$. Consider the family $\mathcal{A} \subseteq S_{2k-3,k}$ which consists of permutations with exactly $k-1$ fixed points and one cycle of length $n-(k-1) = k-2$. Since $k-1 > n/2$, any two permutations of $\mathcal{A}$ must intersect in at least one fixed point, so $\mathcal{A}$ is 1-cycle-intersecting. The size of $\mathcal{A}$ is given by

$$|\mathcal{A}| = \binom{2k-3}{k-1}(k-3)!,$$

which is greater than $|\mathcal{F}_0|$ when $k = 4$. Unfortunately, our numerical computation suggests that this size is smaller than $|\mathcal{F}_0| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} 2k-4 \\ k-1 \end{bmatrix}$ all $k \geqslant 5$.

It is worth noting that the idea in the above construction is that if n and k are close and t is small, then all permutations in $S_{n,k}$ will be t-cycle-intersecting. In general, take $k = n-r$ for some r. By the pigeonhole principle, every permutation in $S_{n,k}$ has at least $n-2r$ fixed points. If $n-4r \geqslant t$, then every two permutations in $S_{n,k}$ must intersect in at least t fixed points. By substituting $r = n-k$ and $t = 1$, the condition $n-4r \geqslant t$ becomes $n \leqslant \frac{4k-1}{3}$. We conjecture that the optimal lower bound for n in Theorem 2 for $t = 1$ is $O(4k/3)$.

Acknowledgments

We would like to thank the anonymous referee for the comments that had helped us make several improvements to this paper, particularly for pointing out the kernel method which has simplified our original proof. This project was supported by the Advanced Fundamental Research Cluster, University of Malaya (UMRG RG238/12AFR).

References

[1] R. Ahlswede and L. H. Khachatrian, The complete intersection theorem for systems of finite sets, *Eur. J. Comb.* **18** (1997), 125–136.

[2] R. Ahlswede and L. H. Khachatrian, The diametric theorem in Hamming spaces – optimal anticodes, *Adv. Appl. Math.* **20** (1998), 429–449.

[3] P. Borg, Extremal t-intersecting sub-families of hereditary families, *J. Lond. Math. Soc., II. Ser.* **79** (2009), 167–185.

[4] P. Borg, On t-intersecting families of signed sets and permutations, *Discrete Math.* **309** (2009), 3310–3317.

[5] P. Borg and F. C. Holroyd, The Erdős-Ko-Rado property of various graphs containing singletons, *Discrete Math.* **309** (2009), 2877–2885.

[6] F. Brunk and S. Huczynska, Some Erdős-Ko-Rado theorems for injections, *Eur. J. Comb.* **31** (2010), 839–860.

[7] P. J. Cameron and C. Y. Ku, Intersecting families of permutations, *Eur. J. Comb.*. **24** (2003), 881–890.

[8] A. Chowdhury and B. Patkós, Shadows and intersections in vector spaces, *J. Comb. Theory, Ser. A* **117** (2010), 1095–1106.

[9] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, *J. Comb. Theory, Ser. A* **22** (1977), 352–360.

[10] D. Ellis, Stability for t-intersecting families of permutations, *J. Comb. Theory, Ser. A* **118** (2011), 208–227.

[11] D. Ellis, E. Friedgut and H. Pilpel, Intersecting families of permutations, *J. Am. Math. Soc.* **24** (2011), 649-682.

[12] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Q. J. Math., Oxf. II. Ser.* **12** (1961), 313–320.

[13] P. Frankl, The Erdős-Ko-Rado theorem is true for $n = ckt$, *Col. Soc. Math. J. Bolyai* **18** (1978), 365–375.

[14] P. Frankl and R. M. Wilson, The Erdős-Ko-Rado theorem for vector spaces, *J. Comb. Theory, Ser. A* **43** (1986), 228–236.

[15] C. Godsil and K. Meagher, A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations, *Eur. J. Comb.* **30** (2009), 404–414.

[16] A. Hajnal and B. Rothschild, A generalisation of the Erdős-Ko-Rado theorem on finite set systems, *J. Comb. Theory, Ser. A* **15** (1973), 359–362.

[17] A. J. W. Hilton and C. L. Spencer, A graph-theoretical generalisation of Berge's analogue of the Erdős-Ko-Rado theorem, *Trends in Graph Theory, Birkhauser Verlag, Basel, Switzerland* (2006), 225–242.

[18] F. C. Holroyd, C. Spencer and J. Talbot, Compression and Erdős-Ko-Rado graphs, *Discrete Math.* **293** (2005), 155–164.

[19] F. C. Holroyd and J. Talbot, Graphs with the Erdős-Ko-Rado property, *Discrete Math.* **293** (2005), 165–176.

[20] G. Hurlbert and V. Kamat, Erdős-Ko-Rado theorems for chordal graphs and trees, *J. Comb. Theory, Ser. A* **118** (2011), 829–841.

[21] P. Keevash, Shadows and intersections: Stability and new proofs, *Adv. Math.* **218** (2008) 1685–1703.

[22] C. Y. Ku and I. Leader, An Erdős-Ko-Rado theorem for partial permutations, *Discrete Math.* **306** (2006), 74–86.

[23] C. Y. Ku and D. Renshaw, Erdős-Ko-Rado theorems for permutations and set partitions, *J. Comb. Theory, Ser. A* **115** (2008), 1008–1020.

[24] C. Y. Ku and T. W. H. Wong, Intersecting families in the alternating group and direct product of symmetric groups, *Electron. J. Comb.* **14** (2007), #R25.

[25] C. Y. Ku and K. B. Wong, An analogue of Hilton-Milner theorem for set partitions, *J. Comb. Theory, Ser. A* **120** (2013), 1508–1520.

[26] C. Y. Ku and K. B. Wong, On cross-intersecting families of set partitions, *Electron. J. Comb.* **19**(4) (2012), #P49.

[27] C. Y. Ku and K. B. Wong, On r-cross intersecting families of sets, *Far East J. Math. Sci.* **75** (2013), 295–300.

[28] C. Y. Ku and K. B. Wong, An analogue of the Erdős-Ko-Rado theorem for weak compositions, *Discrete Math.* **313** (2013) 2463-2468.

[29] B. Larose and C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, *Eur. J. Comb.* **25** (2004), 657–673.

[30] Y.-S. Li and Jun Wang, Erdős-Ko-Rado-type theorems for colored sets, *Electron. J. Comb.* **14** (2007) #R1.

[31] E. H. Lieb, Concavity properties and a generating functions of Stirling numbers, *J. Comb. Theory* **5** (1968), 203–206.

[32] K. Meagher and L. Moura, Erdős-Ko-Rado theorems for uniform set-partition systems, *Electron. J. Comb.* **12** (2005), #R40.

[33] K. Meagher and A. Purdy, An Erdős-Ko-Rado theorem for multisets, *Electron. J. Comb.* **18** (2011), #P220.

[34] A. Moon, An analogue of the Erdős-Ko-Rado theorem for the Hamming schemes $H(n, q)$, *J. Comb. Theory, Ser. A* **32**(3) (1982), 386–390.

[35] J. Wang and S. J. Zhang, An Erdős-Ko-Rado-type theorem in Coxeter groups, *Eur. J. Comb.* **29** (2008), 1112–1115.

[36] R. M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, *Combinatorica* **4** (1984), 247–257.

[37] R. Woodroofe, Erdős-Ko-Rado theorems for simplicial complexes, *J. Comb. Theory, Ser. A* **118** (2011), 1218–1227.