Some identities involving the partial sum of q-binomial coefficients

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Abstract

We give some identities involving sums of powers of the partial sum of q-binomial coefficients, which are q-analogues of Hirschhorn's identities [*Discrete Math.* 159 (1996), 273–278] and Zhang's identity [*Discrete Math.* 196 (1999), 291–298].

Keywords: binomial coefficients, q-binomial coefficients, q-binomial theorem

1 Introduction

In [2], Calkin proved the following curious identity:

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^{3} = n\cdot 2^{3n-1} + 2^{3n} - 3n\binom{2n}{n}2^{n-2}.$$

Hirschhorn [5] established the following two identities on sums of powers of binomial partial sums:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j} = n\cdot 2^{n-1} + 2^{n}, \tag{1}$$

and

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^{2} = n\cdot 2^{2n-1} + 2^{2n} - \frac{n}{2}\binom{2n}{n}. \tag{2}$$

In [7], Zhang proved the following alternating form of (2):

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2=\begin{cases}1, & \text{if } n=0,\\ 2^{2n-1}, & \text{if } n \text{ is even and } n\neq 0,\\ -2^{2n-1}-(-1)^{(n-1)/2}\binom{n-1}{(n-1)/2}, & \text{if } n \text{ is odd.}\end{cases} \quad (3)$$

Several generalizations are given in [6, 8, 9]. Later, Guo *et al.* [4] gave the following q-identities:

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{j=0}^{k}\begin{bmatrix}2n\\ j\end{bmatrix}_q\right)^2=\left(\sum_{k=0}^{2n}\begin{bmatrix}2n\\ k\end{bmatrix}_q\right)\left(\sum_{k=0}^{n}\begin{bmatrix}2n\\ 2k\end{bmatrix}_q\right),$$

and

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{j=0}^{k}\begin{bmatrix}2n+1\\ j\end{bmatrix}_q\right)^2=-\left(\sum_{k=0}^{n}\begin{bmatrix}2n+1\\ 2k\end{bmatrix}_q\right)\left(\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q\right)$$
$$-\sum_{k=0}^{n}(-1)^k\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2-2\sum_{0\leqslant i<j\leqslant n}(-1)^i\begin{bmatrix}2n+1\\ i\end{bmatrix}_q\begin{bmatrix}2n+1\\ j\end{bmatrix}_q.$$

Here and in what follows, $\begin{bmatrix}n\\ k\end{bmatrix}_q$ is the q-binomial coefficient defined by

$$\begin{bmatrix}n\\ k\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leqslant k\leqslant n,\\ 0, & \text{otherwise,}\end{cases}$$

where $(z;q)_n=(1-z)(1-zq)\cdots(1-zq^{n-1})$ is the q-shifted factorial for $n\geqslant 0$.

The purpose of this paper is to study q-analogues of (1)–(2) and establish a new q-version of (3). Our main results may be stated as follows.

Theorem 1. *For any positive integer n and any non-zero integer m, we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\begin{bmatrix}n\\ j\end{bmatrix}_q q^{mk+\binom{j}{2}}=\frac{(-q^m,q)_n-q^{m(n+1)}(-1,q)_n}{1-q^m}, \quad (4)$$

and

$$\sum_{k=0}^{n}q^{-k}\left(\sum_{i=0}^{k}\begin{bmatrix}n\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{j=0}^{k}\begin{bmatrix}n\\ j\end{bmatrix}_q q^{\binom{j}{2}+2(1-n)j}\right)$$
$$=\frac{\left((-q^{-1};q)_n-q^{-(n+1)}(-1;q)_n\right)(-q^{2(1-n)};q)_n}{1-q^{-1}}-\sum_{i=0}^{n-1}\frac{1-q^{n-i}}{1-q}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{i}{2}-\frac{3n^2}{2}+\frac{n}{2}+1}. \quad (5)$$

Theorem 2. *For any non-negative integer n, we have*

$$\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n+1\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{j=0}^{k}\begin{bmatrix}2n+1\\ j\end{bmatrix}_q q^{\binom{2n-j+1}{2}}\right)$$
$$= -q^{2n^2+n}(-q^{-2n};q)_{4n+1}-\sum_{i=0}^{n}(-1)^i\begin{bmatrix}2n+1\\ i\end{bmatrix}_{q^2} q^{2\binom{i}{2}}, \qquad (6)$$

and

$$\sum_{k=0}^{2n+2}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n+2\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{i=0}^{k}\begin{bmatrix}2n+2\\ i\end{bmatrix}_q q^{\binom{2n+2-i}{2}}\right)=q^{2n^2+3n+1}(-q^{-1-2n};q)_{4n+3}. \qquad (7)$$

Letting $q\to 1$ and using L'Hôpital's rule and some familiar identities, we easily find that the identities (4)–(5) and (6)–(7) are q-analogues of (1)–(2) and (3) respectively.

In Sections 2 and 3, we will give proofs of Theorems 1.1 and 1.2 respectively by using the q-binomial theorem and generating functions.

2 Proof of Theorem 1.1

To give our proof of Theorem 1.1, we need to establish a result, which is a q-analogue of Chang and Shan's identity (see [3]).

Lemma 3. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1}q^{-k}\left(\sum_{i=0}^{k}\begin{bmatrix}n\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{j=k+1}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q q^{\binom{j}{2}+2(1-n)j}\right)=\sum_{i=0}^{n-1}\frac{1-q^{n-i}}{1-q}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{i}{2}-\frac{3n^2}{2}+\frac{n}{2}+1}.$$

Proof. According to the q-binomial theorem (see [1]), we have for all complex numbers z and q with $|z|<1$ and $|q|<1$, there holds

$$(z,q)_n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\ k\end{bmatrix}_q q^{\binom{k}{2}}z^k \qquad (8)$$

and

$$\frac{1}{(z,q)_n}=\sum_{i\geqslant 0}\begin{bmatrix}n+i-1\\ i\end{bmatrix}_q z^i.$$

It follows that

$$(-z;q)_n\frac{1}{1-z}=\left(\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q q^{\binom{i}{2}}z^i\right)\left(\sum_{i=0}^{\infty}z^i\right),$$
$$(-zq^n;q)_n\frac{1}{1-zq}=\left(\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q q^{\binom{i}{2}+ni}z^i\right)\left(\sum_{i=0}^{\infty}q^iz^i\right),$$

and

$$(-z;q)_{2n}\frac{1}{(z;q)_2} = \left(\sum_{i=0}^{2n} {2n \brack i}_q q^{\binom{i}{2}} z^i\right)\left(\sum_{i=0}^{\infty} {1+i \brack i}_q z^i\right).$$

Therefore, for any non-negetive integer k with $k \leqslant n-1$, the coefficient of z^k in $(-z;q)_n\frac{1}{1-z}$ is

$$\sum_{i=0}^{k} {n \brack i}_q q^{\binom{i}{2}},$$

the coefficient of z^{n-k-1} in $(-zq^n;q)_n\frac{1}{(1-zq}$ is

$$\sum_{i=k+1}^{n} {n \brack i}_q q^{\binom{n-i}{2}+n(n-i)+i-k-1}$$

and the coefficient of z^{n-1} in $(-z;q)_{2n}\frac{1}{(z;q)_2}$ is

$$\sum_{i=0}^{n-1} {2n \brack i}_q \frac{1-q^{n-i}}{1-q} q^{\binom{i}{2}}.$$

Using the fact

$$(-z;q)_n\frac{1}{1-z}\cdot(-zq^n;q)_n\frac{1}{1-zq} = (-z;q)_{2n}\frac{1}{(z;q)_2},$$

equating the coefficients of z^{n-1} and after some simplifications, we obtain Lemma 2.1. □

Proof of Theorem 1.1. We first prove (4).

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{k} {n \brack j}_q q^{mk+\binom{j}{2}} &= \sum_{j=0}^{n} {n \brack j}_q q^{\binom{j}{2}} \sum_{k=j}^{n} q^{mk} \\
&= \frac{\sum_{j=0}^{n} {n \brack j}_q q^{\binom{j}{2}+mj} - q^{m(n+1)} \sum_{j=0}^{n} {n \brack j}_q q^{\binom{j}{2}}}{1-q^m} \\
&= \frac{(-q^m,q)_n - q^{m(n+1)}(-1,q)_n}{1-q^m},
\end{aligned}$$

where in the last step, we have used (8).

We next show (5). By (8), we have

$$\sum_{j=0}^{n} {n \brack j}_q q^{\binom{j}{2}+2(1-n)j} = (-q^{2(1-n)};q)_n,$$

and taking $m=-1$ in (4), we obtain

$$\sum_{k=0}^{n} q^{-k} \sum_{j=0}^{k} {n \brack j}_q q^{\binom{j}{2}} = \frac{(-q^{-1},q)_n - q^{-(n+1)}(-1,q)_n}{1-q^{-1}}.$$

Hence, by Lemma 2.1, we get

$$\begin{aligned}
&\sum_{k=0}^{n} q^{-k} \left(\sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} \right) \left(\sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}+2(1-n)j} \right) \\
&= \sum_{k=0}^{n} q^{-k} \left(\sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} \right) \left((-q^{2(1-n)};q)_n - \sum_{j=k+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}+2(1-n)j} \right) \\
&= (-q^{2(1-n)};q)_n \sum_{k=0}^{n} q^{-k} \left(\sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} \right) \\
&\quad - \sum_{k=0}^{n-1} q^{-k} \left(\sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} \right) \left(\sum_{j=k+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}+2(1-n)j} \right) \\
&= \frac{\left((-q^{-1};q)_n - q^{-(n+1)}(-1;q)_n\right)(-q^{2(1-n)},q)_n}{1-q^{-1}} - \sum_{i=0}^{n-1} \frac{1-q^{n-i}}{1-q} \begin{bmatrix} 2n \\ i \end{bmatrix}_q q^{\binom{i}{2}-\frac{3n^2}{2}+\frac{n}{2}+1}.
\end{aligned}$$

3 Proof of Theorem 1.2

In order to prove the Theorem 1.2, we need the following result, which gives a q-analogue of alternating sums of Chang and Shan's identity.

Lemma 4. *For any non-negative integer n, we have*

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{i=0}^{k} \begin{bmatrix} 2n+1 \\ i \end{bmatrix}_q q^{\binom{i}{2}} \right) \left(\sum_{j=k+1}^{2n+1} \begin{bmatrix} 2n+1 \\ j \end{bmatrix}_q q^{\binom{2n-j+1}{2}} \right) = \sum_{i=0}^{n} (-1)^i \begin{bmatrix} 2n+1 \\ i \end{bmatrix}_{q^2} q^{2\binom{i}{2}}.$$

Proof. By (8), we find that

$$(z;q)_n \frac{1}{1+z} = \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} (-z)^i \right) \left(\sum_{i=0}^{\infty} (-z)^i \right),$$

$$(-z;q)_n \frac{1}{1-z} = \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} z^i \right) \left(\sum_{i=0}^{\infty} z^i \right),$$

and

$$(z^2;q^2)_n \frac{1}{1-z^2} = \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} q^{2\binom{i}{2}} (-1)^i z^{2i} \right) \left(\sum_{i=0}^{\infty} z^{2i} \right).$$

Therefore, for any non-negetive integer k with $k \leqslant n-1$, the coefficient of z^k in $(z;q)_n \frac{1}{1+z}$ is

$$(-1)^k \sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}},$$

the coefficient of z^{n-k-1} in $(-z;q)_n \frac{1}{1-z}$ is

$$\sum_{i=k+1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{n-i}{2}}$$

and the coefficient of z^{n-1} in $(z^2;q^2)_n \frac{1}{1-z^2}$ is

$$\sum_{i=0}^{(n-1)/2} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} q^{2\binom{i}{2}} [2|(n-1)],$$

where $[2|n]$ is defined by

$$[2|n] = \begin{cases} 1, & \text{if } 2|n, \\ 0, & \text{otherwise.} \end{cases}$$

Using the fact

$$(-z;q)_n \frac{1}{1-z} \cdot (z;q)_n \frac{1}{1+z} = (z^2;q^2)_n \frac{1}{1-z^2},$$

equating the coefficients of z^{n-1} and after some simplifications, we obtain Lemma 3.1. □

Proof of Theorem 1.2. We first prove (6). By (8), we have

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}} &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}} \sum_{k=j}^{n} (-1)^k \\ &= \frac{1}{2} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}} ((-1)^j - (-1)^{n+1}) \\ &= \frac{(-1)^n}{2} (-1,q)_n, \end{aligned} \tag{9}$$

and

$$\sum_{j=0}^{2n+1} \begin{bmatrix} 2n+1 \\ j \end{bmatrix}_q q^{\binom{2n-j+1}{2}} = q^{2n^2+n} (-q^{-2n};q)_{2n+1}.$$

Replacing n by $2n+1$ in (9), we obtain

$$\sum_{k=0}^{2n+1} (-1)^k \left(\sum_{i=0}^{k} \begin{bmatrix} 2n+1 \\ i \end{bmatrix}_q q^{\binom{i}{2}} \right) = -(-q;q)_{2n}.$$

Hence, by Lemma 3.1, we get

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n+1\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{j=0}^{k}\begin{bmatrix}2n+1\\ j\end{bmatrix}_q q^{\binom{2n-j+1}{2}}\right)\\
&=\sum_{k=0}^{2n+1}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n+1\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(q^{2n^2+n}(-q^{-2n};q)_{2n+1}-\sum_{j=k+1}^{2n+1}\begin{bmatrix}2n+1\\ j\end{bmatrix}_q q^{\binom{2n-j+1}{2}}\right)\\
&=-q^{2n^2+n}(-q^{-2n};q)_{4n+1}-\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n+1\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{j=k+1}^{2n+1}\begin{bmatrix}2n+1\\ j\end{bmatrix}_q q^{\binom{2n-j+1}{2}}\right)\\
&=-q^{2n^2+n}(-q^{-2n};q)_{4n+1}-\sum_{i=0}^{n}(-1)^i\begin{bmatrix}2n+1\\ i\end{bmatrix}_{q^2} q^{2\binom{i}{2}}.
\end{aligned}$$

We next show (7). By (8), we have

$$\sum_{j=0}^{2n}\begin{bmatrix}2n\\ j\end{bmatrix}_q q^{\binom{2n-j}{2}}=q^{2n^2-n}(-q^{1-2n};q)_{2n},$$

and replacing n by $2n$ in (9), we obtain

$$\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)=(-q;q)_{2n-1}.$$

Hence, by the fact

$$\sum_{k=0}^{2n-1}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{i=k+1}^{2n}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{2n-i}{2}}\right)=0$$

which follows easily from the substitution $k\to 2n-1-k$, we have

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(\sum_{i=0}^{k}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{2n-i}{2}}\right)\\
&=\sum_{k=0}^{2n}(-1)^k\left(\sum_{i=0}^{k}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{i}{2}}\right)\left(q^{2n^2-n}(-q^{1-2n};q)_{2n}-\sum_{i=k+1}^{2n}\begin{bmatrix}2n\\ i\end{bmatrix}_q q^{\binom{2n-i}{2}}\right)\\
&=q^{2n^2-n}(-q^{1-2n};q)_{4n-1}.
\end{aligned}$$

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