Consecutive up-down patterns in up-down permutations

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Abstract

In this paper, we study the distributions of consecutive pattern matches of an up-down permutations τ in the set of up-down permutations. In particular, we study the distribution of consecutive pattern matches of the five up-down permutations of length four, 1324, 1423, 2314, 2413, and 3412 in up-down permutations. We show that such generating functions the distributions of consecutive pattern matches of such τ in the set of up-down permutations can be expressed in terms of the number of generalized maximum packings of τ. We then provide some systematic methods to compute number of generalized packings for such τ.

1 Introduction

If $\sigma = \sigma_1 \dots \sigma_n$ a permutation in the symmetric group S_n, then we let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Ris(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. Let $\mathbb{N} = \{0, 1, \dots, \}$ denote the natural numbers, $\mathbb{P} = \{1, 2, \dots\}$ denote the set of positive integers, $\mathbb{E} = \{0, 2, 4, \dots\}$ denote the set of even numbers in $\mathbb{N}$, and $[n] = \{1, 2, \dots, n\}$ for $n \in \mathbb{P}$. We say that $\sigma = \sigma_1 \dots \sigma_n$ is an *up-down permutation* if $Des(\sigma) = [n-1] \cap \mathbb{E}$. That is, we have

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 \dots .$$

We let $\mathcal{A}_n$ denote the set of up-down permutations in S_n.

Given a sequence $\sigma = \sigma_1 \dots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i-th smallest integer that appears in σ by i. For example, if $\sigma = 2754$, then $\text{red}(\sigma) = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j \in S_j$ and a permutation $\sigma = \sigma_1 \dots \sigma_n$, we say that τ *occurs* in σ if there are $1 \leq i_1 < \dots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$ and we say that σ has a *τ-match starting at position i* in σ if $\text{red}(\sigma_i \sigma_{i+1} \dots \sigma_{i+j-1}) = \tau$.

We say that σ *avoids* τ if there are no occurrences of τ in σ. We let τ-mch(σ) denote the number of τ-matches in σ. We let $A_{n,\tau}(x) = \sum_{\sigma \in A_n} x^{\tau\text{-mch}(\sigma)}$.

These definitions are easily extended to sets of permutations $\Upsilon \subseteq S_j$. That is, we say that Υ occurs in σ if there are $1 \leq i_1 < \ldots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \ldots \sigma_{i_j}) \in \Upsilon$ and we say that σ has a Υ-match starting at position i in σ if $\text{red}(\sigma_i \sigma_{i+1} \ldots \sigma_{i+j-1}) \in \Upsilon$. We say that σ avoids Υ if there are no occurrences of Υ in σ. We let Υ-mch(σ) denote the number of τ-matches in σ. We let $A_{n,\Upsilon}(x) = \sum_{\sigma \in A_n} x^{\Upsilon\text{-mch}(\sigma)}$.

There have been several papers that have studied the number of up-down permutations $\sigma \in A_n$ which avoid a given pattern. For example, in [17, 12], it was proved that if $\tau \in S_3$, then the number of up-down permutations $\sigma \in A_{2n}$ which avoid τ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. In [16], it was shown that the number of $\sigma \in A_{2n}$ which avoids 1234 or 2143 is $\frac{2(3n)!}{n!(n+1)!(n+2)!}$. There has been somewhat less work on the distribution of τ-matches in up-down permutations. Carlitz [5] found the generating function the number of rises in the peaks of the up-down permutations where a rise in the peaks of an up-down permutations is just 213-match in σ.

The main goal of this paper is to study the generating functions

$$A_\tau(t,x) = 1 + \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma \in A_{2n}} x^{\tau\text{-mch}(\sigma)} \tag{1}$$

and

$$B_\tau(t,x) = \sum_{n \geq 1} \frac{t^{2n-1}}{(2n-1)!} \sum_{\sigma \in A_{2n-1}} x^{\tau\text{-mch}(\sigma)}. \tag{2}$$

in the case were $\tau \in A_4$. Note that there are 5 permutations in A_4, namely,

$$\tau^{(1)} = 1324, \tau^{(2)} = 2314, \tau^{(3)} = 2413, \tau^{(4)} = 1423, \text{ and } \tau^{(5)} = 3412.$$

If $\sigma \in A_n$, then $\tau^{(i)}$-matches can only start at odd positions. If $\sigma \in A_{2n}$ and $\tau^{(i)}\text{-mch}(\sigma) = n-1$, then we say that σ is a *maximum packing for* $\tau^{(i)}$. Thus if $\sigma \in A_{2n}$ is a maximum packing for $\tau^{(i)}$, then σ has $\tau^{(i)}$-matches starting at positions $1, 3, \ldots, 2n-3$. We let $\mathcal{MP}_{2n,\tau^{(i)}}$ denote the set of maximum packings for $\tau^{(i)}$ in A_{2n} and we let $\text{mp}_{2n,\tau^{(i)}} = |\mathcal{MP}_{2n,\tau^{(i)}}|$. We shall see that it follows from results of Harmse and Remmel [11] that

$$\begin{aligned} \text{mp}_{2n,\tau^{(1)}} &= \text{mp}_{2n,\tau^{(3)}} = C_{n-1} \text{ and} \\ \text{mp}_{2n,\tau^{(2)}} &= \text{mp}_{2n,\tau^{(4)}} = \text{mp}_{2n\tau^{(5)}} = 1, \end{aligned}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Our main theorem will show that for each $i \in [5]$, the generating functions $A_{\tau^{(i)}}(t,x)$ and $B_{\tau^{(i)}}(t,x)$ can be expressed in terms of what we call generalized maximum packings for $\tau^{(i)}$. That is, we say that $\sigma \in S_{2n}$ is a *generalized maximum packing for* $\tau^{(i)}$ if we can break σ into consecutive blocks $\sigma = B_1 \ldots B_k$ such that

1. for all $1 \leq j \leq k$, B_j is either an increasing sequence of length 2 or $\text{red}(B_j)$ is maximum packing for $\tau^{(i)}$ of length $2s$ for some $s \geq 2$ and

2. for all $1 \leq j \leq k-1$, the last element of B_j is less than the first element of B_{j+1}.

Note that if σ is a generalized maximum packing for $\tau^{(i)}$, there is only one possible block structure. That is, if $\sigma = \sigma_1 \ldots \sigma_{2n} \in S_{2n}$ is a generalized maximum packing for $\tau^{(i)}$, our conditions force that $\sigma_{2j-1} < \sigma_{2j}$ for $i = j, \ldots, n$. Then it is easy to see that $\sigma_{2j-1}\sigma_{2j}$ and $\sigma_{2j+1}\sigma_{2j+2}$ are in the same block if and only if $\sigma_{2j} > \sigma_{2j+1}$.

If σ is a generalized maximum packing for $\tau^{(i)}$ of length $2n$ with block structure $B_1 \ldots B_k$, then we define the weight $w(B_j)$ of block B_j to be 1 if B_j has size 2 and to be $(x-1)^s$ if B_j has length $2s+2$ where $s \geq 1$. Then we define the weight $w(\sigma)$ of σ to be $(-1)^{k-1} \prod_{j=1}^{k} w(B_j)$. For example,

$$\sigma = 1\ 2\ 3\ 5\ 4\ 7\ 6\ 9\ 8\ 10\ 11\ 12\ 13\ 14\ 15\ 17\ 16\ 18$$

is a generalized maximum packing for $\tau^{(1)} = 1324$ where $B_1 = 1\ 2$, $B_2 = 3\ 5\ 4\ 7\ 6\ 9\ 8\ 10$, $B_3 = 11\ 12$, $B_4 = 13\ 14$ and $B_5 = 15\ 17\ 16\ 18$. Thus $w(B_1) = w(B_3) = w(B_4) = 1$, $w(B_2) = (x-1)^2$ and $w(B_5) = (x-1)$ so that $w(\sigma) = (-1)^4(x-1)^3 = (x-1)^3$. We let $\mathcal{GMP}_{2n,\tau^{(i)}}$ denote the set of $\sigma \in S_{2n}$ which are generalized maximum packings for $\tau^{(i)}$ and we let

$$GMP_{2n,\tau^{(i)}}(x) = \sum_{\sigma \in \mathcal{GMP}_{2n,\tau^{(i)}}} w(\sigma). \tag{3}$$

We say that $\sigma \in S_{2n+1}$ is a *generalized maximum packing for* $\tau^{(i)}$ if we can break σ into consecutive blocks $\sigma = B_1 \ldots B_k$ such that

1. for all $1 \leq j < k$, B_j is either an increasing sequence of length 2 or $\mathrm{red}(B_j)$ is maximum packing of $\tau_{(i)}$ of length $2s$ for some $s \geq 2$,

2. B_k is a block of size 1, and

3. for all $1 \leq j \leq k-1$, the last element of B_j is less than the first element of B_{j+1}.

If σ is a generalized maximum packing for $\tau^{(i)}$ of length $2n+1$ with block structure $B_1 \ldots B_k$, then we define the weight $w(B_j)$ of block B_j to be 1 if B_i has size 1 or 2 and to be $(x-1)^s$ if B_j has length $2s+2$ where $s \geq 1$. Then we define the weight $w(\sigma)$ of σ to be $(-1)^{k-1} \prod_{i=1}^{k} w(B_i)$. For example,

$$\sigma = 1\ 2\ 3\ 5\ 4\ 7\ 6\ 9\ 8\ 10\ 11\ 12\ 13\ 14\ 15\ 17\ 16\ 18\ 19$$

is a generalized maximum packing for $\tau^{(1)} = 1324$ where $B_1 = 1\ 2$, $B_2 = 3\ 5\ 4\ 7\ 6\ 9\ 8\ 10$, $B_3 = 11\ 12$, $B_4 = 13\ 14$ and $B_5 = 15\ 17\ 16\ 18$, and $B_6 = 19$. Thus $w(B_1) = w(B_3) = w(B_4)W(B_6) = 1$, $w(B_2) = (x-1)^2$ and $w(B_5) = (x-1)$ so that $w(\sigma) = (-1)^5(x-1)^3 = -(x-1)^3$. We let $\mathcal{GMP}_{2n+1,\tau^{(i)}}$ denote the set of $\sigma \in S_{2n+1}$ which are generalized maximum packings for $\tau^{(i)}$. We then let

$$GMP_{2n+1,\tau^{(i)}}(x) = \sum_{\sigma \in \mathcal{GMP}_{2n+1,\tau^{(i)}}} w(\sigma). \tag{4}$$

In general, it is much more difficult to compute $GMP_{2n,\tau^{(i)}}(x)$ and $GMP_{2n+1,\tau^{(i)}}(x)$ than to compute $\mathrm{mp}_{2n,\tau^{(i)}}$ and $\mathrm{mp}_{2n+1,\tau^{(i)}}$. Indeed, we do not have a closed expression for $GMP_{2n,\tau^{(i)}}(x)$ or $GMP_{2n+1,\tau^{(i)}}(x)$ as a function of n for any i. However, we will show that for $i \in \{1,2,4\}$, $GMP_{n,\tau^{(i)}}(x)$ can be computed via simple recursions. For example, we shall show that $GMP_{2,\tau^{(1)}}(x) = 1$, $GMP_{4,\tau^{(1)}}(x) = x-2$, for $2n > 4$,

$$GMP_{2n,\tau^{(1)}}(x) = C_{n-1}(x-1)^{n-1} - GMP_{2n-2,\tau^{(1)}}(x) - \sum_{k=2}^{n-1} C_{k-1}(x-1)^{k-1} GMP_{2n-2k,\tau^{(1)}}(x).$$

Moreover, we shall show that $GMP_{2n+1,\tau^{(1)}}(x) = -GMP_{2n,\tau^{(1)}}(x)$ for $n \geq 1$.

Our main theorem is the following.

Theorem 1. *For $i = 1, \ldots, 5$,*

$$A_{\tau^{(i)}}(t,x) = \frac{1}{1 - \sum_{n\geq 1} GMP_{2n,\tau^{(i)}}(x)\frac{t^{2n}}{(2n)!}}. \tag{5}$$

$$B_{\tau^{(i)}}(t,x) = \frac{\sum_{n\geq 1} GMP_{2n-1,\tau^{(i)}}(x)\frac{t^{2n-1}}{(2n-1)!}}{1 - \sum_{n\geq 1} GMP_{2n,\tau^{(i)}}(x)\frac{t^{2n}}{(2n)!}}. \tag{6}$$

We shall prove Theorem 1 by applying the so-called homomorphism method which has been developed in series of papers [1, 3, 4, 11, 13, 19, 20, 22, 24, 25]. In particular, we shall show that the generating functions in Theorem 1 arise by applying certain ring homomorphisms defined in the ring of symmetric functions Λ in infinitely many variable to simple symmetric function identities. For example, let h_n denote n-th homogeneous symmetric function in Λ and e_n denote the n-th elementary symmetric function in Λ. That is, h_n and e_n are defined by the generating functions

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t} \tag{7}$$

and

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t) \tag{8}$$

Then we shall show that (5) arises by applying a ring homomorphism Θ to the simple symmetric function identity that

$$H(t) = \frac{1}{E(-t)}.$$

The basic idea of the homomorphism method is to show that

$$(2n)!\theta(h_{2n}) = A_{2n,\tau^{(i)}}(x) = \sum_{\sigma \in A_n} x^{\tau^{(i)}\text{-mch}(\sigma)} \tag{9}$$

for an appropriate chosen ring homomorphism θ. One proves (9) by interpreting the left-hand side of (9) in terms of a signed weighted sum of filled brick tabloids and then applying an appropriate sign-reversing weight-preserving involution to show that the combinatorial interpretation of $(2n)!\theta(h_{2n})$ reduces to the desired generating function. The situation in this paper is a bit different from previous examples of the homomorphism method in that it requires two involutions of the show that our combinatorial interpretation of $(2n)!\theta(h_{2n})$ reduces to the right-hand of (9). Equation (6) is proved in a similar manner except that we apply θ to a more complicated symmetric function identity.

The outline of the paper is as follows. In section 2, we shall provide the necessary background in symmetric functions that are necessary for our proofs. In section 3, we shall prove Theorem 1. In section 4, we shall show how to compute $\mathrm{mp}_{n,\tau^{(i)}}$ for $i = 1, 2, 3, 4, 5$. In section 5, we shall develop recursions for $GMP_{n,\tau^{(i)}}(x)$ for $i = 1, 2, 4$. The simplest case is $GMP_{n,\tau^{(1)}}(x)$. In that case, we show that $GMP_{2n,\tau^{(1)}}(0) = (-1)^{n-1}C_n$ for $n \geq 1$ and $GMP_{2n,\tau^{(1)}}(x)|_x = (-1)^n\binom{2n}{n-2}$. Using these facts, we can compute the generating functions for the number of up-down permutations with no $\tau^{(1)}$-matches or with exactly one $\tau^{(1)}$-match. For example, we shall show that

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} N_{2n,\tau^{(1)}} = \frac{1}{1 + \sum_{n\geq 1}(-1)^n C_n \frac{t^{2n}}{(2n)!}}$$

where $N_{2n,\tau^{(1)}}$ is the number of $\sigma \in A_{2n}$ with no $\tau^{(1)}$-matches. Finally, in section 6, we shall study the distribution of double rise pairs and double descent pairs in up-down permutations. That is, if $\sigma = \sigma_1 \ldots \sigma_n \in A_n$ is an up-down permutation, then we say that a pair $(2i-1)(2i)$ is a *double rise pair* if both $\sigma_{2i-1} < \sigma_{2i+1}$ and $\sigma_{2i} < \sigma_{2i+2}$. Thus $(2i-1)(2i)$ is a double rise pair in σ if and only if there is a 1324 match starting at position $2i-1$ in σ. We say that a pair $(2i-1)2i)$ is a *double descent pair* if both $\sigma_{2i-1} > \sigma_{2i+1}$ and $\sigma_{2i} > \sigma_{2i+2}$. Thus $(2i-1)2i)$ is a double descent pair in σ if and only if there is a $D = \{2413, 3412\}$ match starting at position $2i-1$ in σ.

2 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs of the generating functions (5) and (6).

Let Λ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \ldots$ with coefficients in the field of complex numbers $\mathbb{C}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition, that is, λ is a finite sequence of weakly increasing nonnegative integers. Let $\ell(\lambda)$ denote the number of nonzero integers in λ. If the sum of these integers is n, we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions says that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ or, equivalently, that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ. Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, then $\{h_\lambda : \lambda \text{ is a partition}\}$ is also a basis for Λ. Since $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ, we can specify a ring homomorphism θ on Λ by simply defining $\theta(e_n)$ for all $n \geq 0$.

A *brick tabloid* of shape (n) and type $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a filling of a row of n squares of cells with bricks of lengths $\lambda_1, \ldots, \lambda_k$ such that bricks to not overlap. One brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$ is displayed below.

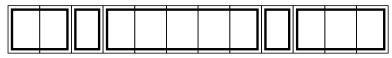

Figure 1: A brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$.

Let $\mathcal{B}_{\lambda,n}$ denote the set of all λ-brick tabloids of shape (n) and let $B_{\lambda,n} = |\mathcal{B}_{\lambda,n}|$. Eğecioğlu and Remmel proved in [7] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{10}$$

Next we define a class of symmetric functions $p_{n,\nu}$ which have a relationship with e_λ that is analogous to the relationship between h_n and e_λ. These functions were first introduced in [15] and [19]. Let ν be a function which maps the set of nonnegative integers into the field F. Recursively define $p_{n,\nu} \in \Lambda_n$ by setting $p_{0,\nu} = 1$ and letting

$$p_{n,\nu} = (-1)^{n-1}\nu(n)e_n + \sum_{k=1}^{n-1} (-1)^{k-1} e_k p_{n-k,\nu}$$

for all $n \geq 1$. By multiplying series, this means that

$$\left(\sum_{n\geq 0} (-1)^n e_n t^n\right)\left(\sum_{n\geq 1} p_{n,\nu} t^n\right) = \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} p_{n-k,\nu}(-1)^k e_k\right) t^n = \sum_{n\geq 1} (-1)^{n-1}\nu(n) e_n t^n,$$

where the last equality follows from the definition of $p_{n,\nu}$. Therefore,

$$\sum_{n\geq 1} p_{n,\nu} t^n = \frac{\sum_{n\geq 1} (-1)^{n-1}\nu(n) e_n t^n}{\sum_{n\geq 0} (-1)^n e_n t^n}$$

or, equivalently,

$$1 + \sum_{n\geq 1} p_{n,\nu} t^n = \frac{1 + \sum_{n\geq 1} (-1)^n (e_n - \nu(n) e_n) t^n}{\sum_{n\geq 0} (-1)^n e_n t^n}. \tag{11}$$

When taking $\nu(n) = 1$ for all $n \geq 1$, (11) becomes

$$1 + \sum_{n\geq 1} p_{n,1} t^n = 1 + \frac{\sum_{n\geq 1} (-1)^{n-1} e_n t^n}{\sum_{n\geq 0} (-1)^n e_n t^n} = \frac{1}{\sum_{n\geq 0} (-1)^n e_n t^n} = 1 + \sum_{n\geq 1} h_n t^n$$

which implies $p_{n,1} = h_n$. Other special cases for ν give well-known generating functions. For example, if $\nu(n) = n$ for $n \geq 1$, then $p_{n,\nu}$ is the power symmetric function $\sum_i x_i^n$. By taking $\nu(n) = (-1)^k \chi(n \geq k+1)$ for some $k \geq 1$, $p_{n,(-1)^k\chi(n\geq k+1)}$ is the Schur function corresponding to the partition $(1^k, n)$.

This definition of $p_{n,\nu}$ is desirable because of its expansion in terms of elementary symmetric functions. The coefficient of e_λ in $p_{n,\nu}$ has a nice combinatorial interpretation similar to that of the homogeneous symmetric functions. Suppose T is a brick tabloid of shape (n) and type λ and that the final brick in T has length ℓ. Define the weight of a brick tabloid $w_\nu(T)$ to be $\nu(\ell)$ and let

$$w_\nu(B_{\lambda,n}) = \sum_{\substack{T \text{ is a brick tabloid} \\ \text{of shape } (n) \text{ and type } \lambda}} w_\nu(T).$$

When $\nu(n) = 1$ for $n \geq 1$, $B_{\lambda,n}$ and $w_\nu(B_{\lambda,n})$ are the same. By the recursions found in the definition of $p_{n,\nu}$, it may be shown that

$$p_{n,\nu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_\nu(B_{\lambda,n}) e_\lambda$$

in almost the exact same way that (10) was proved in [7].

3 The proof of Theorem 1.

In this section, we shall prove Theorem 1.

We start out by proving

$$A_{\tau^{(i)}}(t,x) = \frac{1}{1 - \sum_{n\geq 1} GMP_{2n,\tau^{(i)}}(x) \frac{t^{2n}}{(2n)!}}. \tag{12}$$

Define a ring homomorphism θ from Λ into $\mathbb{Q}(x)$ by setting $\theta(e_0) = 1$, $\theta(e_{2n+1}) = 0$ for all $n \geq 1$, and

$$\theta(e_{2n}) = \frac{(-1)^{n-1}}{n!} GMP_{2n,\tau^{(i)}}(x) \tag{13}$$

for all $n \geq 1$.

Then we claim that $\theta(h_{2n-1}) = 0$ and

$$(2n)!\theta(h_{2n}) = \sum_{\sigma \in A_{2n}} x^{\tau^{(i)}\text{-mch}(\sigma)}. \tag{14}$$

for all $n \geq 1$. Note that by (10),

$$\theta(h_{2n-1}) = \sum_{\mu \vdash 2n-1} (-1)^{2n-1-\ell(\mu)} B_{\mu,2n-1} \theta(e_\mu).$$

Clearly if μ is partition of $2n-1$, then μ must have an odd part so that $\theta(e_\mu)=0$. Thus $\theta(h_{2n-1})=0$ for all $n\geq 1$. Note also that

$$\theta(h_{2n}) = \sum_{\mu\vdash 2n} (-1)^{2n-\ell(\mu)} B_{\mu,2n}\theta(e_\mu) \tag{15}$$

so that there is no loss if we restrict the sum on the right-hand side of (14) to partitions μ where every part of μ is even, i.e., to partitions of the form 2λ where λ is a partition of n and $2\lambda=(2\lambda_1,\ldots,2\lambda_{\ell(\lambda)})$ if $\lambda=(\lambda_1,\ldots,\lambda_{\ell(\lambda)})$. Thus

$$\begin{aligned}
&(2n)!\theta(h_{2n})\\
&= (2n)!\sum_{\lambda\vdash n}(-1)^{2n-\ell(\lambda)}B_{2\lambda,2n}\theta(e_{2\lambda})\\
&= (2n)!\sum_{\lambda\vdash n}(-1)^{2n-\ell(\lambda)}B_{2\lambda,2n}\prod_{j=1}^{\ell(\lambda)}\frac{(-1)^{2\lambda_j-1}}{(2\lambda_j)!}GMP_{2\lambda_j,\tau^{(i)}}(x)\\
&= \sum_{\lambda\vdash n}\binom{2n}{2\lambda_1,\ldots,2\lambda_{\ell(\lambda)}}B_{2\lambda,2n}\prod_{j=1}^{\ell(\lambda)}GMP_{2\lambda_j,\tau^{(i)}}(x)\\
&= \sum_{\lambda\vdash n}\sum_{T=(2b_1,\ldots,2b_{\ell(\lambda)})\in\mathcal{B}_{2\lambda,2n}}\binom{2n}{2b_1,\ldots,2b_{\ell(\lambda)}}\prod_{j=1}^{\ell(\lambda)}GMP_{2b_j,\tau^{(i)}}(x).
\end{aligned} \tag{16}$$

Next we want to give a combinatorial interpretation to the right-hand side of (15). We start with a brick tabloid $T=(2b_1,\ldots,2b_{\ell(\lambda)})$ of type 2λ. Then the binomial coefficient $\binom{2n}{2b_1,\ldots,2b_{\ell(\lambda)}}$ allows us to pick a set partition $\vec{U}=(U_1,\ldots,U_{\ell(\lambda)})$ of $\{1,\ldots,2n\}$ where $|U_i|=2b_i$ for $i=1,\ldots,\ell(\lambda)$. Next we use the factor $\prod_{j=1}^{\ell(\lambda)}GMP_{2b_j,\tau^{(i)}}(x)$ to choose a sequence of permutations $\vec{\sigma}=(\sigma^{(1)},\ldots,\sigma^{(\ell(\lambda))})$ such that $\sigma^{(j)}\in A_{2\lambda_j}$ is a generalized maximum packing for $\tau^{(i)}$ for $j=1,\ldots,\ell(\lambda)$. Then for each j, we let $\alpha^{(j)}$ be the sequence that arises by replacing the r-th largest element of $\sigma^{(j)}$ by the r-th largest element of U_j and then placing these elements in the cells of brick $2b_j$ from left to right. For example, we have pictured this process for $\tau^{(1)}=1324$ where the underlying brick tableau $T=(2,8,6)$. We have also indicated the block structure in each brick by underlying those elements in a common block. The weight of such a triple $(T,\vec{U},\vec{\sigma})$, $w(T,\vec{U},\vec{\sigma})$, is $\prod_{j=1}^{\ell(\lambda)}w(\sigma^{(j)})$. We can interpret $w(T,\vec{U},\vec{\sigma})$ by placing a weight 1 on top of each block of size 2 that ends a brick and a weight of -1 on each block of size 2 which does not end a brick. For blocks of size ≥ 4, we place an $(x-1)$ at start of each $\tau^{(i)}$-match in the block and, in addition, we add a factor of -1 to the first match in the block if the block is not the last block in a brick. Thus the RHS of (16) can be interpreted as a the sum of the weights of all triples (T,α,L) such that

1. $T=(d_1,\ldots,d_k)$ is brick tabloid of shape $(2n)$ where each brick d_j has even length,
2. α is a permutation of S_{2n} such that in each brick d_j, the sequence of elements in brick d_j reduces to a permutation in $\mathcal{GMP}_{d_j,\tau^{(i)}}$,

3. $L : \{1, \ldots, 2n\} \to \mathbb{Q}[x]$ assigns a label to each cell in T where $L(j)$ denotes the label of j-th cell of T, reading from left to right, and

   (a) $L(j) = 1$ if j is the second cell of a block of size 2 or j does not start a $\tau^{(i)}$-match in a block of size ≥ 4,

   (b) $L(j) = 1$ is j is the first cell of block of size 2 which ends a brick and $L(j) = -1$ is j is the first cell of block of size 2 which does not end a brick,

   (c) $L(j) = (x-1)$ is j is the first cell of block of size ≥ 4 which ends a brick and $L(j) = -(x-1)$ is j is the first cell of block of size ≥ 4 which does not end a brick, and

   (d) $L(j) = (x-1)$ if j is a cell which starts a $\tau^{(i)}$-match in a block of size ≥ 6 which is not the first cell in its block.

We define the weight of (T, α, L), $w(T, \alpha, L)$ to be $\prod_{j=1}^{2n} L(j)$. For example, in Figure 2, $T = (2, 8, 6)$, $\alpha = 1\ 3\ 4\ 5\ 7\ 10\ 9\ 12\ 11\ 13\ 2\ 8\ 6\ 14\ 15\ 16$, and L is the labeling where all the cells which do not have explicit label in them are assumed to have label 1.

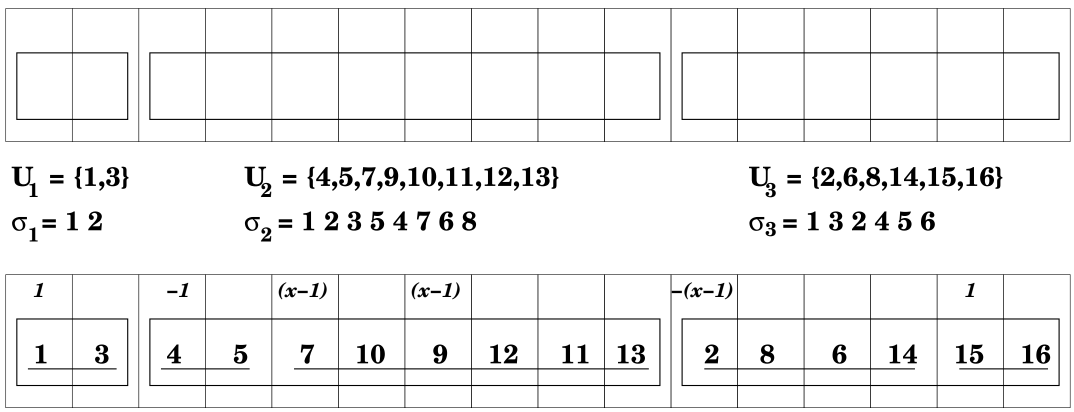

Figure 2: An elements of $\mathcal{T}_{16,\tau^{(1)}}$.

We let $\mathcal{T}_{2n,\tau^{(i)}}$ denote the set of all such triples constructed in this way. It then follows that

$$(2n)!\theta(h_{2n}) = \sum_{(T,\alpha,L)\in\mathcal{T}_{2n,\tau^{(i)}}} w(T, \alpha, L) \tag{17}$$

Next we will define two involutions I and J which will show that RHS of (17) is equal to the RHS of (14). We define $I : \mathcal{T}_{2n,\tau^{(i)}} \to \mathcal{T}_{2n,\tau^{(i)}}$ as follows. Suppose that we are given a triple (T, α, L) where $T = (d_1, \ldots, d_k)$. Then read the bricks from left to right until you find the first brick d_j such that either (i) the generalized maximum packing corresponding to the elements in d_j consists of more than one block or (ii) the generalized maximum packing corresponding to the elements in d_j consists of a single block and the last element of d_j is less than the first element of the following brick d_{j+1}. In case (i), split d_j into two bricks d^* and d^{**} where d^* contains the cells of the first block in the generalized maximum packing corresponding to the elements in d_j and d^{**} contains the remaining cells of d_j. We keep all the labels the same except that we change the label on the first cell of d^* from -1 to 1 if the first block of d_j is of size 2 and from $-(x-1)$ to $(x-1)$ if the first block of d_j has size ≥ 4. In case (ii), we combine bricks d_j and d_{j+1} into a single brick d. Note that since the last element of d_j is less than the first element of d_{j+1}, the elements in the new brick d will still reduce of generalized maximum packing. We keep all the labels the same except that we change the label on the first cell of d_j from 1 to -1 if d_j is of size 2 and from $(x-1)$ to $-(x-1)$ if d_j has size ≥ 4. In both cases, we do not change the underlying permutation α. If neither case (i) or case (ii) applies, then $I(T,\alpha,L) = (T,\alpha,L)$. For example, if (T,α,L) is the element of $\mathcal{T}_{16,\tau^{(1)}}$ pictured in Figure 2, then we are in case (ii) since we can combine the first and second bricks so that $I(T,\alpha,L)$ is pictured in Figure 3.

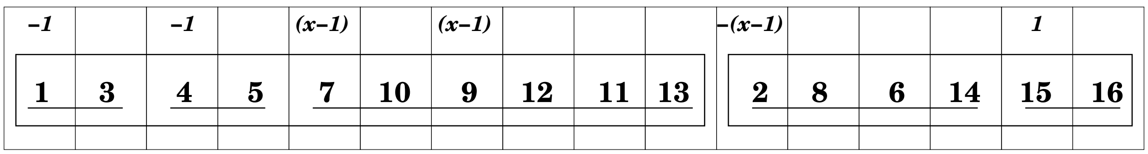

Figure 3: The image of (T,α,L) in Figure 2 under I.

It is easy to see that if $I(T,\alpha,L) = (T',\alpha,L') \neq (T,\alpha,L)$, then $I(T',\alpha,L') = (T,\alpha,L)$ and $w(T,\alpha,L) = -w(T',\alpha,L')$. Hence I shows that

$$\begin{aligned}(2n)!\theta(h_{2n}) &= \sum_{(T,\alpha,L)\in\mathcal{T}_{2n,\tau^{(i)}}} w(T,\alpha,L) \\ &= \sum_{(T,\alpha,L)\in\mathcal{T}_{2n,\tau^{(i)}},I(T,\alpha,L)=(T,\alpha,L)} w(T,\alpha,L). \qquad (18)\end{aligned}$$

Thus we must examine the fixed points of I. Clearly, if (T,α,L) is a fixed point of I, then the elements of each brick d in T must reduce to a generalized maximum packing of $\tau^{(i)}$ which consists of a single block. Second, we must not be able to combine any two bricks so that if $T = (d_1,\ldots,d_k)$, then the last element of d_j is greater than or equal to the first element of d_{j+1} for $j = 1,\ldots,k-1$. But this means that the underlying permutation α is an up-down permutation. It follows that the fixed points I consists of triples (T,α,L) such that

(I) α is an up-down permutation of length $2n$,

(II) $T = (d_1,\ldots,d_k)$ where each d_j has even length and the elements of d_j reduces to a generalized maximum packing of $\tau^{(i)}$ which consists of a single block, and

(III) the label of $L(j)$ of the j-th cell of T is $(x-1)$ if j is the start of $\tau^{(i)}$-match in α and is equal to 1 otherwise.

Next we want to modify our interpretation of the right-hand side of (18) to consists of all triples (T',α,L') such that

(I') α is an up-down permutation of length $2n$,

(II') $T = (d_1, \ldots, d_k)$ where each d_j has even length and the elements of d_j reduces to a generalized maximum packing of $\tau^{(i)}$ which consists of a single block, and

(III') the label of $L(j)$ of the j-th cell of T is either x or -1 if j is the start of $\tau^{(i)}$-match in α and is equal to 1 otherwise.

We let $\mathcal{FI}_{2n,\tau^{(i)}}$ denote the set of triples (T', α, L') satisfying (I')-(III') and for any $(T', \alpha, L') \in \mathcal{FI}_{2n,\tau^{(i)}}$, we define the weight of (T', α, L'), $w(T', \alpha, L')$, to be $\prod_{j=1}^{2n} L'(j)$. For example, Figure 4 pictures an element of $\mathcal{FI}_{16,\tau^{(1)}}$ whose weight is x where again the cells which do not have labels are assumed to have label 1.

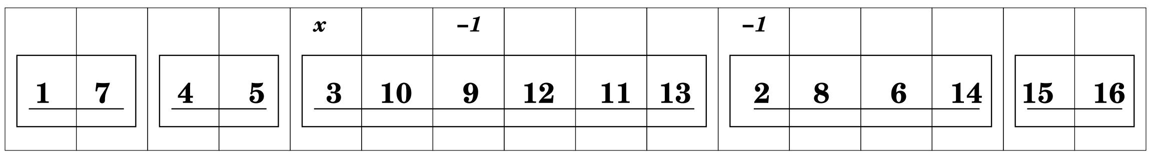

Figure 4: An element of $\mathcal{FI}_{16,\tau^{(1)}}$.

It then follows that

$$(2n)!\theta(h_{2n}) = \sum_{(T',\alpha,L') \in \mathcal{FI}_{2n,\tau^{(i)}}} w(T', \alpha, L'). \tag{19}$$

Next we define an involution $J : \mathcal{FI}_{2n,\tau^{(i)}} \to \mathcal{FI}_{2n,\tau^{(i)}}$. Given an element $(T, \alpha, L) \in \mathcal{FI}_{2n,\tau^{(i)}}$, scan the cells of $T = (d_1, \ldots, d_k)$ from left to right looking for the first brick d_j such that either (A) d_j has a -1 label on one its cells or (B) the elements of d_j and d_{j+1} reduce a generalized maximum packing of $\tau^{(i)}$ which consists of a single block. In case (A), we break d_j into 2 bricks d^* and d^{**} as follows. We know that that there $\tau^{(i)}$-matches starting at cells $1, 3, \ldots, d_j - 3$ of d_j. Let c be the left most cell of d_j which has a -1 label. If c is the first cell of d_j, then $d*$ will consist of the first two cells of d_j and d^{**} will consist of the rest of the cells of d_j. If is not the first cell of d_j, then c is the $2i-1$-th cell of d_j for some $i > 1$. Thus there are $\tau^{(i)}$-matches starting a cell $c-2$ and cell c of d_j and cell $c-2$ is labeled with x. Then we let d^* be the contain the cells of d_j up to and including cell $c+2$ and d^{**} contain the rest of the cells of d_j. In either case, we remove the -1 label from cell c and replace by 1. In case (B), there must a $\tau^{(i)}$-match starting at the second to last cell of d_j that includes the last two cells of d_j and the first two cells of d_{j+1}. We then replace replace the bricks d_j and d_{j+1} by a single brick d and replace the label of 1 on the second to last cell of d_j by -1. In either case, we do not change the underlying permutation α. If neither case (A) or case (B) applies, then we let $J(T, \alpha, L) = (T, \alpha, L)$. For example, if we consider the (T, α, L) as pictured in Figure 4, we cannot combine bricks d_1 and d_2 because α does not have a $\tau^{(1)}$-match starting cell 1 and we cannot combine bricks d_2 and d_3 because α does not have a $\tau^{(1)}$-match starting cell 3. Thus we are in case (B) where $d_j = d_3$ and $c = 7$. Thus we split d_3 at two cells to the right of cell c which means at cell 9. Thus $J(T,\alpha,L) = (T',\alpha,L')$ is pictured in Figure 5. Note that it will automatically be the case that the first action that we can take for (T',α,L') is to combine the two brick that made up the d_3 in (T,α,L).

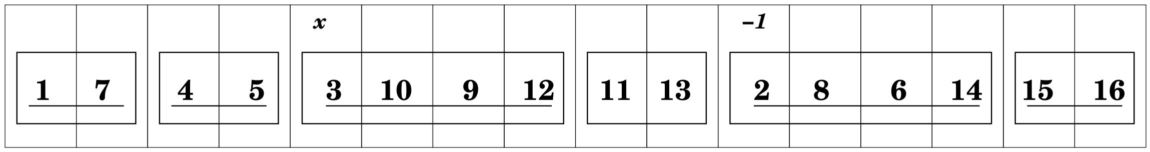

Figure 5: $J(T,\alpha,L)$ for (T,α,L) of Figure 4.

It is easy to see that if $J(T,\alpha,L) = (T',\alpha,L') \neq (T,\alpha,L)$, then $w((T,\alpha,L) = -w(T',\alpha,L')$ and $J(T',\alpha,L') = (T,\alpha,L)$. Thus it follows that

$$(2n)!\theta(h_{2n}) = \sum_{(T,\alpha,L)\in\mathcal{FI}_{2n,\tau^{(i)}},J(T,\alpha,L)=(T,\alpha,L)} w(T,\alpha,L). \tag{20}$$

Thus we must examine the fixed points of J. If $J(T,\alpha,L) = (T,\alpha,L)$, then clearly (T,α,L) can have no cells which have a -1 label. Thus in a brick d of T of size ≥ 4, the start of every $\tau^{(i)}$ starting in d is labeled with an x. Moreover, there there cannot be an $\tau^{(i)}$-matches that involves cells in two different bricks since otherwise we could combine those two bricks since we would be guaranteed that elements of in those two cells would reduce to a generalized maximum packing with a single block. Hence $w(T,\alpha,L) = x^{\tau^{(i)}\text{-mch}(\alpha)}$ Moreover if we are given an $\alpha \in A_{2n}$, we can create a fixed point (T,α,L) of J by defining the bricks $d_1, d_2, \ldots$ inductively. That is, we let d_1 be of size 2 if there is no $\tau^{(i)}$-match in α starting at 1 and d_1 be of size $2s$ if there are $\tau^{(i)}$-matches starting at positions $1, 3, 2s-3$ but not at $2s-1$ in α. Then having defined bricks $d_1, \ldots, d_r$ where d_r ends at cell $c = 2k < 2n$, we let d_{r+1} be of size 2 if there is no $\tau^{(i)}$-match in α starting at $2k+1$ and d_{r+1} be of size $2s$ if there are $\tau^{(i)}$-matches starting at positions $2k+1, 2k+3, 2k+2s-3$ but not at $2k+2s-1$ in α. Hence

$$\begin{aligned}(2n)!\theta(h_{2n}) &= \sum_{(T,\alpha,L)\in\mathcal{FI}_{2n,\tau^{(i)}},J(T,\alpha,L)=(T,\alpha,L)} w(T,\alpha,L) &(21)\\ &= \sum_{\alpha\in A_{2n}} x^{\tau^{(i)}\text{-mch}(\alpha)}. &(22)\end{aligned}$$

It then follows that

$$\begin{aligned}
&1+\sum_{n\geq 1}\frac{t^{2n}}{(2n)!}\sum_{\alpha\in A_{2n}} x^{\tau^{(i)}\text{-mch}(\alpha)}\\
&=\theta(\sum_{n\geq 0} h_n t^n)\\
&=\frac{1}{1+\sum_{n\geq 1}(-t)^n\theta(e_n)}\\
&=\frac{1}{1+\sum_{n\geq 1}(-t)^{2n}\frac{(-1)^{2n-1}}{(2n)!}GMP_{2n,\tau^{(i)}}(x)}\\
&=\frac{1}{1-\sum_{n\geq 1}\frac{t^{2n}}{(2n)!}GMP_{2n,\tau^{(i)}}(x)}.
\end{aligned}$$

which is what we wanted to prove.

Next we want to prove

$$B_{\tau^{(i)}}(t,x)=\frac{\sum_{n\geq 1} GMP_{2n-1,\tau^{(i)}}(x)\frac{t^{2n-1}}{(2n-1)!}}{1-\sum_{n\geq 1} GMP_{2n,\tau^{(i)}}(x)\frac{t^{2n}}{(2n)!}}. \tag{23}$$

To prove (23), we need to define an appropriate weight function $\nu:\{1,2,\ldots\}\rightarrow\mathbb{Q}(x)$. That is, we let $\nu(2i-1)=0$ for all $i\geq 1$ and

$$\begin{aligned}
\nu(2n)&=\frac{\frac{(-1)^{2n-1}}{(2n-1)!}GMP_{2n-1,\tau^{(i)}}(x)}{\frac{(-1)^{2n-1}}{(2n)!}GMP_{2n,\tau^{(i)}}(x)}\\
&=(2n)\frac{GMP_{2n-1,\tau^{(i)}}(x)}{GMP_{2n,\tau^{(i)}}(x)}. \qquad (24)
\end{aligned}$$

We have designed ν so that

$$\nu(2n)\theta(e_{2n})=\frac{(-1)^{2n-1}}{(2n-1)!}GMP_{2n-1,\tau^{(i)}}(x). \tag{25}$$

Then we claim that $\theta(p_{2n+1,\nu})=0$ and

$$(2n+1)!\theta(p_{2n+2,\nu})=\sum_{\sigma\in A_{2n+1}} x^{\tau^{(i)}\text{-mch}(\sigma)}. \tag{26}$$

for all $n\geq 0$. Note that by (10),

$$\theta(p_{2n+1,\nu})=\sum_{\mu\vdash 2n+1}(-1)^{2n+1-\ell(\mu)}w_\nu(B_{\mu,2n+1})\theta(e_\mu).$$

Clearly if μ is partition of $2n+1$, then μ must have an odd part so that $\theta(e_\mu)=0$. Thus $\theta(p_{2n+1,\nu})=0$ for all $n\geq 0$. It also follows that when we want to compute $\theta(p_{2n+2,\nu})$, we can restrict ourselves to considering partitions of the form 2λ where λ is a partition of $n+1$. Thus

$$\begin{aligned}
&(2n+1)!\theta(p_{2n+2,\nu}) \\
&= (2n+1)! \sum_{\lambda \vdash n+1} (-1)^{2n+2-\ell(\lambda)} w_\nu(B_{2\lambda,2n+2})\theta(e_{2\lambda}) \\
&= (2n+1)! \sum_{\lambda \vdash n+1} (-1)^{2n+2-\ell(\lambda)} \sum_{T=(2b_1,\ldots,2b_{\ell(\lambda)})\in\mathcal{B}_{2\lambda,2n+2}} \nu(2b_{\ell(\lambda)})\theta(e_{2b_{\ell(\lambda)}})\times \\
&\qquad \prod_{j=1}^{\ell(\lambda)-1} \frac{(-1)^{2b_j-1}}{(2b_j)!} GMP_{2b_j,\tau^{(i)}}(x) \\
&= (2n+1)! \sum_{\lambda \vdash n+1} (-1)^{2n+2-\ell(\lambda)} \sum_{T=(2b_1,\ldots,2b_{\ell(\lambda)})\in\mathcal{B}_{2\lambda,2n+2}} \frac{(-1)^{2b_{\ell(\lambda)}-1}}{(2b_{\ell(\lambda)}-1)!} GPM_{2b_{\ell(\lambda)}-1,\tau^{(i)}}\times \\
&\qquad \prod_{j=1}^{\ell(\lambda)-1} \frac{(-1)^{2b_j-1}}{(2b_j)!} GMP_{2b_j,\tau^{(i)}}(x) \\
&= \sum_{\lambda \vdash n+1} \sum_{T=(2b_1,\ldots,2b_{\ell(\lambda)})\in\mathcal{B}_{2\lambda,2n+2}} \binom{2n+1}{2b_1,\ldots,2b_{\ell(\lambda)-1},,2b_{\ell(\lambda)}-1} GPM_{2b_{\ell(\lambda)}-1,\tau^{(i)}}\times \\
&\qquad \prod_{j=1}^{\ell(\lambda)-1} GMP_{2b_j,\tau^{(i)}}(x).
\end{aligned} \tag{27}$$

As before, we want to give a combinatorial interpretation to the right-hand side of (27). We start with a brick tabloid $T=(2b_1,\ldots,2b_{\ell(\lambda)})$ of length $2n+2$ and type 2λ. Then the binomial coefficient $\binom{2n+1}{2b_1,\ldots,2b_{\ell(\lambda)-1},2b_{\ell(\lambda)}-1}$ allows us to pick a set partition $\vec{U} = (U_1,\ldots,U_{\ell(\lambda)})$ of $\{1,\ldots,2n\}$ where $|U_i| = 2b_i$ for $i=1,\ldots,\ell(\lambda)-1$ and $|S_{\ell(\lambda)}| = 2b_{\ell(\lambda)} - 1$. Next we use the factor $GMP_{2b_{\ell(\lambda)}-1}(x)\prod_{j=1}^{\ell(\lambda)-1} GMP_{2b_j,\tau^{(i)}}(x)$ to choose a sequence of permutations $\vec{\sigma} = (\sigma^{(1)},\ldots,\sigma^{(\ell(\lambda))})$ such that $\sigma^{(j)} \in A_{2b_j}$ is a generalized maximum packing for $\tau^{(i)}$ for $j=1,\ldots,\ell(\lambda)-1$ and $\sigma^{(\ell(\lambda))} \in A_{2b_{\ell(\lambda)}-1}$ is a generalized maximum packing for $\tau^{(i)}$. Then for each j, we let $\alpha^{(j)}$ be the sequence that arises by replacing the r-th largest element of $\sigma^{(j)}$ by the r-th largest element of U_j and then placing these elements in the cells of brick $2b_j$ from left to right. This means that for the last brick $2b_{\ell(\lambda)}$, we will fill in all but the last cell which we leave blank. For example, we have pictured this process for $\tau^{(1)} = 1324$ where the underlying brick tableau $T=(2,8,6)$. We have also indicated the block structure in each brick by underlying those elements in a common block. The weight of such a triple $(T,\vec{U},\vec{\sigma})$, $w(T,\vec{U},\vec{\sigma})$, is $\prod_{j=1}^{\ell(\lambda)} w(\sigma^{(j)})$. We can interpret $w(T,\vec{U},\vec{\sigma})$ by placing a weight 1 on top of each block of size 1 or 2 that ends a brick and a weight of -1 on each block of size 2 which does not end a brick. For blocks that size ≥ 4, we place an $(x-1)$ at start of each $\tau^{(i)}$-match in the block and, in addition, we add a factor of -1 to the first match in the block if the block is not the last block in its brick. Thus the RHS of (27) can be interpreted as a the sum of the weights of all triples (T, α, L) such that

1. $T = (d_1, \ldots, d_k)$ is brick tabloid of shape $(2n+2)$ where each brick d_j has even length,

2. α is a permutation of S_{2n+1} such that in each brick d_j with $j < k$, the sequence of elements in brick d_j reduces to a permutation in $\mathcal{GMP}_{d_j,\tau^{(i)}}$ and the sequence of elements in brick d_k fill the first $d_k - 1$ cells of brick d_k and reduces to a permutation in $\mathcal{GMP}_{d_k-1,\tau^{(i)}}$

3. $L : \{1, \ldots, 2n+1\} \to \mathbb{Q}[x]$ assigns a label to each cell in T where $L(j)$ denotes the label of j-th cell of T, reading from left to right, and

   (a) $L(j) = 1$ if j is the second cell of a block of size 2 or j does not start a $\tau^{(i)}$-match in a block of size ≥ 4,

   (b) $L(j) = 1$ is j is the first cell of block of size 1 or 2 which ends a brick and $L(j) = -1$ is j is the first cell of block of size 2 which does not end a brick,

   (c) $L(j) = (x-1)$ is j is the first cell of block of size ≥ 4 which ends a brick and $L(j) = -(x-1)$ is j is the first cell of block of size ≥ 4 which does not end a brick, and

   (d) $L(j) = (x-1)$ if j is a cell which starts a $\tau^{(i)}$-match in a block of size ≥ 6 which is not the first cell in its block,

where the weight of (T, α, L), $w(T, \alpha, L)$, equals $\prod_{j=1}^{2n+1} L(j)$. For example, in Figure 6, $T = (2, 8, 6)$, $\alpha = 1\ 4\ 3\ 7\ 5\ 10\ 9\ 12\ 11\ 13\ 2\ 8\ 6\ 14\ 15$, and L is the labeling where all the cells which do not have explicit label in them are assumed to have label 1.

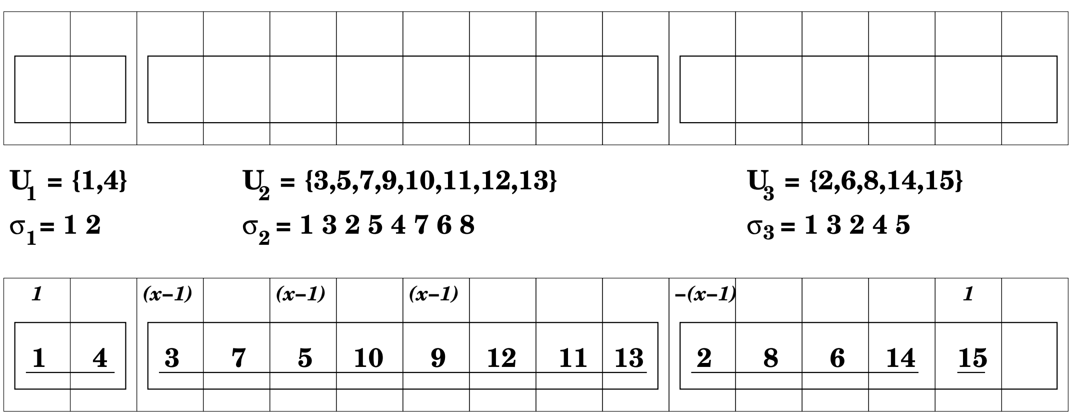

Figure 6: An element of $\mathcal{T}_{15,\tau^{(1)}}$.

We let $\mathcal{T}_{2n+1,\tau^{(i)}}$ denote the set of all such triples constructed in this way. It then follows that

$$(2n+1)!\theta(p_{2n+2,\nu}) = \sum_{(T,\alpha,L)\in\mathcal{T}_{2n+1,\tau^{(i)}}} w(T, \alpha, L) \tag{28}$$

Note that the only differences between the fillings of even length in the proof of (14) and our current fillings is that, in our current fillings, we have forced that last brick ends in a block of size 1 and the last cell of that brick is blank.

Next we will define two involutions I and J which will show that RHS of (28) is equal to the RHS of (23). In fact the definitions of I and J are essentially the same as before. We simply observe that the fact that the last block the final brick is size 1 does not change things. We define $I : \mathcal{T}_{2n+1,\tau^{(i)}} \to \mathcal{T}_{2n+1,\tau^{(i)}}$ as follows. Given a triple (T, α, L), let $T = (d_1, \ldots, d_k)$. Then read the bricks from left to right until you find the first brick d_j such that either (i) the generalized maximum packing corresponding to the elements in d_j consists of more than one block or (ii) the generalized maximum packing corresponding to the elements in d_j consists of a single block and the last element of d_j is less than the first element of the following block d_{j+1}. In case (i), split d_j into two bricks d^* and d^{**} where d^* contains the cells of the first block of the generalized maximum packing corresponding to the elements in d_j and d^{**} contains the remaining cells of d_j. We keep all the labels the same except that we change the label on the first cell of d^* from -1 to 1 if the first block of d_j is of size 2 and from $-(x-1)$ to $(x-1)$ if the first block of d_j has size ≥ 4. In case (ii), we combine bricks d_j and d_{j+1} into a single brick d. Note that since the last element of d_j is less than the first element of d_{j+1}, the elements in the new brick d will still reduce of generalized maximum packing. We keep all the labels the same except that we change the label on the first cell of d_j from 1 to -1 if d_j is of size 2 and from $(x-1)$ to $-(x-1)$ if d_j has size ≥ 4. In both cases, we do not change the underlying permutation α. If neither case (i) or case (ii) applies, then $I(T, \alpha, L) = (T, \alpha, L)$. For example, if (T, α, L) is the element of $\mathcal{T}_{15,\tau^{(1)}}$ pictured in Figure 6, then we are in case (i) since we can combine the first and second bricks or the second and third bricks. Thus we break that last brick in to two bricks where the final brick of size 2 which has a single block of size 1. $I(T, \alpha, L)$ is pictured in Figure 7.

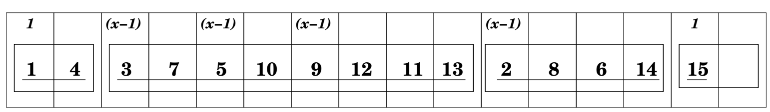

Figure 7: The image of (T, α, L) in Figure 6 under I.

It is easy to see that if $I(T, \alpha, L) = (T', \alpha, L') \neq (T, \alpha, L)$, then $I(T', \alpha, L') = (T, \alpha, L)$ and $w(T, \alpha, L) = -w(T', \alpha, L')$. Hence I shows that

$$\begin{aligned} (2n-1)!\theta(p_{2n+2,\nu}) &= \sum_{(T,\alpha,L)\in\mathcal{T}_{2n+1,\tau^{(i)}}} w(T,\alpha,L) \\ &= \sum_{(T,\alpha,L)\in\mathcal{T}_{2n+1,\tau^{(i)}}, I(T,\alpha,L)=(T,\alpha,L)} w(T,\alpha,L). \end{aligned} \tag{29}$$

Again we must examine the fixed points of I. Clearly, if (T, α, L) is a fixed point of I, then the elements of each brick d in T must reduce to a generalized maximum packing of $\tau^{(i)}$ which consists of a single block. In particular, this means that the last brick of T

must be a size 2 and consist of a block with 1 element. Second, we must not be able to combine any two bricks so that if $T = (d_1, \ldots, d_k)$, then the last element of d_j is greater than or equal to the first element of d_{j+1} for $j = 1, \ldots, k-1$. But this means that the underlying permutation α is an up-down permutation. It follows that the fixed points I consists of triples (T, α, L) such that

(I) α is an up-down permutation of length $2n+1$,

(II) $T = (d_1, \ldots, d_k)$ where each d_j with $j < k$ has even length and the elements of d_j reduces to a generalized maximum packing of $\tau^{(i)}$ which consists of a single block,

(III) d_k is a brick of size 2 which consists of a single block of size 1, and

(IV) the label of $L(j)$ of the j-th cell of T is $(x-1)$ if j is the start of $\tau^{(i)}$-match in α and is equal to 1 otherwise.

Now we can modify our interpretation of the right-hand side of (29) and define the involution J exactly as before since the last element in the last brick never played a role in either the modification or the involution J. Thus J will show that

$$(2n+1)!\theta(p_{2n+2,\nu}) = \sum_{\alpha \in A_{2n+1}} x^{\tau^{(i)}\text{-mch}(\alpha)}. \tag{30}$$

It then follows that

$$\begin{aligned}
&\sum_{n \geq 0} \frac{t^{2n+2}}{(2n+1)!} \sum_{\alpha \in A_{2n+1}} x^{\tau^{(i)}\text{-mch}(\alpha)} \\
&= \theta(\sum_{n \geq 1} p_{n,\nu} t^n) \\
&= \frac{\sum_{n \geq 1} (-1)^{n-1} \nu(n) \theta(e_n) t^n}{\sum_{n \geq 0} (-1)^n \theta(e_n) t^n} \\
&= \frac{\sum_{n \geq 1} (-1)^{2n-1} \frac{(-1)^{2n-1}}{(2n-1)!} GMP_{2n-1,\tau^{(i)}}(x) t^{2n}}{1 - \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} GMP_{2n,\tau^{(i)}}(x)} \\
&= \frac{\sum_{n \geq 1} \frac{t^{2n}}{(2n-1)!} GMP_{2n-1,\tau^{(i)}}(x)}{1 - \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} GMP_{2n,\tau^{(i)}}(x)}.
\end{aligned} \tag{31}$$

If we divide both sides of (31) by t, we obtain that

$$\sum_{n \geq 0} \frac{t^{2n+1}}{(2n+1)!} \sum_{\alpha \in A_{2n+1}} x^{\tau^{(i)}\text{-mch}(\alpha)} = \frac{\sum_{n \geq 1} GMP_{2n-1,\tau^{(i)}}(x) \frac{t^{2n-1}}{(2n-1)!}}{1 - \sum_{n \geq 1} GMP_{2n,\tau^{(i)}}(x) \frac{t^{2n}}{(2n)!}} \tag{32}$$

which is what we wanted to prove.

4 Computing $\mathbf{mp}_{n,\tau^{(i)}}$.

In this section, we shall consider the problem of computing $\mathrm{mp}_{\tau^{(i)},n}$ since we will need such computations to compute $GMP_{\tau^{(i)},n}$.

The problem of computing $\mathrm{mp}_{\tau^{(i)},2n}$ has been studied by Harmse and Remmel [11] in a different context. Harmse and Remmel studied maximum packings in column strict arrays. That is, $\mathcal{F}_{n,k}$ denote the set of all fillings of a $k \times n$ rectangular array with the integers $1, \ldots, kn$ such that that the elements increase from bottom to top in each column. We let (i,j) denote the cell in the i-th row from the bottom and the j-th column from the left of the $k \times n$ rectangle and we let $F(i,j)$ denote the element in cell (i,j) of $F \in \mathcal{F}_{n,k}$.

If F is any filling of a $k \times n$-rectangle with distinct positive integers such that elements in each column increase, reading from bottom to top, then we let $\mathrm{red}(F)$ denote the element of $\mathcal{F}_{n,k}$ which results from F by replacing the i-th smallest element of F by i. For example, Figure 8 demonstrates a filling, F, with its corresponding reduced filling, $\mathrm{red}(F)$.

F =

12	16	22
8	15	17
6	10	13
1	7	5

red(F) =

7	10	12
5	9	11
3	6	8
1	4	2

Figure 8: An example of $F \in \mathcal{F}_{3,4}$ and $\mathrm{red}(F)$.

If $F \in \mathcal{F}_{n,k}$ and $1 \leq c_1 < \cdots < c_j \leq n$, then we let $F[c_1, \ldots, c_j]$ be the filling of the $k \times j$ rectangle where the elements in column a of $F[c_1, \ldots, c_j]$ equal the elements in column c_a in F for $a = 1, \ldots, j$. We can then extend the usual pattern matching definitions from permutations to elements of $\mathcal{F}_{n,k}$ as follows.

Definition 2. *Let P be an element of $\mathcal{F}_{j,k}$ and $F \in \mathcal{F}_{n,k}$ where $j \leq n$. Then we say*

1. *P **occurs** in F if there are $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ such that* $\mathrm{red}(F[i_1, \ldots, i_j]) = P$,
2. *F **avoids** P if there is no occurrence of P in F, and*
3. *there is a **P-match in F starting at position** i if* $\mathrm{red}(F[i, i+1, \ldots, i+j-1]) = P$.

When $k = 1$, then $\mathcal{F}_{n,1} = S_n$, where S_n is the symmetric group, and our definitions reduce to the standard definitions that have appeared in the pattern matching literature.

We let P-mch(F) denote the number of P-matches in F. For example, if we consider the fillings $P \in \mathcal{F}_{3,3}$ and $F, G \in \mathcal{F}_{6,3}$ shown in Figure 9, then it is easy to see that there are no P-matches in F but there is an occurrence of P in F, since $\mathrm{red}(F[1,2,5]) = P$. Also, there are 2 P-matches in G starting at positions 1 and 2, respectively, so P-mch$(G) = 2$.

If $P \in \mathcal{F}_{2,k}$, then we define MP_n^P to be the set of $F \in \mathcal{F}_{n,k}$ with P-mch$(F) = n-1$, i.e. the set of $F \in \mathcal{F}_{n,k}$ with the property that there are P-matches in F starting at positions $1, 2, \ldots, n-1$. We let $\mathrm{mp}_n^P = |MP_n^P|$, and by convention, define $\mathrm{mp}_1^P = 1$. For

P=

3	6	9
2	5	8
1	4	7

F=

4	11	12	16	18	14
2	10	8	13	17	9
1	5	6	3	15	7

G =

4	7	11	16	18	14
2	6	10	13	17	9
1	5	8	12	15	3

Figure 9: Examples of P-matches and occurrences of P.

example, if P is the element of $\mathcal{F}_{2,k}$ that has the elements $1, \ldots, k$ in the first column and the elements $k+1, \ldots, 2k$ in the second column, then it is easy to see that $\mathrm{mp}_n^P = 1$ for all $n \geq 1$ since the only element of $F \in \mathcal{F}_{n,k}$ with P-mch$(F) = n-1$ has the entries $(i-1)k+1, \ldots, (i-1)k+k$ in the i-th column for $i = 1, \ldots, n$.

For each $\tau^{(i)}$, let $P^{(i)}$ denote the element of $\mathcal{F}_{2,2}$ such that the first and third elements of $\tau^{(i)}$ are the first and second elements in row 1 of $P^{(i)}$, respectively, and the second and forth elements of $\tau^{(i)}$ are the first and second elements in row 2 of $P^{(i)}$, respectively. We have pictured $P^{(1)}, \ldots, P^{(5)}$ in Figure 10. It is then easy to see that given maximum packing $F \in \mathcal{F}_{n,2}$ of $P^{(i)}$, one can construct an up-down permutation $\sigma(F) \in A_{2n}$ which is maximum packing for $\tau^{(i)}$ by reading each column from bottom to top and the columns from right to left. Vice versa, given $\sigma = \sigma_1 \ldots, \sigma_{2n} \in A_{2n}$ which is a maximum packing for $\tau^{(i)}$, we can obtain an element $F(\sigma) \in \mathcal{F}_{n,2}$ which is a maximum packing for $P^{(i)}$ by placing $\sigma_1, \sigma_3, \ldots, \sigma_{2n-1}$ in the bottom row of $F(\sigma)$, reading from left to right, and placing $\sigma_2, \sigma_4, \ldots, \sigma_{2n}$ in the top row of $F(\sigma)$, reading from left to right. An example of this correspondence is pictured at the top of Figure 10 for $\tau^{(1)} = 1324$ and $P^{(1)} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$.

5	6	8	9	11	13	15	16
1	2	3	4	7	10	12	14

⟶ 1 5 2 6 3 8 4 9 7 11 10 13 12 15 14 16

1324 ⟶

3	4
1	2

2413 ⟶

4	3
2	1

1423 ⟶

4	3
1	2

3412 ⟶

4	2
3	1

2314 ⟶

3	4
2	1

Figure 10: The correspondence between $\mathcal{MP}_{2n,\tau^{(i)}}$ and $\mathcal{MP}_n^{P^{(i)}}$.

It follows that for $i = 1, \ldots, 5$, $\mathrm{mp}_{2n,\tau^{(i)}} = \mathrm{mp}_n^{P^{(i)}}$. Now Harmse and Remmel [11] proved that for $n \geq 2$,

$$\begin{aligned} \mathrm{mp}_n^{P^{(1)}} &= \mathrm{mp}_n^{P^{(4)}} = C_{n-1} \text{ and} \\ \mathrm{mp}_n^{P^{(2)}} &= \mathrm{mp}_n^{P^{(3)}} = \mathrm{mp}_n^{P^{(5)}} = 1. \end{aligned}$$

Thus we know that

$$\begin{aligned} \mathrm{mp}_{2n,\tau^{(1)}} &= \mathrm{mp}_{2n,\tau^{(4)}} = C_{n-1} \text{ and} \\ \mathrm{mp}_{2n,\tau^{(2)}} &= \mathrm{mp}_{2n,\tau^{(3)}} = \mathrm{mp}_{2n,\tau^{(5)}} = 1. \end{aligned}$$

To compute $\mathrm{mp}_{2n+1,\tau^{(i)}}$, we must exploit some of the techniques used by Harmse and Remmel [11] to compute mp_n^P for $P \in \mathcal{F}_{2,k}$. To help us visualize the order relationships within $P^{(i)}$, we form a directed graph $G_{P^{(i)}}$ on the cells of the 2×2 rectangle by drawing a directed edge from the position of the number j to the position of the number $j+1$ in P for $j = 1, \ldots, 3$. For example, in Figure 11, the graph $G_{P^{(1)}}$ pictured immediately to the right $P^{(1)}$. Then $G_{P^{(i)}}$ determines the order relationships between all the cells in $P^{(i)}$ since $P^{(i)}(r,s) < P^{(i)}(u,v)$ if there is a directed path from cell (r,s) to cell (u,v) in $G_{P^{(i)}}$. Now suppose that $F \in \mathcal{MP}_n^{P^{(i)}}$ where $n \geq 3$. Because there is a $P^{(i)}$-match starting in column j, we can superimpose $G_{P^{(i)}}$ on the cells in columns j and $j+1$ to determine the order relations between the elements in those two columns. If we do this for every pair of columns, j and $j+1$ for $j = 1, \ldots, n-1$, we end up with a directed graph on the cells of the $2 \times n$ rectangle which we will call $G_{n,P^{(i)}}$. For example, Figure 11, $G_{6,P^{(1)}}$ is pictured in the second row. It is then easy to see that if $F \in \mathcal{MP}_n^{P^{(i)}}$ and there is a directed path from cell (r,s) to cell (u,v) in $G_{n,P^{(i)}}$, then it must be the case that $F(r,s) < F(u,v)$. Note that $G_{n,P^{(i)}}$ will always be a directed acyclic graph with no multiple edges.

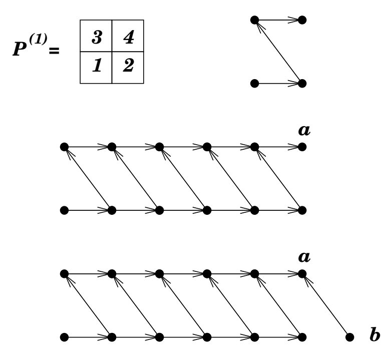

Figure 11: The graphs $G_{n,P^{(1)}}$ and $G^+_{n,P^{(1)}}$.

Harmse and Remmel [11] proved that the problem of computing $\mathrm{mp}_n^{P^{(i)}}$ for a column strict tableau $P^{(i)}$ of shape 2^2 can be reduced to finding the number of linear extensions of a certain poset associated with $P^{(i)}$. That is, the graph $G_{n,P^{(i)}}$ induces a poset $\mathcal{W}_{n,P^{(i)}} = (\{(i,j) : 1 \leq i \leq 2 \;\&\; 1 \leq j \leq n\}, <_W)$ on the cells of the $2 \times n$ rectangle by defining $(i,j) <_W (s,t)$ if and only if there is a directed path from (i,j) to (s,t) in $G_{n,P^{(i)}}$. Harmse and Remmel proved that there is a 1:1 correspondence between the elements of

$\mathcal{MP}_n$ and the linear extensions of $\mathcal{W}_{n,P^{(i)}}$. That is, if $F \in \mathcal{MP}_n^{P^{(i)}}$, then it is easy to see that $(a_1,b_1),\ldots,(a_{kn},b_{kn})$ where $F(a_i,b_i) = i$ is a linear extension of $\mathcal{W}_{P,n}$. Vice versa, if $(a_1,b_1),\ldots,(a_{kn},b_{kn})$ is a linear extension of $\mathcal{W}_{P,n}$, then one can define F so that $F(a_i,b_i) = i$ and it will automatically be the case that $F \in \mathcal{MP}_n^{P^{(i)}}$.

We can define a similar poset for maximum packings of $\tau^{(i)}$ of length $2n+1$. Note that in a maximum packing $F \in \mathcal{MP}_n^{P^{(i)}}$, the element a in the top right-hand corner of F corresponds to the last element of $\sigma(F)$ so that, to account for the last element in a permutation $\alpha = \alpha_1 \ldots \alpha_{2n+1} \in A_{2n+1}$ which has $\tau^{(i)}$-matches starting at positions $1, 3, \ldots 2n-3$, we must add an extra element b to graph $G_{n,P^{(i)}}$ with a directed arrow from b to a since we know that $\alpha_{2n} > \alpha_{2n+1}$. We let $G^+_{n,P^{(i)}}$ denote this extended graph. For example, the graph $G^+_{6,P^{(1)}}$ is pictured in the third line of Figure 11. It follows that $\mathrm{mp}_{2n+1,\tau^{(i)}}$ equals the number of linear extensions of $W_{G^+_{n,P^{(i)}}}$.

Next we state a simple lemma about directed acyclic graphs with no multiple edges due to Harmse and Remmel [11] which will allow us to replace $G_{n,P^{(i)}}$ and $G^+_{n,P^{(i)}}$ by a simpler acyclic directed graphs which contains the same information about their associated posets $W_{G_{n,P^{(i)}}}$ and $W_{G^+_{n,P^{(i)}}}$. Given a directed acyclic graph $G = (V,E)$ with no multiple edges, let $Con(G)$ equal the set of all pairs $(i,j) \in V \times V$ such that there is a directed path in G from vertex i to vertex j.

Lemma 3. *Let $G = (V,E)$ be a directed acyclic graph with no multiple edges. Let H be the subgraph of G that results by removing all edges $e = (i,j) \in E$ such that there is a directed path from i to j in G that does not involve e. Then $Con(G) = Con(H)$.*

First consider the problem of computing $\mathrm{mp}_{2n+1,P^{(1)}}$ for $n \geq 2$. In this case, let a be the rightmost element in the top row of $G_{n,P^{(1)}}$. Since there is a directed path in $G_{n,P^{(1)}}$ from every element other than a to a, it must be the case that a is the last element in any linear extension of $W_{G_{n,P^{(1)}}}$ and, hence, in any $F \in \mathcal{MP}_n^{P^{(1)}}$, $F(a) = 2n$. Note that same thing happens in $G^+_{n,P^{(1)}}$. That is, there is a directed path in $G^+_{n,P^{(1)}}$ from every element other than a to a. Thus it must be the case that a is the last element in any linear extension of $W_{G^+_{n,P^{(1)}}}$ so that a would be assigned the label $2n+1$ in any linear extension. But then it easy to see that b can be assigned any element in $\{1,\ldots,2n\}$. Thus once we pick the value assigned b, then the number of linear extensions of $G^+_{n,P^{(1)}}$ just reduces to the number of linear extension of $G_{n,P^{(1)}}$ which is C_{n-1}. Thus $\mathrm{mp}_{2n+1,\tau^{(1)}} = (2n)C_{n-1}$.

In Figure 12, we have pictured the graphs of $G_{6,P^{(2)}}$ and $G^+_{6,P^{(2)}}$ on the left in the second and third lines, respectively. In both cases, one can use Lemma 3 to show that we can remove all the vertical arrows in the graphs except the leftmost vertical arrow with out loosing an information about the number of linear extensions. Let $\overline{G}_{n,P^{(2)}}$ and $\overline{G}^+_{n,P^{(2)}}$ denote the resulting graphs obtained from $G_{n,P^{(2)}}$ and $G^+_{n,P^{(2)}}$, respectively. In this case, it is easy to see that in a linear extension of $G_{n,P^{(2)}}$, the rightmost top element a must be the largest element $2n$ since there is a directed path in $G_{n,P^{(2)}}$ from every element other than a to a. The same thing happens in $G^+_{n,P^{(2)}}$, namely $2n+1$ must be assigned to a since there is a directed path in $G^+_{n,P^{(2)}}$ from every element other than a to a. But then

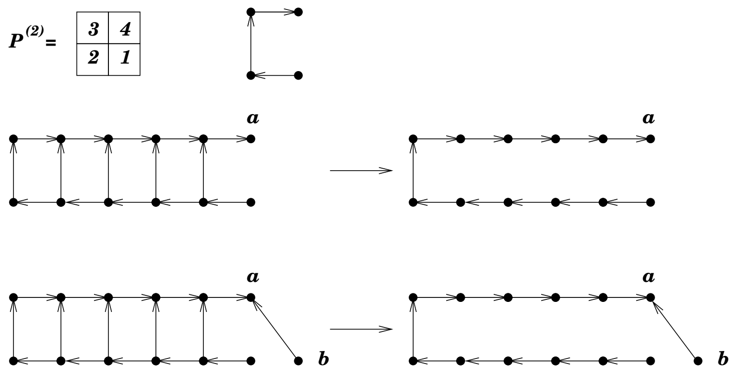

Figure 12: The graphs $G_{n,P^{(2)}}$ and $G^{+}_{n,P^{(2)}}$.

it easy to see that b can be assigned to any element in $\{1,\ldots,2n\}$. Thus once we pick a value that is assigned to b, then the number of linear extensions of $G^{+}_{n,P^{(2)}}$ just reduces to the number of linear extension of $G_{n,P^{(2)}}$ which is clearly just 1. Thus $\mathrm{mp}_{2n+1,\tau^{(2)}} = 2n$.

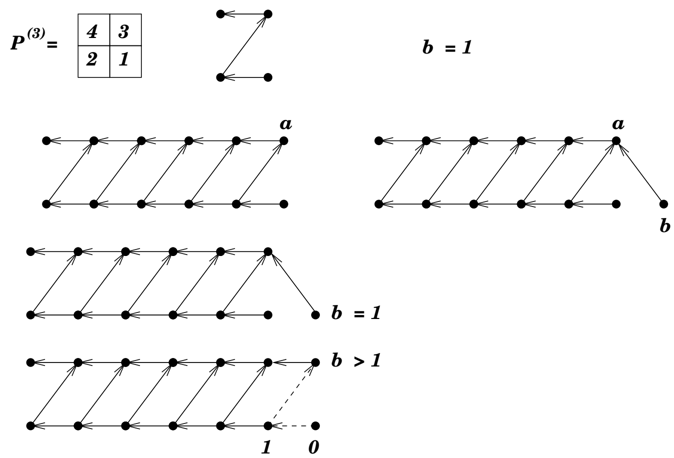

Figure 13: The graphs $G_{n,P^{(3)}}$ and $G^{+}_{n,P^{(3)}}$.

In Figure 13, we have pictured the graphs of $G_{6,P^{(3)}}$ and $G^{+}_{6,P^{(3)}}$ on the second line. Now consider the element b in $G^{+}_{n,P^{(3)}}$. If we assign b the value 1, then there is no restriction on the linear extension of the remaining elements so that we get a total of C_{n-1} linear extensions in that case since $\mathrm{mp}_{2n,\tau^{(3)}} = C_{n-1}$. However, if $b > 1$, then it is easy to see that rightmost bottom element must be the first element in any linear extension since there is a directed path from that element to any other element which not equal to b. Thus the rightmost bottom element must be assigned to 1. It then follows that we can extend the graph $G^{+}_{n,P^{(3)}}$ to graph $G^{++}_{n,P^{(3)}}$ by adding a new element 0 and adding new directed edges connecting 0 to 1 and 1 to b. This process is pictured on line 4 of Figure 13. It is easy to see that the number of linear extensions of $G^{+}_{n,P^{(3)}}$ where $b > 1$ is just the number of linear extensions of $G^{++}_{n,P^{(3)}}$ which is the same as the number of linear extensions of $G_{n+1,P^{(3)}}$. Since the number of linear extensions of $G_{n+1,P^{(3)}}$ is C_n, it follows that $\mathrm{mp}_{2n+1,P^{(3)}} = C_{n-1} + C_n$.

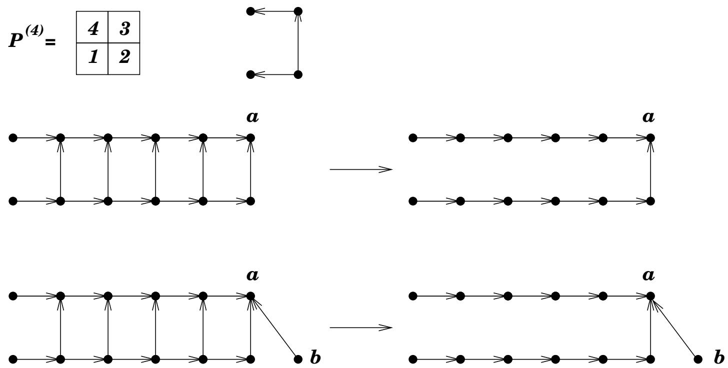

Figure 14: The graphs $G_{n,P^{(4)}}$ and $G^+_{n,P^{(4)}}$.

In Figure 14, we have pictured the graphs of $G_{6,P^{(4)}}$ and $G^+_{6,P^{(4)}}$ on the left in the second and third lines, respectively. In both cases, one can use Lemma 3 to show that we can remove all the vertical arrows in the graphs except the rightmost vertical arrow with out loosing an information about the number of linear extensions $W_{G_{6,P^{(4)}}}$ and $W_{G^+_{6,P^{(4)}}}$. Let $\overline{G}_{n,P^{(4)}}$ and $\overline{G}^+_{n,P^{(4)}}$ denote the resulting graphs obtained from $G_{n,P^{(2)}}$ and $G^+_{n,P^{(2)}}$, respectively. In this case, it is easy to see that the element that in a linear extension of $W_{G_{n,P^{(4)}}}$, the rightmost top element a of $G_{n,P^{(4)}}$ must be $n+1$-st element in the linear extension of $W_{G_{6,P^{(4)}}}$ since there are n elements x for which there are directed paths in $\overline{G}_{n,P^{(4)}}$ from x to a and there are $n-1$ elements y such that there is a directed path from a to y in $\overline{G}_{n,P^{(4)}}$. The same thing happens in $\overline{G}^+_{n,P^{(4)}}$, namely a must be the $n+2$-nd element in any linear extension of $W_{G^+_{6,P^{(4)}}}$ since there are $n+1$ elements x for which there are directed paths in $\overline{G}^+_{n,P^{(4)}}$ from x to a and there are $n-1$ elements y such that there is a directed path from a to y in $\overline{G}^+_{n,P^{(4)}}$. Hence we can assign b to be any element from $1, \ldots, n+1$. Once we pick a value for b, then the number of linear extensions of $\overline{G}^+_{n,P^{(4)}}$ just reduces to the number of linear extension of $\overline{G}_{n,P^{(4)}}$ which is clearly just 1. Thus $\mathrm{mp}_{2n+1,\tau^{(4)}} = n+1$.

In Figure 15, we have pictured the graphs of $G_{6,P^{(5)}}$ and $G^+_{6,P^{(5)}}$ on the left in the second and third lines, respectively. In the case of $G^+_{n,P^{(5)}}$, it is easy to see that the rightmost top element a must be the third element in the linear extension of $W_{G^+_{6,P^{(5)}}}$. Thus we have two linear extension depending how we order the two elements connected to a. Hence $\mathrm{mp}_{2n+1,P^{(5)}} = 2$.

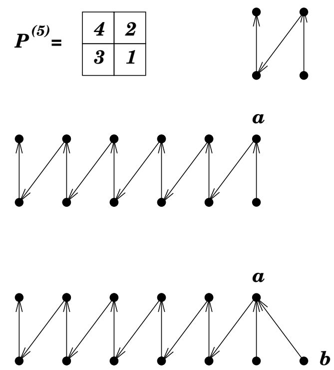

Figure 15: The graphs $G_{n,P^{(5)}}$ and $G^+_{n,P^{(5)}}$.

5 Computing $GMP_{n,\tau^{(i)}}$.

In this section, we shall study the problem of computing $GMP_{n,\tau^{(i)}}(x)$ for $n \geq 1$ and $i = 1, \dots, 5$. First it is easy to see that for any i,

$$\begin{aligned} GMP_{1,\tau^{(i)}}(x) &= 1, \\ GMP_{2,\tau^{(i)}}(x) &= 1, \\ GMP_{3,\tau^{(i)}}(x) &= -1, \text{ and} \\ GMP_{4,\tau^{(i)}}(x) &= x-2. \end{aligned}$$

That is, there is only one generalized maximum packing of size 1 which consist of block of size 1 and weight 1. Similarly, there is only one generalized maximum packing of size 2 which consist of block of size 2 and weight 1. There is only one generalized maximum packings of size 3, namely, 123 where 12 is block of weight 1 and 3 is block of size one. Thus $GMP_{3,\tau^{(i)}} = w(123) = -1$. There are two generalized maximum packings of size 4, namely, 1234 which consist of two blocks of size 2 and has weight -1 and $\tau^{(i)}$ which consist of single block with weight $x-1$. Thus $GMP_{4,\tau^{(i)}} = x-2$.

In general, we do not know how to find closed formulas for $GMP_{n,\tau^{(i)}}(x)$ as function of n, but there are several cases where there are simple recursions for computing $GMP_{n,\tau^{(i)}}(x)$.

The easiest case is for $\tau^{(1)} = 1324$. In this case it is easy to see from the form of the graphs $G_{2n,P^{(1)}}$ that any maximum packing $\sigma \in \mathcal{MP}_{2n,\tau^{(1)}}$ must start with 1 and end with $2n$. It follows that if $\sigma = \sigma_1 \dots \sigma_{2n} \in \mathcal{GMP}_{2n,\tau^{(1)}}$ with block structure $B_1 \dots B_k$, then for each $i < k$, the elements in $B_1 \dots B_i$ is a rearrangement of the elements $\{1, \dots, \sum_{j=1}^{k} |B_i|\}$. That is, for each $1 \leq i < k$, we know that the last element of of B_i which is the largest element in B_i is less than the first element of B_{i+1} which is the smallest element in B_{i+1}. Thus all the elements of B_i are smaller than any element in B_{i+1}.

Now suppose that $n \geq 2$ and $\sigma = \sigma_1 \dots \sigma_{2n} \in \mathcal{GMP}_{2n,\tau^{(1)}}$. There are three possibilities.

Case 1. σ consists of a single block.

In this case σ is a maximum packing of $\tau^{(1)}$ and $w(\sigma) = (x-1)^{n-1}$. Since $\mathrm{mp}_{2n,\tau^{(i)}} = C_{n-1}$, the contribution of the permutations in case 1 to $GMP_{2n,\tau^{(1)}}(x)$ is $C_{n-1}(x-1)^{n-1}$.

Case 2. σ has block structure $B_1 \dots B_k$ where $k \geq 2$ and B_1 is of size 2.
In this case $B_1 = 12$ and has weight -1 and $\mathrm{red}(B_2 \dots B_k)$ is a generalized maximum packing of size $2n-2$. Hence the contribution of the permutations in case 2 to $GMP_{2n,\tau^{(1)}}(x)$ is $-GMP_{2n-2,\tau^{(i)}}(x)$.

Case 3. σ has block structure $B_1 \dots B_k$ where $k \geq 2$ and B_1 has size greater than 2.
In this case $B_1 = 1 \dots 2k$ is a maximum packing for $\tau^{(i)}$ of size $2k$ for some $2 \leq k \leq n-1$ which has weight $-(x-1)^{k-1}$ and $\mathrm{red}(B_2 \dots B_k)$ is a generalized maximum packing of size $2n-2k$. If $2 \leq k \leq n-1$, then there are C_{k-1} choices for B_1. Hence the contribution of the permutations in case 2 to $GMP_{2n,\tau^{(1)}}(x)$ is $-\sum_{k=2}^{n-1} C_{k-1}(x-1)^{k-1} GMP_{2n-2k,\tau^{(i)}}(x)$.

Thus for $n \geq 2$,

$$GMP_{2n,\tau^{(1)}}(x) = C_{n-1}(x-1)^{n-1} - GMP_{2n-2,\tau^{(i)}}(x) - \sum_{k=2}^{n-1} C_{k-1}(x-1)^{k-1} GMP_{2n-2k,\tau^{(i)}}(x). \tag{33}$$

Note that in this case it is easy to compute $GMP_{2n+1,\tau^{(1)}}(x)$. That is, since a generalized maximum packing $\sigma \in A_{2n+1}$ has block structure $B_1 \dots B_k$ where B_k has size 1 and $B_1 \dots B_{k-1}$ reduces to a generalized maximum packing for $\tau^{(1)}$ of size $2n$, we know that the last element of B_{k-1} is the largest element in $B_1 \dots B_{k-1}$ and hence the element in B_k must be $2n+1$. Thus in this case $GMP_{2n+1,\tau^{(1)}}(x) = -GMP_{2n,\tau^{(1)}}(x)$. It follows that

$$B_{\tau^{(1)}}(t,x) = \frac{t - \sum_{n\geq 1} GMP_{2n,\tau^{(1)}}(x) \frac{t^{2n+1}}{(2n+1)!}}{1 - \sum_{n\geq 1} GMP_{2n,\tau^{(1)}}(x) \frac{t^{2n}}{(2n)!}}. \tag{34}$$

Here is list of the first few values of $GMP_{2n,\tau^{(1)}}(x)$.

$$\begin{aligned}
GMP_{2,\tau^{(1)}}(x) &= 1 \\
GMP_{4,\tau^{(1)}}(x) &= -2 + x \\
GMP_{6,\tau^{(1)}}(x) &= 5 - 6x + 2x^2 \\
GMP_{8,\tau^{(1)}}(x) &= -14 + 28x - 20x^2 + 5x^3 \\
GMP_{10,\tau^{(1)}}(x) &= 42 - 120x + 135x^2 - 70x^3 + 14x^4 \\
GMP_{12,\tau^{(1)}}(x) &= -132 + 495x - 770x^2 + 616x^3 - 252x^4 + 42x^5 \\
GMP_{14,\tau^{(1)}}(x) &= 429 - 2002x + 4004x^2 - 4368x^3 + 2730x^4 - 924x^5 + 132x^6 \\
GMP_{16,\tau^{(1)}}(x) &= -1430 + 8008x - 19656x^2 + 27300x^3 - 23100x^4 + \\
&\quad 11880x^5 - 3432x^6 + 429x^7
\end{aligned}$$

Plugging these values into the generating functions (14) and (23) and using Mathematica, we have computed the following table of values of $A_{n,\tau^{(1)}}(x) = \sum_{\sigma\in A_n} x^{\tau^{(1)}\text{-mch}(\sigma)}$.

$A_{1,\tau^{(1)}}$	1
$A_{2,\tau^{(1)}}$	1
$A_{3,\tau^{(1)}}$	2
$A_{4,\tau^{(1)}}$	$4+x$
$A_{5,\tau^{(1)}}$	$12+4x$
$A_{6,\tau^{(1)}}$	$35+24x+2x^2$
$A_{7,\tau^{(1)}}$	$142+118x+12x^2$
$A_{8,\tau^{(1)}}$	$546+672x+162x^2+5x^3$
$A_{9,\tau^{(1)}}$	$2816+3968x+1112x^2+40x^3$
$A_{10,\tau^{(1)}}$	$13482+24660x+11145x^2+1220x^3+14x^4$
$A_{11,\tau^{(1)}}$	$84764+170996x+87200x^2+10666x^3+168x^4$

In this case, we can find explicit generating functions for the number of σ in A_n which have no $\tau^{(1)}$-matches. That is, we claim that $G_{2m,\tau^{(1)}}(0) = (-1)^{m-1}C_m$ for $m \geq 1$. This is clear for $m = 1$ and $m = 2$. Now if $n > 2$ and we assume that $G_{2m,\tau^{(1)}}(0) = (-1)^{m-1}C_m$ for $m < n$, then by (33), we have

$$\begin{aligned}
&GMP_{2n,\tau^{(1)}}(0) \\
&= C_{n-1}(-1)^{n-1} - GMP_{2n-2,\tau^{(i)}}(0) - \sum_{k=2}^{n-1} C_{k-1}(x-1)^{k-1} GMP_{2n-2k,\tau^{(i)}}(0) \\
&= C_{n-1}(-1)^{n-1} - (-1)^{n-2}C_{n-1} - \sum_{k=2}^{n-1} C_{k-1}(-1)^{k-1}(-1)^{n-k-1}C_{n-k} \\
&= (-1)^{n-1}(\sum_{k=1}^{n} C_{k-1}C_{n-k}) = (-1)^{n-1}C_n
\end{aligned}$$

Thus if we let $N_{n,\tau^{(i)}} = A_{n,\tau^{(i)}}(0)$ be the number of $\sigma \in A_n$ with no $\tau^{(i)}$-matches, then by setting $x = 0$ in (14) and (23) we obtain that

$$1 + \sum_{n\geq 1} N_{2n,\tau^{(1)}} \frac{t^{2n}}{(2n)!} = \frac{1}{1 + \sum_{n\geq 1}(-1)^n C_n \frac{t^{2n}}{(2n)!}} \tag{35}$$

and

$$\sum_{n\geq 0} N_{2n+1,\tau^{(1)}} \frac{t^{2n+1}}{(2n+1)!} = \frac{t + \sum_{n\geq 1}(-1)^n C_n \frac{t^{2n+1}}{(2n+1)!}}{1 + \sum_{n\geq 1}(-1)^n C_n \frac{t^{2n}}{(2n)!}}. \tag{36}$$

We also claim that

$$GMP_{2n,\tau^{(1)}}(x)|_x = (-1)^n \binom{2n}{n-2} \text{ for } n \geq 2. \tag{37}$$

That is, $GMP_{4,\tau^{(1)}}(x)|_x = 1$ so the formula holds for $n = 2$. For $n > 2$, we have that

$$GMP_{2n,\tau^{(1)}}(x) = C_{n-1}(x-1)^{n-1} - GMP_{2n2,\tau^{(1)}}(x) - \sum_{k=1}^{n-2} C_k(x-1)^k GMP_{2n-2k-2,\tau^{(1)}}(x).$$

Taking the coefficient of x on both sides, we see that

$$GMP_{2n,\tau^{(1)}}(x)|_x = \frac{1}{n}\binom{2n-2}{n-1}(n-1)(-1)^{n-2} - (-1)^{n-1}\binom{2n-2}{n-3} -$$
$$\sum_{k=1}^{n-2}(C_k(x-1)^k|)_x(GMP_{2n-2k-2,\tau^{(1)}}(x))|_{x^0} -$$
$$\sum_{k=1}^{n-2}(C_k(x-1)^k)|_{x^0}(GMP_{2n-2k-2,\tau^{(1)}}(x))|_x.$$

Since $GMP_{2,\tau^{(1)}}(x)|_x = 0$, we see that

$$GMP_{2n,\tau^{(1)}}(x)|_x$$
$$= (-1)^n\binom{2n-2}{n-2} + (-1)^n\binom{2n-2}{n-3} -$$
$$\sum_{k=1}^{n-2}\frac{1}{k+1}\binom{2k}{k}k(-1)^{k-1}(-1)^{n-k-2}\frac{1}{n-k}\binom{2n-2k-2}{n-k-1} -$$
$$\sum_{k=1}^{n-3}\frac{1}{k+1}\binom{2k}{k}(-1)^k(-1)^{n-k-1}\binom{2n-2k-2}{n-k-3}.$$

Observe that $\binom{2n-1}{n-2} = \binom{2n-2}{n-2} + \binom{2n-2}{n-3}$, $\frac{1}{k+1}\binom{2k}{k}k = \binom{2k}{k-1}$, and $\frac{1}{k+1}\binom{2k}{k} = \frac{1}{k-1}\binom{2k}{k-1}$. Thus if we multiply both sides of our previous equation by $(-1)^n$ and use these observations, we see that

$$(-1)^n GMP_{2n,\tau^{(1)}}(x)|_x = \binom{2n-1}{n-2} + \sum_{k=1}^{n-2}\frac{1}{n-k}\binom{2k}{k-1}\binom{2n-2k-2}{n-k-1} +$$
$$\sum_{k=1}^{n-3}\frac{1}{k}\binom{2k}{k}\binom{2n-2k-2}{n-k-3}.$$

Replacing k by $n-k-1$ in the first sum we see that

$$\begin{aligned}
&\sum_{k=1}^{n-2} \frac{1}{n-k}\binom{2k}{k-1}\binom{2n-2k-2}{n-k-1} \\
&= \sum_{k=1}^{n-2} \frac{1}{k+1}\binom{2k}{k}\binom{2n-2k-2}{n-k-2} \\
&= \sum_{k=1}^{n-2} \frac{1}{k}\binom{2k}{k-1}\binom{2n-2k-2}{n-k-2} \\
&= \frac{1}{n-2}\binom{2n-4}{n-3} + \sum_{k=1}^{n-3} \frac{1}{k}\binom{2k}{k-1}\binom{2n-2k-2}{n-k-2}.
\end{aligned}$$

Thus

$$\begin{aligned}
(-1)^n GMP_{2n,\tau^{(1)}}(x)|_x &= \binom{2n-1}{n-2} + \frac{1}{n-2}\binom{2n-4}{n-3} + \\
&\sum_{k=1}^{n-3} \frac{1}{k}\binom{2k}{k}\left(\binom{2n-2k-2}{n-k-3} + \binom{2n-2k-2}{n-k-2}\right) \\
&= \binom{2n-1}{n-2} + \frac{1}{n-2}\binom{2n-4}{n-3} + \sum_{k=1}^{n-3} \frac{1}{k}\binom{2k}{k}\binom{2n-2k-1}{n-k-2}.
\end{aligned}$$

Thus to complete our induction, we must show that

$$\binom{2n-1}{n-3} = \frac{1}{n-2}\binom{2n-4}{n-3} + \sum_{k=1}^{n-3} \frac{1}{k}\binom{2k}{k}\binom{2n-2k-1}{n-k-2}. \tag{38}$$

Using the Simplify command in Mathematica on the RHS of (38) will yield that

$$\begin{aligned}
&\frac{1}{n-2}\binom{2n-4}{n-3} + \sum_{k=1}^{n-3} \frac{1}{k}\binom{2k}{k}\binom{2n-2k-1}{n-k-2} \\
&= \frac{2(2n-1)}{(n+2)(n+1)(n-2)}\left((9-6n)\binom{2n-4}{n-3} + (2+n^2)\binom{2n-3}{n-3}\right).
\end{aligned}$$

Thus we need only verify that

$$\binom{2n-1}{n-3} = \frac{2(2n-1)}{(n+2)(n+1)(n-2)}\left((9-6n)\binom{2n-4}{n-3} + (2+n^2)\binom{2n-3}{n-3}\right) \tag{39}$$

which can be directly verified using Mathematica.

We can now use the fact that $GMP_{2n,\tau^{(1)}}(x)|_x = (-1)^n \binom{2n}{n-2}$ for $n \geq 2$ to compute the generating function of the number of $\sigma \in A_n$ with exactly one $\tau^{(1)}$-match. That is, let

$$\begin{aligned}
R(t) &= \sum_{n\geq 1} (-1)^{n-1} C_n \frac{t^{2n}}{(2n)!}, \\
S(t) &= \sum_{n\geq 2} (-1)^n \binom{2n}{n-2} \frac{t^{2n}}{(2n)!}, \\
U(t) &= \sum_{n\geq 1} (-1)^{n-1} C_n \frac{t^{2n+1}}{(2n+1)!}, \text{ and} \\
V(t) &= \sum_{n\geq 2} (-1)^n \binom{2n}{n-2} \frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Then we know that

$$\begin{aligned}
A_{\tau^{(1)}}(t,x) &= \frac{1}{1-R(t)-xS(t)+O(x^2)} \text{ and} \\
B_{\tau^{(1)}}(t,x) &= \frac{t-U(t)-xV(t)+o(x^2)}{1-R(t)-xS(t)+O(x^2)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
A_{\tau^{(1)}}(t,x)|_x &= \left(1+\sum_{n\geq 1}(R(t)+xS(t))^n\right)|x \\
&= \sum_{n\geq 1} nS(t)(R(t))^{n-1} \\
&= S(t)\frac{d}{dt}\frac{1}{1-R(t)} \\
&= \frac{S(t)\frac{d}{dt}R(t)}{(1-R(t))^2}. \qquad (40)
\end{aligned}$$

Similarly, one can compute that

$$\begin{aligned}
B_{\tau^{(1)}}(t,x)|_x &= \frac{(t-U(t))S(t)\frac{d}{dt}R(t)}{(1-R(t))^2} + \frac{V(t)}{1-R(t)} \\
&= \frac{V(t)(1-R(t))+(t-U(t))S(t)\frac{d}{dt}R(t)}{(1-R(t))^2}. \qquad (41)
\end{aligned}$$

Next we consider $GMP_{n,\tau^{(2)}}(x)$. It is clear from the graph $G_{n,P^{(2)}}$ that the unique maximum packing $\sigma = \sigma_1 \ldots \sigma_{2n} \in A_{2n}$ for $\tau^{(2)}$ have $\sigma_2\sigma_4 \ldots \sigma_{2n} = (n+1)(n+2)\ldots(2n)$ and $\sigma_1\sigma_3 \ldots \sigma_n = n(n-1)(n-2)\ldots 1$. It follows that in a generalized maximum packing $\alpha \in S_{2n}$ with block structure $B_1 \ldots B_k$ that the last element of each block B_i is the largest element in the block. Hence if B_k is size two then, its two elements must be $(2n-1)$ and $(2n)$ since they must be larger than all the largest elements in each block B_i. In that case $B_1 \dots B_{k-1}$ is just a generalized maximum packing for $\tau^{(2)}$ of size $2n-2$ whose weight is $-w(\sigma)$.

If $B_k = \sigma_{2n-2j+1} \dots \sigma_{2n}$ reduces to a maximum packing for $\tau^{(2)}$ of size $2j$ where $2 \leq j \leq n-1$, then it must be the case $\sigma_{2n-2j+1} < \sigma_{2n-2j+2} < \sigma_{2n-2j+4} < \cdots < \sigma_{2n}$ must be the $j+1$ largest elements from $\{1, \dots, 2n\}$ since they will be larger that all the remaining elements of B_k and larger than the largest element of B_i for $i \neq k$. It follows that first element of block B_k is $(2n-j)$. Our conditions to a generalized maximum packing for $\tau^{(2)}$ do not impose any relations between the remaining elements of B_k, namely $\sigma_{2n-2j+3}, \sigma_{2n-2j+5}, \dots, \sigma_{2n-1}$, and the elements in blocks $B_1, \dots, B_{k-1}$. Thus we have $\binom{2n-j-1}{j-1}$ ways to choose those elements. Once we have chosen those elements, then $B_1 \dots B_{k-1}$ must reduce to a generalized maximum packing for $\tau^{(2)}$ of size $2n-2j$. Thus by classifying the elements of $\mathcal{GMP}_{2n,\tau^{(2)}}$ by the size of the last block, we see that for $n \geq 2$,

$$GMP_{2n,\tau^{(2)}}(x) = (x-1)^{n-1} - GMP_{2n-2,\tau^{(2)}}(x) - \sum_{j=2}^{n-1} \binom{2n-j-1}{j-1} GMP_{2n-2j,\tau^{(2)}}(x). \quad (42)$$

Note that in this case it is easy to compute $GMP_{2n+1,\tau^{(2)}}(x)$. That is, since a generalized maximum packing $\sigma \in A_{2n+1}$ has block structure $B_1 \dots B_k$ where B_k has size 1 and $B_1 \dots B_{k-1}$ reduces to a generalized maximum packing for $\tau^{(2)}$ of size $2n$, we know that the last element of B_{k-1} is the largest element in $B_1 \dots B_{k-1}$ and hence the element in B_k must be $2n+1$. Thus in this case $GMP_{2n+1,\tau^{(2)}}(x) = -GMP_{2n,\tau^{(2)}}(x)$.

It follows that

$$B_{\tau^{(2)}}(t,x) = \frac{t - \sum_{n \geq 1} GMP_{2n,\tau^{(2)}}(x) \frac{t^{2n+1}}{(2n+1)!}}{1 - \sum_{n \geq 1} GMP_{2n,\tau^{(2)}}(x) \frac{t^{2n}}{(2n)!}}. \quad (43)$$

Here is list of the first few values of $GMP_{2n,\tau^{(1)}}(x)$.

$$\begin{aligned}
GMP_{2,\tau^{(2)}}(x) &= 1 \\
GMP_{4,\tau^{(2)}}(x) &= x-2 \\
GMP_{6,\tau^{(2)}}(x) &= 6-6x+x^2 \\
GMP_{8,\tau^{(2)}}(x) &= -23+36x-15x^2+x^3 \\
GMP_{10,\tau^{(2)}}(x) &= 106-229x+160x^2-37x^3+x^4 \\
GMP_{12,\tau^{(2)}}(x) &= -567+1574x-1566x^2+650x^3-93x^4+x^5 \\
GMP_{14,\tau^{(2)}}(x) &= 3434-11706x+15248x^2-9310x^3+2572x^4-238x^5+x^6 \\
GMP_{16,\tau^{(2)}}(x) &= -23137+93831x-151933x^2+123814x^3-52136x^4+ \\
&\quad 10175x^5-616x^6+x^7
\end{aligned}$$

In this case the sequence $((-1)^{n-1} GMP_{2n,\tau^{(2)}}(0))_{n \geq 1}$ which starts out with $1, 2, 6, 23, 106, 567, 23137, \dots$ seems to be sequence A125273 in the Online Encyclopedia of Integer Sequences (OEIS). Unfortunately, there seems to be no exact formula for this sequence. The sequence $((-1)^n GMP_{2n,\tau^{(2)}}(x)|_x)_{n\geq 1}$ which starts out $1, 6, 36, 1574, 11706, 933831, \ldots$ does not appear in the OEIS.

Plugging these values into the generating functions (14) and (23) and using Mathematica, we have computed the following table of values of $A_{n,\tau^{(2)}}(x) = \sum_{\sigma\in A_n} x^{\tau^{(2)}\text{-mch}(\sigma)}$.

$A_{1,\tau^{(2)}}$	1
$A_{2,\tau^{(2)}}$	1
$A_{3,\tau^{(2)}}$	2
$A_{4,\tau^{(2)}}$	$4+x$
$A_{5,\tau^{(2)}}$	$12+4x$
$A_{6,\tau^{(2)}}$	$36+24x+x^2$
$A_{7,\tau^{(2)}}$	$148+118x+6x^2$
$A_{8,\tau^{(2)}}$	$593+680x+111x^2+x^3$
$A_{9,\tau^{(2)}}$	$3128+4032x+768x^2+8x^3$
$A_{10,\tau^{(2)}}$	$15676+25691x+8680x^2+473x^3+x^4$
$A_{11,\tau^{(2)}}$	$101094+180134x+68326x^2+4228x^3+10x^4$

Next we consider $GMP_{2n,\tau^{(4)}}(x)$. It is clear from the graph $G_{n,P^{(4)}}$ that the unique maximum packing $\sigma = \sigma_1 \ldots \sigma_{2n} \in A_{2n}$ for $\tau^{(4)}$ has $\sigma_2\sigma_4 \ldots \sigma_{2n} = (2n)(2n-1)\ldots(n+1)$ and $\sigma_1\sigma_3 \ldots \sigma_{2n-1} = 12\ldots n$. It follows that in a generalized maximum packing $\alpha \in S_{2n}$ with block structure $B_1 \ldots B_k$ that the first element of each block B_i is the smallest element in the block. Hence if B_1 is size two then, its two elements must be 1 and 2 since they must be smaller than all the smallest elements in each block B_i for $i > 1$. In that case $B_2 \ldots B_k$ is just a generalized maximum packing for $\tau^{(4)}$ of size $2n-2$ whose weight is $-w(\sigma)$.

If $B_1 = \sigma_1 \ldots \sigma_{2j}$ reduces to a maximum packing for $\tau^{(4)}$ of size $2j$ where $2 \leq j \leq n-1$, then it must be the case $\sigma_1 < \sigma_3 < \sigma_5 < \cdots < \sigma_{2j-1} < \sigma_{2j}$ must be the $j+1$ smallest elements from $\{1,\ldots,2n\}$ since they will be smaller that all the remaining elements of B_1 and smaller than the smallest element of B_i for $i > 1$. It follows that last element of block B_1 is $(j+1)$. Our conditions to a generalized maximum packing for $\tau^{(4)}$ do not impose any relations between the remaining elements of B_1, namely $\sigma_2, \sigma_4, \ldots, \sigma_{2j-2}$, and the elements in blocks $B_2, \ldots, B_k$. Thus we have $\binom{2n-j-1}{j-1}$ ways to choose those elements. Once we have chosen those elements, then $B_2 \ldots B_k$ must reduce to a generalized maximum packing for $\tau^{(4)}$ of size $2n-2j$. Thus by classifying the elements of $\mathcal{GMP}_{2n,\tau^{(4)}}$ by the size of the first block, we see that for $n \geq 2$,

$$GMP_{2n,\tau^{(4)}}(x) = (x-1)^{n-1} - GMP_{2n-2,\tau^{(4)}}(x) - \sum_{j=2}^{n-1} \binom{2n-j-1}{j-1} GMP_{2n-2j,\tau^{(4)}}(x). \quad (44)$$

This is the same recursion as (42) and it implies that $A_{\tau^{(2)}}(t,x) = A_{\tau^{(4)}}(t,x)$. This is no surprise since even length alternating permutations are closed under reverse complement. That is, if $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, then the reverse of σ, σ^r, is defined to be $\sigma^r = \sigma_n \ldots \sigma_1$ and the complement of σ, σ^c, is defined by $\sigma^c = (n+1-\sigma_1)\dots(n+1-\sigma_n)$. The reverse-complement of σ is $(\sigma^r)^c$. It is easy to check that $\sigma \in A_{2n}$ if and only if $(\sigma^r)^c \in A_{2n}$ and that $(2314^r)^c = 1423$. Thus the map which send $\sigma \in A_{2n}$ to $(\sigma^r)^c$ shows that $A_{\tau^{(2)}}(t,x) = A_{\tau^{(4)}}(t,x)$.

It is not the case that $B_{\tau^{(2)}}(t,x) = B_{\tau^{(4)}}(t,x)$ since for $n \geq 2$, the number of maximum packings for $\tau^{(2)}$ of length $2n+1$ is $2n$ while the number of maximum packings for $\tau^{(4)}$ of length $2n+1$ is $(n+1)$. Nevertheless, we can still develop a recursion for $GMP_{2n+1,\tau^{(4)}}(x)$ for $n \geq 2$. That is, since any generalized maximum packing for $\tau^{(4)}$ of size $2n+1$ has block structure $B_1 \dots B_k$ where B_k has size 1 and $B_1 \dots B_{k-1}$ reduces to a generalized maximum packing for $\tau^{(4)}$ of size $2n$, it will still be the case that the first element in each block is the smallest element.

Hence if B_1 is size two then, its two elements must be 1 and 2 since they must be smaller than all the smallest elements in each block B_i. In that case $B_2 \dots B_k$ is just a generalized maximum packing for $\tau^{(4)}$ of size $2n-1$ whose weight is $-w(\sigma)$.

If $B_1 = \sigma_1 \dots \sigma_{2j}$ reduces to a maximum packing for $\tau^{(4)}$ of size $2j$ where $2 \leq j \leq n-1$, then it will still be the case that $\sigma_1 < \sigma_3 < \sigma_5 < \cdots < \sigma_{2j-1} < \sigma_{2j}$ are the $j+1$ smallest elements from $\{1,\dots,2n\}$ and the last element of B_1 is $j+1$. If $j = n$, then it is easy to see that we have n choices of σ_{2n+1} and once we choose σ_{2n+1}, then the order of the rest of the element is completely determined. Thus generalized maximum packings for $\tau^{(4)}$ which have two blocks B_1B_2 contribute a weight of $-n(x-1)^{n-1}$ to $GMP_{2n+1,\tau^{(4)}}$. If $2 \leq j \leq n-1$, then our conditions to a generalized maximum packing for $\tau^{(4)}$ do not impose any relations between the remaining elements of B_1, namely $\sigma_2, \sigma_4, \dots, \sigma_{2j-2}$, and the elements in blocks $B_2, \dots, B_k$. Thus we have $\binom{2n-j}{j-1}$ ways those choose those elements. Once we have chosen those elements, then $B_2 \dots B_k$ must reduce to a generalized maximum packing for $\tau^{(4)}$ of size $2n+1-2j$. Thus by classifying the elements of $\mathcal{GMP}_{2n+1,\tau^{(4)}}$ by the size of the first block, we see that for $n \geq 2$,

$$GMP_{2n+1,\tau^{(4)}}(x) = -n(x-1)^{n-1} - GMP_{2n-1,\tau^{(4)}}(x) - \sum_{j=2}^{n-1} \binom{2n-j}{j-1} GMP_{2n+1-2j,\tau^{(4)}}(x). \tag{45}$$

Here is list of the first few values of $GMP_{2n+1,\tau^{(4)}}(x)$.

$$\begin{aligned}
GMP_{1,\tau^{(4)}}(x) &= 1 \\
GMP_{3,\tau^{(4)}}(x) &= -1 \\
GMP_{5,\tau^{(4)}}(x) &= 3-2x \\
GMP_{7,\tau^{(4)}}(x) &= -10+12x-3x^2 \\
GMP_{9,\tau^{(4)}}(x) &= 42-74x+37x^2-4x^3 \\
GMP_{11,\tau^{(4)}}(x) &= -210+498x-394x^2+110x^3-5x^4 \\
GMP_{13,\tau^{(4)}}(x) &= 1199-3596x+3946x^2-1872x^3+330x^4-6x^5 \\
GMP_{15,\tau^{(4)}}(x) &= -7670+27908x-39356x^2+26604x^3-8476x^4+996x^5-7x^6
\end{aligned}$$

In this case, the sequence $\{GMP_{2n+1,\tau^{(4)}}(0)\}_{n\geq 0}$ seems to be sequence A125274 in the OEIS. Unfortunately, there is no exact formuls for the elements in this sequence.

Plugging these values into the generating functions (14) and (23) and using Mathematica, we have computed the following table of values of $A_{n,\tau^{(4)}}(x) = \sum_{\sigma \in A_n} x^{\tau^{(4)}\text{-mch}(\sigma)}$.

$A_{1,\tau^{(4)}}$	1
$A_{2,\tau^{(4)}}$	1
$A_{3,\tau^{(4)}}$	2
$A_{4,\tau^{(4)}}$	$4+x$
$A_{5,\tau^{(4)}}$	$13+3x$
$A_{6,\tau^{(4)}}$	$36+24x+x^2$
$A_{7,\tau^{(4)}}$	$165+103x+4x^2$
$A_{8,\tau^{(4)}}$	$593+680x+111x^2+x^3$
$A_{9,\tau^{(4)}}$	$3507+3832x+592x^2+5x^3$
$A_{10,\tau^{(4)}}$	$15676+25691x+8680x^2+473x^3+x^4$
$A_{11,\tau^{(4)}}$	$113387+179369x+58016x^2+3014x^3+6x^4$

The key to our ability to develop recursions for $GMP_{n,\tau^{(i)}}(x)$ in the case where $i \in \{1,2,4\}$ is due to the fact that $\tau^{(1)}$, $\tau^{(2)}$, and $\tau^{(4)}$ either start with 1 or end with 4. This allowed us to develop recursions based in either size of the first block or the size of the last block in a generalized maximum packing. Neither $\tau^{(3)} = 2413$ nor $\tau^{(5)} = 3412$ start with 1 or end with 4 so that we have not been able to find any simple recursions for $GMP_{n,\tau^{(3)}}$ or $GMP_{n,\tau^{(5)}}$.

J. Harmse [10] computed the following initial values of $GMP_{n,\tau^{(3)}}$ and $GMP_{n,\tau^{(5)}}$ by computing the number of linear extension of the posets associated with the various block structures of generalized maximal packings.

Here is list of the first few values of $GMP_{2n,\tau^{(3)}}(x)$.

$$\begin{aligned}
GMP_{1,\tau^{(3)}}(x) &= 1 \\
GMP_{2,\tau^{(3)}}(x) &= 1 \\
GMP_{3,\tau^{(3)}}(x) &= -1 \\
GMP_{4,\tau^{(3)}}(x) &= -2+x \\
GMP_{5,\tau^{(3)}}(x) &= 3-2x \\
GMP_{6,\tau^{(3)}}(x) &= 9-10x+2x^2 \\
GMP_{7,\tau^{(3)}}(x) &= -18+24x-7x^2 \\
GMP_{8,\tau^{(3)}}(x) &= -74+132x-64x^2+5x^3 \\
GMP_{9,\tau^{(3)}}(x) &= 190-376x+213x^2-26x^3 \\
GMP_{10,\tau^{(3)}}(x) &= 974-2394x+1927x^2-520x^3+14x^4 \\
GMP_{11,\tau^{(3)}}(x) &= -3078+8180x-7287x^2+2282x^3-98x^4 \\
GMP_{12,\tau^{(3)}}(x) &= -17688+54228x-59393x^2+26807x^3-3997x^4+42x^5
\end{aligned}$$

Plugging these values into the generating functions (14) and (23) and using Mathematica, we have computed the following table of values of $A_{n,\tau^{(3)}}(x) = \sum_{\sigma \in A_n} x^{\tau^{(3)}\text{-mch}(\sigma)}$.

$A_{1,\tau^{(3)}}$	1
$A_{2,\tau^{(3)}}$	1
$A_{3,\tau^{(3)}}$	2
$A_{4,\tau^{(3)}}$	$4 + x$
$A_{5,\tau^{(3)}}$	$13 + 3x$
$A_{6,\tau^{(3)}}$	$39 + 20x + 2x^2$
$A_{7,\tau^{(3)}}$	$178 + 87x + 7x^2$
$A_{8,\tau^{(3)}}$	$710 + 552x + 118x^2 + 5x^3$
$A_{9,\tau^{(3)}}$	$4168 + 3146x + 603x^2 + 19x^3$
$A_{10,\tau^{(3)}}$	$29774 + 21666x + 5370x^2 + 2697x^3 + 14x^4$
$A_{11,\tau^{(3)}}$	$149030 + 152170x + 27000x2 + 25536x^3 + 56x^4$

Here is list of the first few values of $GMP_{2n,\tau^{(5)}}(x)$.

$$\begin{aligned}
GMP_{1,\tau^{(5)}}(x) &= 1\\
GMP_{2,\tau^{(5)}}(x) &= 1\\
GMP_{3,\tau^{(5)}}(x) &= -1\\
GMP_{4,\tau^{(5)}}(x) &= -2 + x\\
GMP_{5,\tau^{(5)}}(x) &= 4 - 3x\\
GMP_{6,\tau^{(5)}}(x) &= 14 - 14x + x^2\\
GMP_{7,\tau^{(5)}}(x) &= -39 + 44x - 6x^2\\
GMP_{8,\tau^{(5)}}(x) &= -168 + 252x - 86x^2 + x^3\\
GMP_{9,\tau^{(5)}}(x) &= 594 - 1002 + 416x^2 - 7x^3\\
GMP_{10,\tau^{(5)}}(x) &= 3352 - 6704x + 3782x^2 - 430x^3 + x^4\\
GMP_{11,\tau^{(5)}}(x) &= -13814 + 30264x - 19404x^2 = 2962x^3 - 9x^4\\
GMP_{12,\tau^{(5)}}(x) &= -91038 + 224751x - 180196x^2 + 48387x^3 - 1906x^4 + x^5
\end{aligned}$$

Plugging these values into the generating functions (14) and (23) and using Mathematica, we have computed the following table of values of $A_{n,\tau^{(5)}}(x) = \sum_{\sigma\in A_n} x^{\tau^{(5)}\text{-mch}(\sigma)}$.

$A_{1,\tau^{(5)}}$	1
$A_{2,\tau^{(5)}}$	1
$A_{3,\tau^{(5)}}$	2
$A_{4,\tau^{(5)}}$	$4+x$
$A_{5,\tau^{(5)}}$	$14+2x$
$A_{6,\tau^{(5)}}$	$44+16x+x^2$
$A_{7,\tau^{(5)}}$	214_56x+2x^2
$A_{8,\tau^{(5)}}$	$896+448x+40x^2+x^3$
$A_{9,\tau^{(5)}}$	$5610+2190x+134x^2+2x^3$
$A_{10,\tau^{(5)}}$	$29392+18496x+2552x^2+80x^3+x^4$
$A_{11,\tau^{(5)}}$	$224878+116776x+11880x^2+256x^3+2x^4$

6 Double rise pairs and double descent pairs.

Suppose that $\sigma = \sigma_1 \dots \sigma_n \in A_n$. Then we say that $(2i-1)(2i)$ is a double rise (double descent) pair in σ if both $\sigma_{2i-1} < \sigma_{2i+1}$ and $\sigma_{2i} < \sigma_{2i+2}$ ($\sigma_{2i-1} > \sigma_{2i+1}$ and $\sigma_{2i} > \sigma_{2i+2}$). It is easy to see that $(2i-1)(2i)$ is a double rise pair if and only if $\text{red}(\sigma_{2i-1}\sigma_{2i}\sigma_{2i+1}\sigma_{2i+2}) = 1324$ so that the number of double rise pairs in σ is just the number of 1324-matches in σ. Similarly, is $(2i-1)(2i)$ is a double descent pair if and only if $\text{red}(\sigma_{2i-1}\sigma_{2i}\sigma_{2i+1}\sigma_{2i+2}) \in \{3412, 2413\}$ so that if $D = \{3412, 2413\}$, then the number of double descents pairs in σ is just the number of D-matches in σ.

In general, if $\Upsilon \subseteq A_4$, we say that $\sigma \in A_{2n}$ is a maximum packing for Υ if $\Upsilon\text{-mch}(\sigma) = n-1$. We say that $\sigma \in S_{2n}$ is a *generalized maximum packing for* Υ if we can break σ into consecutive blocks $\sigma = B_1 \dots B_k$ such that

1. for all $1 \leq j \leq k$, B_j is either an increasing sequence of length 2 or $\text{red}(B_j)$ is maximum packing for Υ of length $2s$ for some $s \geq 2$ and
2. for all $1 \leq j \leq k-1$, the last element of B_j is less than the first element of B_{j+1}.

Similarly, we say that $\sigma \in A_{2n+1}$ is a maximum packing for Υ if $\Upsilon\text{-mch}(\sigma) = n-1$. We say that $\sigma \in S_{2n+1}$ is a *generalized maximum packing for* Υ if we can break σ into consecutive blocks $\sigma = B_1 \dots B_k$ such that

1. for all $1 \leq j < k$, B_j is either an increasing sequence of length 2 or $\text{red}(B_j)$ is maximum packing for Υ of length $2s$ for some $s \geq 2$,
2. B_k is block of size 1, and
3. for all $1 \leq j \leq k-1$, the last element of B_j is less than the first element of B_{j+1}.

We then have the following theorem.

Theorem 4. *Let $\Upsilon \subseteq A_4$. Then*

$$\begin{aligned} A_\Upsilon(t,x) &= 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma \in A_{2n}} x^{\Upsilon\text{-mch}(\sigma)} \\ &= \frac{1}{1 - \sum_{n\geq 1} GMP_{2n,\Upsilon}(x) \frac{t^{2n}}{(2n)!}} \end{aligned}$$

and

$$\begin{aligned} B_\Upsilon(t,x) &= \sum_{n\geq 1} \frac{t^{2n-1}}{(2n-1)!} \sum_{\sigma \in A_{2n-1}} x^{\Upsilon\text{-mch}(\sigma)} \\ &= \frac{\sum_{n\geq 1} GMP_{2n-1,\Upsilon}(x) \frac{t^{2n-1}}{(2n-1)!}}{1 - \sum_{n\geq 1} GMP_{2n,\Upsilon}(x) \frac{t^{2n}}{(2n)!}}. \end{aligned}$$

That is, if $\Upsilon \subseteq A_4$, the proof of Theorem 1 goes through without change if we replace maximum packings for $\tau^{(i)}$ with maximum packings for Υ and generalized maximum packings for $\tau^{(i)}$ by generalized maximum packings for Υ throughout the proof.

Thus if we can compute $GMP_{n,D}(x)$, we would have the generating function for the distribution of double descents in A_n. We can compute $\mathrm{mp}_{2n,D}$ and $\mathrm{mp}_{2n+1,D}$. That is, we have the following theorem.

Theorem 5. *For all $n \geq 1$,*

$$mp_{2n,D} = C_n \text{ and} \tag{46}$$
$$mp_{2n+1,D} = C_{n+1}. \tag{47}$$

Proof. It is easy to see that $\mathrm{mp}_{2n,D}$ equals the number of $F \in \mathcal{F}_{2,n}$ such that for each $i < n$, there is either a $P^{(2)}$-match or a $P^{(5)}$-match starting in column i. Let the reverse of F equal F^r where the first row of F^r is $F(1,n), F(1,n-1)\ldots, F(1,1)$ reading from left to right and the second row of F^r is $F(2,n), F(2,n-1)\ldots, F(2,1)$ reading from left to right. For example,

$$(P^{(3)})^r = P^{(1)} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$$

and

$$(P^{(5)})^r = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

It is easy to see that $F \in \mathcal{F}_{2,n}$ has the property that for each $i < n$, there is either a $P^{(2)}$-match or a $P^{(5)}$-match starting at column i if and only if $F^r \in \mathcal{F}_{2,n}$ has the property that for each $i < n$, there is either a $(P^{(2)})^r$-match or a $(P^{(5)})^r$-match starting at column i. But note that $(P^{(3)})^r$ and $(P^{(5)})^r$ are the two standard tableaux of shape (2,2). Thus F^r has the property that for each $i < n$, there is either a $(P^{(2)})^r$-match or a $(P^{(5)})^r$-match starting at column i if and only if F^r is standard tableau of shape (n,n). But it follows from the Frame-Robinson-Thrall hook formula [8] for the number of standard tableaux of a given shape λ that the number of standard tableaux of shape (n,n) is the Catalan number C_n. Thus $\mathrm{mp}_{2n,u} = C_n$.

It follows that there is a graph G_D associated with D which consists of the graph pictured on the right in the first line of Figure 16. Then we can construct graphs $G_{D,n}$ and $G^+_{D,n}$ using G_D in the same way that we constructed graphs $G_{n,P^{(i)}}$ and $G^+_{n,P^{(i)}}$ from $G_{P^{(i)}}$. For example, the graphs $G_{D,6}$ and $G^+_{D,6}$ are pictured on line 2 of Figure 16. Then $\mathrm{mp}_{2n,D}$ is the number of linear extensions of the poset determined by $G_{n,D}$ and $\mathrm{mp}_{2n+1,D}$ is the number of linear extensions of the poset determined by $G^+_{n,D}$.

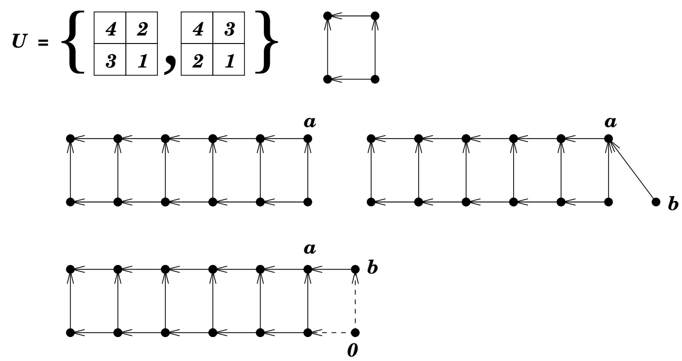

Figure 16: The graphs $G_{n,D}$ and $G^+_{n,D}$.

We claim that the number of linear extensions of the poset determined by $G^+_{n,D}$ is just C_{n+1}. Note that in $G_{n,D}$, the element in bottom right-hand corner must be the first element in any linear extension of the poset determined by $G_{n,D}$. Now create a new graph $G^{++}_{n,D}$ by adding a new element 0 and new directed edges connecting 0 to the element in the bottom right hand corner of $G^+_{n,D}$ and 0 to a which is the element in the top right corner of $G^+_{n,D}$ to form a graph $G^{++}_{n,D}$. It is easy to see that the number of linear extensions of the poset determined by $G^+_{n,D}$ is equal to the number of linear extensions of the poset determined by $G^{++}_{n,D}$. However the number of linear extensions of the poset determined by $G^{++}_{n,D}$ is just the number of linear extensions of the poset determined by $G_{n+1,D}$ which is C_{n+1}. $\square$

Unfortunately elements of $\mathcal{MP}_{2n,D}$ do not end in 1 or $2n$ so that there does not seem to be any way to develop simple recursions for $GMP_{2n,D}(x)$ or $GMP_{2n+1,D}(x)$. Nevertheless,

J. Harmse [10] computed the following initial values. of $GMP_{n,D}(x)$

$$\begin{aligned}
GMP_{1,D}(x) &= 1\\
GMP_{2,D}(x) &= 1\\
GMP_{3,D}(x) &= -1\\
GMP_{4,D}(x) &= 2x-3\\
GMP_{5,D}(x) &= 6-5x\\
GMP_{6,D}(x) &= 24-28x+5x^2\\
GMP_{7,D}(x) &= -64+84x-21x^2\\
GMP_{8,D}(x) &= -369+648x-294x^2+14x^3\\
GMP_{9,D}(x) &= 1288-2439x+1236x^2-84x^3\\
GMP_{10,D}(x) &= 8970-20792x+15189x^2-3408x^3+42x^4\\
GMP_{11,D}(x) &= -31121+73723x-54978x^2+12705x^3-330x^4\\
GMP_{12,D}(x) &= -323736+933223x-937838x^2+369138x^3-40920x^4+132x^5
\end{aligned}$$

Plugging these values into the generating functions (46) and (46) and using Mathematica, we have computed the following table of values of $A_{n,D}(x) = \sum_{\sigma \in A_n} x^{D\text{-mch}(\sigma)}$.

$A_{1,D}$	1
$A_{2,D}$	1
$A_{3,D}$	2
$A_{4,D}$	$3+2x$
$A_{5,D}$	$11+5x$
$A_{6,D}$	$24+32x+5x^2$
f$A_{7,D}$	$125+133x+14x^2$
$A_{8,D}$	$345+760x+266x^2+14x^3$
$A_{9,D}$	$1341+4359x+1194x^2+42x^3$
$A_{10,D}$	$7890+24928x+15609x^2+2052x^3+42x^4$
$A_{11,D}$	$17752+162570x+115401x^2+2937x^3+132x^4$

The techniques that we have developed in this paper can be extended to find generating functions for distribution of τ-matches in A_n where $\tau \in A_{2j}$ is an up-down minimal overlapping permutation. Here $\tau \in A_{2j}$ is an up-down minimal overlapping permutation if the smallest i such that there exists a $\sigma \in A_{2i}$ such that τ-mch$(\sigma) = 2$ is $4j-2$. Also the techniques that we have developed in this paper can be generalized to find to find generating functions for consecutive matches in generalized k-Euler permutations. That is, let $E_n^{(k)} = \{\sigma \in S_n : Des(\sigma) = \{kj : j \geq 1\} \cap [n-1]\}$. Then we can generalize the results of this paper to study the distribution of τ-matches in $E_n^{(k)}$ where $\tau \in E_{2k}^{(k)}$. Some of this work will appear in the forthcoming Ph.D. thesis of Adrian Duane.

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