Extremal problems for the p-spectral radius of graphs

Liying Kang*
Department of Mathematics
Shanghai University
Shanghai, PR China
lykang@shu.edu.cn

Vladimir Nikiforov
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, USA
vnikifrv@memphis.edu

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Abstract

The p-spectral radius of a graph G of order n is defined for any real number $p \geqslant 1$ as

$$\lambda^{(p)}(G) = \max\left\{ 2 \sum_{\{i,j\} \in E(G)} x_i x_j : x_1, \ldots, x_n \in \mathbb{R} \text{ and } |x_1|^p + \cdots + |x_n|^p = 1 \right\}.$$

The most remarkable feature of $\lambda^{(p)}$ is that it seamlessly joins several other graph parameters, e.g., $\lambda^{(1)}$ is the Lagrangian, $\lambda^{(2)}$ is the spectral radius and $\lambda^{(\infty)}/2$ is the number of edges. This paper presents solutions to some extremal problems about $\lambda^{(p)}$, which are common generalizations of corresponding edge and spectral extremal problems.

Let $T_r(n)$ be the r-partite Turán graph of order n. Two of the main results in the paper are:

(I) Let $r \geqslant 2$ and $p > 1$. If G is a K_{r+1}-free graph of order n, then

$$\lambda^{(p)}(G) < \lambda^{(p)}(T_r(n)),$$

unless $G = T_r(n)$.

(II) Let $r \geqslant 2$ and $p > 1$. If G is a graph of order n, with

$$\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n)),$$

then G has an edge contained in at least cn^{r-1} cliques of order $r+1$, where c is a positive number depending only on p and r.

Keywords: extremal problems; Turán problems; spectral radius; clique number; extremal problems; saturation problems.

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1 Introduction

In this paper we study extremal problems for the p-spectral radius $\lambda^{(p)}$ of graphs, so first let us recall the definition of $\lambda^{(p)}$. Suppose that G is a graph of order n. The quadratic form of G is defined for any vector $[x_i] \in \mathbb{R}^n$ as

$$P_G([x_i]) := 2 \sum_{\{i,j\} \in E(G)} x_i x_j.$$

Now, for any real number $p \geqslant 1$, the *p-spectral radius* of G is defined as

$$\lambda^{(p)}(G) = \max \left\{ 2 \sum_{\{i,j\} \in E(G)} x_i x_j : x_1, \ldots, x_n \in \mathbb{R} \text{ and } |x_1|^p + \cdots + |x_n|^p = 1 \right\}.$$

Note that $\lambda^{(p)}$ is a multifaceted parameter, as $\lambda^{(1)}(G)$ is the Lagrangian of G, $\lambda^{(2)}(G)$ is its spectral radius, and $\lim_{p\to\infty} \lambda^{(p)}(G) = 2e(G)$. The p-spectral radius has been introduced for uniform hypergraphs by Keevash, Lenz, and Mubayi in [10], and subsequently studied in [9], [18], [19], and [20].

The problems studied in this paper originate from the following general one:

What is the maximum $\lambda^{(p)}(G)$ of a graph G of order n, not containing a given subgraph H?

Similar questions for the maximum number of edges $e(G)$ and a fixed subgraph H are called Turán problems and are central in classical extremal graph theory, as known, e.g., from [1], Ch. 6. In fact, we shall build a parallel extremal theory for $\lambda^{(p)}$, which extends the classical theory, given that $\lim_{p\to\infty} \lambda^{(p)}(G) = 2e(G)$; thus, the classical extremal theory is a limiting case of the extremal theory for $\lambda^{(p)}$. More important, our main focus will be on forbidden subgraphs H whose order grows with n, as this approach gives more insight and leads to definite results like Theorem 6 below.

To begin with, recall that the Turán graph $T_r(n)$ is the complete r-partite graph of order n, with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. The prominence of $T_r(n)$ in extremal graph theory has been established by the ground-breaking result of Turán [22]:

Theorem A *If G is a K_{r+1}-free graph of order n, then $e(G) < e(T_r(n))$, unless $G = T_r(n)$.*

A very similar result has been proved for the spectral radius $\lambda^{(2)}$ in [13]:

Theorem B *If G is a K_{r+1}-free graph of order n, then $\lambda^{(2)}(G) < \lambda^{(2)}(T_r(n))$, unless $G = T_r(n)$.*

Our starting point is a common generalization of Theorems A and B, stated as follows.

Theorem 1. *Let $r \geqslant 2$ and $p > 1$. If G is K_{r+1}-free graph of order n, then $\lambda^{(p)}(G) < \lambda^{(p)}(T_r(n))$, unless $G = T_r(n)$.*

Like Turán's theorem in extremal graph theory, Theorem 1 motivates a lot of related results, some of which we shall study in this and a forthcoming paper. In particular, our results answer important instances of the following broad question:

Which subgraphs are necessary present in a graph G of sufficiently large order n if

$$\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n))?$$

As we shall see, here the range of the difference $f(n) = \lambda^{(p)}(G) - \lambda^{(p)}(T_r(n))$ determines different problems: when $f(n) = o\left(n^{1-2/p}\right)$ we have what are called *saturation problems*, and when $f(n) = O(n^{1-2p})$, we have *Erdős-Stone type problems.*

We also shall study *stability problems,* which concern near-maximal graphs without forbidden subgraphs. More precisely a stability problem can be stated as:

Suppose that H is a graph which is necessary present in any graph G of sufficiently large order n, with $\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n))$. What is the structure of a graph G of order n if

$$\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n)) - o\left(n^{2-2/p}\right),$$

but G contains no H?

Many extremal problems along the above lines have been successfully solved for $\lambda^{(2)}$, the classical spectral radius; see [17] for a survey and references. However, $\lambda^{(2)}$ belongs to the realm of Linear Algebra and its study builds on proven solid ground. By contrast, linear-algebraic methods are irrelevant for the study of $\lambda^{(p)}$ in general, and in fact no efficient general methods are known for it. Thus the study of $\lambda^{(p)}$ for $p \neq 2$ is far more complicated than of $\lambda^{(2)}$. One of the aims on the present paper is to find out if specific applications of the spectral radius $\lambda^{(2)}$ can be extended to $\lambda^{(p)}$ in general. So far, most attempts have been successful, but there are many basic unanswered questions, see [20] for some examples.

It should be noted that extremal problems for $\lambda^{(p)}$ of hypergraphs have been studied in [9], [10], [19], and [20], but 2-graphs are better understood, so it is worthwhile to delve into deeper extremal theory. Another line has been investigated in [18], where the emphasis is on hereditary properties.

2 Turán type theorems for $\lambda^{(p)}(G)$

It is not hard to see that if $n \geqslant r > q$, then $\lambda^{(p)}(T_r(n)) > \lambda^{(p)}(T_q(n))$ for every $p \geqslant 1$. This observation entails the following reformulation of Theorem 1.

Theorem 2. *Let $r \geqslant 2$ and $p > 1$. If G is a graph of order n, with clique number ω, then $\lambda^{(p)}(G) < \lambda^{(p)}(T_\omega(n))$, unless $G = T_\omega(n)$.*

As already mentioned, $\lim_{p\to\infty} \lambda^{(p)}(G) = 2e(G)$, so Turán's Theorem A can be recovered in full detail from Theorem 1.

Let us note that particular relations between the clique number ω of a graph G and $\lambda^{(p)}(G)$ have been long known. For example, the result of Motzkin and Straus [11] (see Theorem E below) establishes the fundamental fact that $\lambda^{(1)}(G) = 1 - 1/\omega$; later it has been used by Wilf [23] to derive the bound

$$\lambda^{(2)}(G) \leqslant (1 - 1/\omega)\, n;$$

and in [12] it was used for the stronger inequality

$$\lambda^{(2)}(G) \leqslant \sqrt{2\,(1 - 1/\omega)\, e\,(G)}.$$

Note that the last two results are explicit, while being almost tight. It turns out that the approach of Motzkin and Straus helps to deduce similar explicit results for $\lambda^{(p)}(G)$ and any $p \geqslant 1$ as well.

Theorem 3. *Let $r \geqslant 2$ and $p \geqslant 1$. If G is a K_{r+1}-free graph of order n, then*

$$\lambda^{(p)}(G) \leqslant \left(1 - \frac{1}{r}\right)^{1/p} (2e\,(G))^{1-1/p}\,, \tag{1}$$

and

$$\lambda^{(p)}(G) \leqslant \left(1 - \frac{1}{r}\right) n^{2-2/p}. \tag{2}$$

If $p > 1$, equality holds in (2) if and only if $r|n$ and $G = T_r\,(n)$.

In particular, Theorem 3 implies that if G is a graph of order n, with clique number ω, then

$$\lambda^{(p)}(G) \leqslant (1 - 1/\omega)^{1/p}\,(2e\,(G))^{1-1/p}$$

and

$$\lambda^{(p)}(G) \leqslant (1 - 1/\omega)\, n^{2-2/p}.$$

A natural question is how good bounds (1) and (2) are compared to the bound in Theorem 1, which is attained for every n. It turns out that bounds (1) and (2) are never too far from the best possible one, as seen in the following several estimates.

Theorem 4. *Let $T_r\,(n)$ be the r-partite Turán graph of order n. Then*

$$\lambda^{(1)}(T_r\,(n)) = 1 - 1/r, \tag{3}$$

and for every $p > 1$,

$$2e\,(T_r\,(n)) \leqslant \lambda^{(p)}(T_r\,(n))\, n^{2/p} \leqslant 2e\,(T_r\,(n)) \left(1 + \frac{r}{pn^2}\right), \tag{4}$$

$$\left(1 - \frac{1}{r}\right) n^2 - \frac{r}{4} \leqslant \lambda^{(p)}(T_r\,(n))\, n^{2/p} \leqslant \left(1 - \frac{1}{r}\right) n^2. \tag{5}$$

3 Saturation problems

Theorem 1 implies that if G is a graph of order n, with $\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n))$, then G contains a K_{r+1}. We shall show that, in fact, much larger supergraphs of K_{r+1} can be found in G. Such problems are usually called saturation problems.

3.1 Joints

In [5] Erdős proved that if $r \geqslant 2$, and G is a graph of sufficiently large order n, with $e(G) > e(T_r(n))$, then G has an edge that is contained in at least $n^{r-1}/(10r)^{6r}$ cliques of order $r+1$. This fact is fundamental, so to study its consequences, the following definition was given in [3]:

An r-joint of size t is a collection of t distinct r-cliques sharing an edge.

A 3-joint is also called a *book*. Books have been studied extensively in extremal and Ramsey graph theory. Note that books are determined by their size alone, while for $r > 3$ there are many non-isomorphic r-joints of the same size.

We write $\mathrm{js}_r(G)$ for the maximum size of an r-joint in a graph G. The following theorem enhances Theorem 1, insofar that from the same premises it implies the existence of subgraphs whose order grows with n. For this reason we shall use it as a starting point for several other extensions.

Theorem 5. *Let $r \geqslant 2$ and $p > 1$. If G is a graph of order n, with*

$$\lambda^{(p)}(G) \geqslant \lambda^{(p)}(T_r(n)),$$

then

$$\mathrm{js}_{r+1}(G) > \frac{n^{r-1}}{r^{r^6 p/(p-1)}},$$

unless $G = T_r(n)$.

Let us note that the order of n^{r-1} is obviously best possible, but the coefficient $r^{-r^6 p/(p-1)}$ is far from being optimal. Nevertheless, this small coefficient makes the statement valid for all n, and for larger n it can be somewhat increased.

3.2 Color critical subgraphs

Call a graph *k-color critical*, if it is k-colorable, but it can be made $(k-1)$-colorable by removing a particular edge. For example, books are 3-color critical graphs.

Simonovits [21] has proved that if F is an $(r+1)$-color critical graph, then $F \subset G$ for every graph G of sufficiently large order n, with $e(G) > e(T_r(n))$.

This statement can be generalized considerably. Indeed, given the integers $r \geqslant 2$ and $t \geqslant 2$, let $K_r^+(t)$ be the complete r-partite graph with each part of size t, and with an edge added to its first part. The study of $K_r^+(t)$ in connection to the Turán theorem has been initiated by Erdős [4], [6], but a definite result has been obtained only in [16]:

Theorem C *Let* $r \geqslant 2$ *and* $c \leqslant c_0(r)$ *be a sufficiently small positive number. If* G *is a graph of sufficiently large order* n, *with* $e(G) > e(T_r(n))$, *then* G *contains a* $K_r^+(\lfloor c \log n \rfloor)$.

This type of result is indeed a neat generalization of Simonovits's result, for any $(r+1)$-color critical graph is a subgraph of $K_r^+(\lfloor c \log n \rfloor)$ if n is large enough. In [15] a similar theorem has been proved also for the spectral radius $\lambda^{(2)}$:

Theorem D *Let* $r \geqslant 2$ *and* $c \leqslant c_0(r)$ *be sufficiently small positive number. If* G *is a graph of sufficiently large order* n, *with* $\lambda^{(2)}(G) > \lambda^{(2)}(T_r(n))$, *then* G *contains a* $K_r^+(\lfloor c \log n \rfloor)$.

We give a common generalization of Theorems C and D in the following theorem.

Theorem 6. *Let* r, p, c, *and* n *satisfy*

$$r \geqslant 2, \quad p > 1, \quad 0 < c \leqslant r^{-(r+8)r}/2, \quad \text{and} \quad \log n \geqslant 2p/(cp - c).$$

If G *is a graph of order* n, *with* $\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n))$, *then* G *contains a* $K_r^+(\lfloor c \log n \rfloor)$.

Let us emphasize that in Theorem 6 c may depend on n, e.g., if c is a slowly decaying function of n, like $c = 1/\log\log n$, the conclusion is meaningful for sufficiently large n.

It should be noted that the authors of [10], in their Corollary 2, prove a similar theorem, where instead of $K_r^+(\lfloor c \log n \rfloor)$ they take a fixed $(r+1)$-color critical subgraph. However, they claim that their statement generalizes Theorem D as well, which is false, as the order of $K_r^+(\lfloor c \log n \rfloor)$ grows with n. In fact the change from a fixed $(r+1)$-color critical graph to $K_r^+(\lfloor c \log n \rfloor)$ is a major difference, requiring a longer proof, with more advanced techniques and more delicate calculations.

3.3 An abstract saturation theorem

The proofs of Theorems 5 and 6, and of several stability results in a forthcoming paper, will be deduced from a fairly general, multiparameter statement, which reads as follows.

Theorem 7. *Let the numbers* p, γ, A, R, *and* n *satisfy*

$$1 < p \leqslant 2, \quad 0 < 4\gamma < A < 1, \quad R \geqslant 0, \quad \text{and} \quad n > \frac{4(R+1)p}{\gamma(p-1)} A^{-p/(\gamma p - \gamma)}.$$

If G *is a graph of order* n, *with*

$$\lambda^{(p)}(G)\, n^{2/p-1} \geqslant An - R/n \quad \text{and} \quad \delta(G) \leqslant (A - \gamma)\, n,$$

then there exists an induced subgraph $H \subset G$ *of order* $k > A^{-p/(Ap-A)}n$, *with*

$$\lambda^{(p)}(H)\, k^{2/p-1} > Ak \quad \text{and} \quad \delta(H) > (A - \gamma)\, k.$$

This theorem seems overly complicated, but its meaning and usage are straightforward. It will be applied to prove the existence of certain subgraphs. The starting point will be some known statement ensuring that if G is a graph of sufficiently large order n, with

$$\lambda^{(p)}(G)\,n^{2/p-1} > An \quad \text{and} \quad \delta(G) > (A-\gamma)\,n,$$

then G contains a subgraph F.

Now, suppose that G is of sufficiently large order n, but $\lambda^{(p)}(G)\,n^{2/p-1} \geqslant An - O(1)$, and $\delta(G) \leqslant (A-\gamma)\,n$, so the requirement for the existence of F are not met at all. In this case Theorem 7 helps to mend the situation, as it guarantees that there is an induced subgraph $H \subset G$ of relatively large order k, satisfying

$$\lambda^{(p)}(H)\,k^{2/p-1} > Ak \quad \text{and} \quad \delta(H) > (A-\gamma)\,k.$$

Now, if n is large enough, then k is large enough, and so $F \subset H \subset G$, as desired.

Let us note that, in any concrete case, the choice of γ, A and R is determined by the type of the subgraph F.

In the remaining part of the paper we prove Theorems 1-7.

4 Proofs

4.1 Notation and preliminaries

In our proofs we shall use a number of classical inequalities: the Power Mean inequality (PM inequality), the Bernoulli and the Maclaurin inequalities; for more details on these inequalities we refer the reader to [7].

For graph notation and concepts undefined here, we refer the reader to [2]. In particular, given a graph G, we write:

- $V(G)$ for the vertex set of G and $v(G)$ for $|V(G)|$;
- $E(G)$ for the edge set of G and $e(G)$ for $|E(G)|$;
- $\Gamma_G(u)$ for the set of neighbors of a vertex u (we drop the subscript if G is understood);
- $\delta(G)$ for the minimum degree of G;
- $k_r(G)$ for the number of r-cliques of G;
- $G-u$ for the graph obtained by removing the vertex $u \in V(G)$.

If G is a graph of order n and $V(G)$ is not defined explicitly, it is assumed that $V(G) := \{1,\ldots,n\}$.

4.1.1 Some facts about the p-spectral radius

All required facts about the p-spectral radius of graphs are given below. Additional reference material can be found in [9], [19], and [20].

Let G be a graph of order n. A vector $[x_i] \in \mathbb{R}^n$ such that $|x_1|^p + \cdots + |x_n|^p = 1$ and $\lambda^{(p)}(G) = P_G([x_i])$ is called an *eigenvector* to $\lambda^{(p)}(G)$. It is easy to see,that there is always a non-negative eigenvector to $\lambda^{(p)}(G)$. If $p > 1$, by Lagrange's method, one can show that

$$\lambda^{(p)}(G)\, x_k^{p-1} = \sum_{i \in \Gamma(k)} x_i. \tag{6}$$

for each $k = 1, \ldots, n$. Equation (6) is called the *eigenequation* of $\lambda^{(p)}(G)$ for the vertex k.

In the following three bounds it is assumed that $p \geqslant 1$. First, by Maclaurin's and the PM inequalities we find the absolute maximum of $\lambda^{(p)}(G)$ with respect to n :

$$\lambda^{(p)}(G) \leqslant 2 \sum_{1 \leqslant i < j \leqslant n} x_i x_j < (n-1)\, n \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leqslant (n-1)\, n \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{2/p} = \frac{n-1}{n^{2/p-1}}. \tag{7}$$

Second, we find a bound with respect to $e(G)$:

$$\lambda^{(p)}(G) \leqslant 2e(G)^{1-1/p} \left(\sum_{1 \leqslant i < j \leqslant n} x_i^p x_j^p \right)^{1/p} \leqslant 2e(G)^{1-1/p} \left(\frac{n-1}{2n} \right)^{1/p} \leqslant (2e(G))^{1-1/p}. \tag{8}$$

In the other direction, taking the n-vector $\mathbf{x} = \left(n^{-1/p}, \ldots, n^{-1/p} \right)$, we obtain a useful lower bound

$$\lambda^{(p)}(G) \geqslant P_G(\mathbf{x}) = 2e(G)\, n^{-2/p}. \tag{9}$$

Note that if $1 \leqslant p < 2$, then bound (9) may not be tight for some regular graphs, but for $p \geqslant 2$ it is always tight for regular graphs; in fact, as mentioned earlier,

$$\lim_{p \to \infty} \lambda^{(p)}(G)\, n^{2/p} = \lim_{p \to \infty} \lambda^{(p)}(G) = 2e(G).$$

It is worth noting that using the PM inequality, one can find that $\lambda^{(p)}(G)\, n^{2/p}$ is nonincreasing in p, that is to say, if $p > q \geqslant 1$, then

$$\lambda^{(q)}(G)\, n^{2/q} \geqslant \lambda^{(p)}(G)\, n^{2/p}. \tag{10}$$

4.2 Proof of Theorem 1

Since a statement similar to Theorem 1 has been claimed in [10], Corollary 2, we need to make a comment here. The proof given below reduces Theorem 1 to r-partite graphs, for which we already gave an independent proof in [9]. The same reduction, albeit more complicated, has been carried out in [10] as well, but these authors provide no proof for r-partite graphs, so their proof of Theorem 1 is essentially incomplete. Unfortunately, this omission is not negligible, as the proof for r-partite graphs is much longer and more involved than the reduction of Theorem 1 to r-partite graphs.

Next, we state the main ingredient of our proof, which is a particular instance of a result in [9] about the p-spectral radius of k-partite uniform hypergraphs.

Theorem 8. *Let $r \geqslant 2$, and $p > 1$. If G is an r-partite graph of order n, then*

$$\lambda^{(p)}(G) < \lambda^{(p)}(T_r(n)),$$

unless $G = T_r(n)$.

Thus, to prove Theorem 1, all we need is that the maximum $\lambda^{(p)}(G)$ of a K_{r+1}-free graph G of order n is attained on an r-partite graph. Reductions of this kind have been pioneered by Zykov [24] and Erdős, but to spectral problems they have been first applied by Guiduli, in an unpublished proof of the spectral Turán theorem. Another noteworthy application of the same techniques is for the spectral radius of the signless Laplacian of K_{r+1}-free graphs in [8]. Thus we proceed with a reduction lemma for $\lambda^{(p)}(G)$ of a K_{r+1}-free graph G.

Lemma 9. *Let $p \geqslant 1$. If G is a K_{r+1}-free graph of order n, then there exists an r-partite graph H of order n such that $\lambda^{(p)}(H) \geqslant \lambda^{(p)}(G)$.*

Proof. Let $\mathbf{x} = [x_i]$ be a nonnegative eigenvector to $\lambda^{(p)}(G)$. For each $v \in V(G)$, set

$$D_G(v, \mathbf{x}) := \sum_{i \in \Gamma_G(v)} x_i.$$

We shall prove that there exists a complete r-partite graph H such that $V(H) = V(G)$ and $D_H(v, \mathbf{x}) \geqslant D_G(v, \mathbf{x})$ for any $v \in V(G)$. This proof will be carried out by induction on r. Let $u \in V(G)$ satisfy

$$D_G(u, \mathbf{x}) := \max\{D_G(v, \mathbf{x}) : v \in V(G)\},$$

and set $U := \Gamma_G(u)$ and $W := V(G) \backslash \Gamma_G(u)$. To start the induction let $r := 2$; hence G is triangle-free, and so $e(G[U]) = 0$. We shall show that the complete bipartite graph H with bipartition $V(H) = U \cup W$ is as required. Indeed, if $v \in U$, then $\Gamma_G(v) \subset W$, and so

$$D_H(v, \mathbf{x}) = \sum_{i \in W} x_i \geqslant \sum_{i \in \Gamma_G(v)} x_i = D_G(v, \mathbf{x}).$$

On the other hand, if $v \in W$, then $D_H(v, \mathbf{x}) = D_G(u, \mathbf{x}) \geqslant D_G(v, \mathbf{x})$. Hence the graph H is as required.

Now, let $r > 2$ and assume that the assertion is true for r' whenever $2 \leqslant r' < r$. First note that $G[U]$ is a K_r-free graph; hence, by the induction assumption there exists a complete $(r-1)$-partite graph F with $V(F) = U$ and $D_F(v, \mathbf{x}) \geqslant D_{G[U]}(v, \mathbf{x})$ for any vertex $v \in U$. Let $V(F) = V_1 \cup \cdots \cup V_{r-1}$ be the partition of $V(F)$ into independent sets and let H be the complete r-partite graph with partition

$$V(H) = V_1 \cup \cdots \cup V_{r-1} \cup W = V(G).$$

We shall prove that H is as required. Indeed, on the one hand, if $v \in U$, then

$$D_H(v, \mathbf{x}) = D_F(v, \mathbf{x}) + \sum_{i \in W} x_i \geqslant D_{G[U]}(v, \mathbf{x}) + \sum_{i \in \Gamma_G(v) \cap W} x_i = D_G(v, \mathbf{x}).$$

On the other hand, if $v\in W$, then $D_H(v,\mathbf{x})=D_G(u,\mathbf{x})\geqslant D_G(v,\mathbf{x})$. Hence, H is a complete r-partite graph such that $D_H(v)\geqslant D_G(v)$ for any $v\in V(G)$. This completes the induction step, and the existence of H is proved.

To finish the proof of the lemma, note that

$$\begin{aligned}\lambda^{(p)}(H)&\geqslant 2\sum_{\{i,j\}\in E(H)}x_ix_j=\sum_{i\in V(H)}x_iD_H(i,\mathbf{x})\geqslant\sum_{i\in V(H)}x_iD_G(i,\mathbf{x})=2\sum_{\{i,j\}\in E(G)}x_ix_j\\&=\lambda^{(p)}(G).\end{aligned}$$

□

4.3 Proofs of Theorems 3 and 4

We use below the result of Motzkin and Straus [11], that can be stated as:

Theorem E *If G is a K_{r+1}-free graph of order n, and $x_1,\ldots,x_n$ are nonnegative numbers such that $x_1+\cdots+x_n=1$, then*

$$2\sum_{\{i,j\}\in E(G)}x_ix_j\leqslant 1-\frac{1}{r}. \tag{11}$$

The conditions for equality in (11) are well known, but we shall omit them. Instead we just note that if $K_r\subset G$, one may choose a vector $(x_1,\ldots,x_n)$ so that equality holds in (11).

We often shall use the following bound on the number of edges of the Turán graph $T_r(n)$,

$$2e(T_r(n))\geqslant\left(1-\frac{1}{r}\right)n^2-\frac{r}{4}. \tag{12}$$

Indeed, let $n=rs+t$, where s and t are nonnegative integers and $0\leqslant t<s$. It is known that

$$e(T_r(n))=\binom{r}{2}\frac{n^2-t^2}{r^2}+\binom{t}{2};$$

hence,

$$2e(T_r(n))=2\binom{r}{2}\frac{n^2-t^2}{r^2}+2\binom{t}{2}=\left(1-\frac{1}{r}\right)n^2-\frac{t(r-t)}{r}\geqslant\left(1-\frac{1}{r}\right)n^2-\frac{r}{4}.$$

Proof of Theorem 3. The proof of inequality (1) has been given many times, but it is short, so for reader's sake we shall give it again. Let $[x_i]$ be a nonnegative eigenvector to $\lambda^{(p)}(G)$. The PM inequality implies that

$$\lambda^{(p)}(G)=2\sum_{\{i,j\}\in E(G)}x_ix_j\leqslant 2e(G)^{1-1/p}\left(\sum_{\{i,j\}\in E(G)}x_i^px_j^p\right)^{1/p}. \tag{13}$$

Note that $x_1^p + \cdots + x_n^p = 1$, and G is K_{r+1}-free, so the Motzkin-Straus result implies that

$$\sum_{\{i,j\}\in E(G)} x_i^p x_j^p \leqslant \frac{r-1}{2r}.$$

Plugging this back in (13), we obtain (1).

To prove (2), we use (1) and the concise Turán theorem, which implies that

$$2e(G) \leqslant \left(1-\frac{1}{r}\right) n^2.$$

Now, if equality holds, i.e., if

$$\lambda^{(p)}(G) = \left(1-\frac{1}{r}\right) n^{2-2/p},$$

then we should have

$$2e(G) = \left(1-\frac{1}{r}\right) n^2,$$

and this can happen only if $r|n$ and $G = T_r(n)$. □

Proof of Theorem 4. Equality (3) follows from the Motzkin-Straus' Theorem E and the fact the $K_r \subset T_r(n)$. The lower bound in (4) follows by (9). To prove the upper bound in (4) let us start with

$$\lambda^{(p)}(T_r(n)) \leqslant \left(1-\frac{1}{r}\right)^{1/p} (2e(T_r(n)))^{1-1/p}.$$

Next, using (12) and Bernoulli's inequality, we get the estimate

$$2e(T_r(n)) > \left(1-\frac{1}{r}\right) n^2 - \frac{r}{4} > \frac{\left(1-\frac{1}{r}\right) n^2}{1+\frac{r}{n^2}} > \frac{\left(1-\frac{1}{r}\right) n^2}{\left(1+\frac{r}{pn^2}\right)^p}.$$

This implies

$$\frac{(2e(T_r(n)))^{1/p}}{n^{2/p}} \left(1+\frac{r}{pn^2}\right) > \left(1-\frac{1}{r}\right)^{1/p},$$

and the upper bound in (4) follows.

The bound in (5) comes from (9) and (8). □

4.4 Proof of Theorem 7

The proof of Theorem 7 goes along lines, which are familiar from Theorem 5 in [14], but the arguments and calculations are more complicated. To clarify the structure of the proof we have extracted two of its essential points into Lemmas 10 and 11.

Lemma 10. *Let the numbers p, A, γ, R, and n satisfy*

$$1 < p \leqslant 2, \quad 0 < \gamma < A < 1, \quad R \geqslant 0, \quad \text{and} \quad n \geqslant 4R/\gamma.$$

Let G be a graph of order n, with

$$\lambda^{(p)}(G)\, n^{2/p-1} \geqslant An - \frac{R}{n} \quad \text{and} \quad \delta(G) \leqslant (A-\gamma)\, n.$$

If $[x_i]$ is a nonnegative eigenvector to $\lambda^{(p)}(G)$, then the value $\sigma := \min\{x_1^p, \ldots, x_n^p\}$ satisfies

$$\sigma \leqslant \frac{1-\gamma/2}{n}. \tag{14}$$

Proof. Let $k \in V(G)$ be a vertex of degree $\delta = \delta(G)$ and set for short $\lambda^{(p)}(G) = \lambda$. Applying the PM inequality to the right side of the eigenequation for x_k, we get

$$\begin{aligned}\lambda\sigma^{1-1/p} \leqslant \lambda x_k^{p-1} = \sum_{i\in\Gamma(k)} x_i \leqslant \delta^{1-1/p}\left(\sum_{i\in\Gamma(k)} x_i^p\right)^{1/p} = \delta^{1-1/p}\left(1 - \sum_{i\notin\Gamma(k)} x_i^p\right)^{1/p} \\ \leqslant \delta^{1-1/p}\left(1-(n-\delta)\,\sigma\right)^{1/p}.\end{aligned}$$

After some algebra, this inequality reduces to

$$\frac{\lambda^p\sigma^{p-1}}{\delta^{p-1}} + (n-\delta)\,\sigma \leqslant 1.$$

In view of (9) $\lambda n^{2/p-1} \geqslant 2e(G)/n \geqslant \delta$; hence

$$\left(\frac{\lambda n^{2/p-1}}{\delta}\right)^{p-1} \geqslant 1,$$

and so,

$$\frac{\lambda^{p-1}}{\delta^{p-1}} > n^{(1-2/p)(p-1)}.$$

Also $\sigma \leqslant 1/n$ and since $p-2 \leqslant 0$, we see that $\sigma^{p-2} \geqslant n^{2-p}$. Therefore,

$$\frac{\lambda^p\sigma^{p-1}}{\delta^{p-1}} \geqslant \lambda n^{(1-2/p)(p-1)-p+2}\sigma = \lambda n^{2/p-1}\sigma,$$

yielding finally

$$\lambda n^{2/p-1}\sigma + (n-\delta)\,\sigma \leqslant 1.$$

Now, plugging the bounds on $\delta(G)$ and $\lambda^{(p)}(G)\,n^{2/p-1}$, we get

$$\sigma \leqslant \frac{1}{\left(A - \frac{R}{n^2}\right)n + n - (A-\gamma)\,n} = \frac{1}{(1+\gamma)\,n - \frac{R}{n}}.$$

To complete the proof of the lemma we shall check that

$$\frac{1}{1+\gamma - R/n^2} < 1 - \frac{\gamma}{2}.$$

Indeed,

$$\begin{aligned}\left(1-\frac{\gamma}{2}\right)\left(1+\gamma-\frac{R}{n^2}\right) &= 1+\frac{\gamma}{2}(1-\gamma) - \frac{R}{n^2}\left(1-\frac{\gamma}{2}\right) > 1+\frac{\gamma}{2}(1-\gamma) - \frac{R}{n}\left(1-\frac{\gamma}{2}\right)\\ &> 1+\frac{\gamma}{2}(1-\gamma) - \frac{\gamma}{4}\left(1-\frac{\gamma}{2}\right) = 1+\frac{\gamma}{2}\left(\frac{3}{4}-\frac{7}{8}\gamma\right)\\ &> 1+\frac{\gamma}{2}\left(\frac{3}{4}-\frac{7}{16}\right) > 1.\end{aligned}$$

Lemma 10 is proved. □

The next lemma shows that if $\lambda^{(p)}(G)\,n^{2/p-1}$ is large enough, but the minimum degree $\delta(G)$ is not too large, we can remove a vertex u, so that $\lambda^{(p)}(G-u)\,(n-1)^{2/p-1}$ is also large.

Lemma 11. *Let the numbers p, γ, A, R, and n satisfy*

$$1 < p \leqslant 2, \quad 0 < \gamma < A < 1, \quad R \geqslant 0, \quad \text{and} \quad n \geqslant 4R/\gamma.$$

Let G be a graph of order n, with

$$\delta(G) \leqslant (A-\gamma)\,n \quad \text{and} \quad \lambda^{(p)}(G)\,n^{2/p-1} \geqslant An - R/n.$$

If $[x_i]$ is a nonnegative eigenvector to $\lambda^{(p)}(G)$ and u is a vertex with $x_u = \min\{x_1,\ldots,x_n\}$, then

$$\lambda^{(p)}(G-u)\,(n-1)^{2/p-1} \geqslant \left(\frac{n-2}{n-1}\right)^{1-(1-1/p)\gamma} \lambda^{(p)}(G)\,n^{2/p-1}.$$

Proof. Let p, γ, A, R, and n satisfy the requirements, let $\mathbf{x} = [x_i]$ be a nonnegative eigenvector to $\lambda^{(p)}(G)$ and u be a vertex with $x_u := \min\{x_1,\ldots,x_n\}$; set $\sigma := x_k^p$. Obviously, Lemma 10 can be applied here, getting

$$\sigma \leqslant \frac{1-\gamma/2}{n}. \tag{15}$$

Next, set for short $\delta := \delta(G)$, $\lambda_n := \lambda^{(p)}(G)$, and $\lambda_{n-1} := \lambda^{(p)}(G-u)$. Letting $\mathbf{x}'$ be the $(n-1)$-vector obtained from $\mathbf{x}$ by omitting the entry x_k, we see that

$$P_{G-k}(\mathbf{x}') = P_G(\mathbf{x}) - 2x_k \sum_{i\in\Gamma(k)} x_i = \lambda_n - 2x_k\left(\lambda_n x_k^{p-1}\right) = \lambda_n\left(1-2x_k^p\right).$$

On the other hand,

$$P_{G-k}(\mathbf{x}') \leqslant \lambda_{n-1}\|\mathbf{x}'\|_p^2 = \lambda_{n-1}\left(1-x_k^p\right)^{2/p},$$

hence, after some algebra, we find that

$$\lambda_{n-1} \geqslant \frac{1-2\sigma}{(1-\sigma)^{2/p}}\lambda_n. \tag{16}$$

Note that the function

$$f(x) = \frac{1-2y}{(1-y)^{2/p}}$$

is decreasing in y for $0<y<1$, for the derivative of $f(y)$ satisfies

$$f'(y) = \frac{-2(1-y)^{2/p} + \frac{2}{p}(1-2y)(1-y)^{2/p-1}}{(1-y)^{4/p}} = -\frac{2(1-y)^{2/p-1}}{p(1-y)^{4/p}}((p-1)+2y) < 0.$$

Therefore, in view of (15), we find that

$$f(\sigma) \geqslant f\left(\frac{1-\gamma/2}{n}\right).$$

Thus, setting for short $\xi := \gamma/2$, we see that

$$\frac{1-2\sigma}{(1-\sigma)^{2/p}} \geqslant \frac{1-2\frac{1-\xi}{n}}{\left(1-\frac{1-\xi}{n}\right)^{2/p}}.$$

Plugging this back in (16), we find that

$$\begin{aligned}\frac{\lambda_{n-1}(n-1)^{2/p-1}}{\lambda_n n^{2/p-1}} &\geqslant \left(1-\frac{(1-\xi)/n}{1-(1-\xi)/n}\right)\cdot\left(\frac{1-1/n}{1-(1-\xi)/n}\right)^{2/p-1}\\ &= \left(1-\frac{1-\xi}{n-1+\xi}\right)\cdot\left(1-\frac{\xi}{n-1+\xi}\right)^{2/p-1}.\end{aligned}$$

To estimate the latter expression, note that $0<1-\xi<1$ and $0<\xi<1$; hence, Bernoulli's inequality implies that

$$\left(1-\frac{1-\xi}{n-1+\xi}\right) \geqslant \left(1-\frac{1}{n-1+\xi}\right)^{1-\xi},$$

and

$$\left(1-\frac{\xi}{n-1+\xi}\right) \geqslant \left(1-\frac{1}{n-1+\xi}\right)^{\xi}.$$

Thus, we obtain

$$\frac{\lambda_{n-1}\left(n-1\right)^{2/p-1}}{\lambda_n n^{2/p-1}} \geqslant \left(1-\frac{1}{n-1+\xi}\right)^{1-2(1-1/p)\xi} > \left(1-\frac{1}{n-1}\right)^{1-(1-1/p)\gamma},$$

as claimed. Lemma 11 is proved. □

The main idea of the proof of Theorem 7 is to iterate the removal of vertices of smallest entry in eigenvectors to $\lambda^{(p)}$. Every time a vertex is removed, the ratio of $\lambda^{(p)}$ of the remaining graph to its order increases. So the vertex removal must stop before $\lambda^{(p)}$ exceeds its absolute maximum. As this stop happens fairly soon, the order of the remaining graph is fairly large.

Proof of Theorem 7. Let p, γ, A, R and n be as required, and let G be a graph of order n, with

$$\lambda^{(p)}\left(G\right) n^{2/p-1} \geqslant An - R/n \quad \text{and} \quad \delta\left(G\right) \leqslant \left(A-\gamma\right) n.$$

Define a decreasing sequence of graphs $G_n \supset G_{n-1} \supset \cdots$ by the following procedure $\mathcal{P}$:

$G_n := G$;
$i := n$;
while $\delta\left(G_i\right) \leqslant \left(A-\gamma\right) i$ **begin**
1. *Select an eigenvector* $(x_1, \ldots, x_i)$ *to* $\lambda^{(p)}\left(G_i\right)$;
2. *Select a vertex* $u \in V\left(G_i\right)$ *with* $x_u = \min\left\{x_1, \ldots, x_i\right\}$;
3. $G_{i-1} := G_i - u$;
4. $i := i - 1$;
end.

We claim that the following compound statement is true:
(i) at line 1 we always have

$$i > A^{p/(\gamma p-\gamma)} n > 4R/\gamma, \tag{17}$$

(ii) at line 3 we always have

$$\lambda^{(p)}\left(G_{i-1}\right)\left(i-1\right)^{2/p-1} > \left(1-\frac{1}{i-1}\right)^{1-(1-1/p)\gamma} \lambda^{(p)}\left(G_i\right) i^{2/p-1}. \tag{18}$$

Clearly, to prove (18) we may use Lemma 11, which, however, requires that $i > 4R/\gamma$; this is why we have to prove (17) as well. We shall use induction on $d := n - i$. To start the induction let $d = n - n = 0$. Clearly inequality (17) is true for $i = n$. Since, at line 1 we always have $\delta\left(G_i\right) \leqslant \left(A-\gamma\right) i$, after removing the vertex u Lemma 11, together with

(5), implies (18). Now, assume that (17) and (18) hold for $0 \leqslant d \leqslant n-i$; we shall prove them for $d+1 = n-(i-1)$. First, the inductive assumption implies that

$$\frac{\lambda^{(p)}(G_s)\, s^{2/p-1}}{\lambda^{(p)}(G_{s+1})\,(s+1)^{2/p-1}} > \left(1-\frac{1}{s}\right)^{1-(1-1/p)\gamma}$$

for each $s = n-1, \ldots, i$. Hence, multiplying these inequalities for $s = n-1, \ldots, i$, we obtain

$$\lambda^{(p)}(G_i)\, i^{2/p-1} > \left(\frac{i-1}{n-1}\right)^{1-(1-1/p)\gamma} \lambda^{(p)}(G_n)\, n^{2/p-1}.$$

On the other hand, by (7) we have

$$\begin{aligned} i-1 \geqslant \lambda^{(p)}(G_i)\, i^{2/p-1} &> \left(\frac{i-1}{n-1}\right)^{1-(1-1/p)\gamma} \lambda^{(p)}(G_n)\, n^{2/p-1} \\ &> \left(\frac{i-1}{n-1}\right)^{1-(1-1/p)\gamma} \left(An - \frac{R}{n}\right) > \left(\frac{i-1}{n-1}\right)^{1-(1-1/p)\gamma} A\,(n-1). \qquad (19) \end{aligned}$$

In the last derivation we use that $n \geqslant R/A$, which follows from

$$n > \frac{4R}{\gamma} A^{-p/(\gamma p-\gamma)} > \frac{R}{\gamma} > RA^{-1}.$$

From (19), we see that

$$\left(\frac{i}{n}\right)^{(1-1/p)\gamma} > \left(\frac{i-1}{n-1}\right)^{(1-1/p)\gamma} = \frac{i-1}{n-1}\left(\frac{n-1}{i-1}\right)^{1-(1-1/p)\gamma} > A,$$

and so,

$$i > nA^{p/(\gamma p-\gamma)} \geqslant \frac{4\,(R+1)\,p}{\gamma\,(p-1)} A^{-p/(\gamma p-\gamma)} A^{p/(\gamma p-\gamma)} > \frac{4R}{\gamma},$$

implying (17). Therefore, after removing the vertex u, Lemma 11, together with (5), implies that (18) holds as well. This completes the induction step and the proof of *(i)* and *(ii)*.

Finally, let $H := G_i$ and $k := v(H) = i$, where G_i is the last graph generated by $\mathcal{P}$. We shall prove the following three properties of H :

$$\delta(H) > (A-\gamma)\, k, \qquad (20)$$

$$k > A^{p/(\gamma p-\gamma)} n \qquad (21)$$

$$\lambda^{(p)}(H)\, k^{2/p-1} > Ak. \qquad (22)$$

Indeed, inequality (20) is obvious, as this is the loop exit condition. Also inequality (21) holds because of (17). Finally, note that

$$\begin{aligned} \lambda^{(p)}(H)\, k^{2/p-1} &> \left(\frac{k-1}{n-1}\right)^{1-(1-1/p)\gamma} \lambda^{(p)}(G_n)\, n^{2/p-1} \\ &> A\left(\frac{k-1}{n-1}\right)^{1-(1-1/p)\gamma} \left(n - \frac{R}{nA}\right). \end{aligned}$$

To prove (22), we shall show that

$$\left(\frac{k-1}{n-1}\right)^{1-(1-1/p)\gamma}\left(n-\frac{R}{nA}\right)>k, \tag{23}$$

which is equivalent to

$$1-\frac{R}{n^2A}>\frac{k}{n}\left(\frac{n-1}{k-1}\right)^{1-(1-1/p)\gamma}. \tag{24}$$

Assume the latter inequality fails, that is to say,

$$1-\frac{R}{n^2A}\leqslant\frac{k}{n}\left(\frac{n-1}{k-1}\right)^{1-(1-1/p)\gamma}=\frac{k\,(n-1)}{n\,(k-1)}\left(\frac{k-1}{n-1}\right)^{(1-1/p)\gamma}.$$

Using Bernoulli's inequality, we get

$$\begin{aligned}\left(1+\frac{n-k}{n\,(k-1)}\right)\left(\frac{k-1}{n-1}\right)^{(1-1/p)\gamma}&\leqslant\left(1+\frac{n-k}{n\,(k-1)}\right)\left(1-(1-1/p)\,\gamma\frac{n-k}{n-1}\right)\\&<1+\frac{n-k}{n\,(k-1)}-(1-1/p)\,\gamma\frac{n-k}{n-1}.\end{aligned}$$

After some rearrangement we obtain

$$\frac{(p-1)}{p}\gamma<\frac{n-1}{n\,(k-1)}+\frac{R\,(n-1)}{(n-k)\,n^2A}<\frac{1}{(k-1)}+\frac{R}{(n-k)\,nA}. \tag{25}$$

Now, obviously

$$\frac{1}{k-1}<\left(nA^{p/(\gamma p-\gamma)}-1\right)^{-1}<\left(\frac{4\,(R+1)\,p}{\gamma\,(p-1)}-1\right)^{-1}<\left(\frac{2p}{\gamma\,(p-1)}\right)^{-1}=\frac{\gamma\,(p-1)}{2p},$$

and also

$$\begin{aligned}\frac{R}{(n-k)\,nA}&<\frac{R}{An}<\frac{R}{A}\cdot\frac{\gamma\,(p-1)}{4\,(R+1)\,p}\cdot A^{p/(\gamma p-\gamma)}<\frac{1}{A}\cdot\frac{\gamma\,(p-1)}{4p}A^{1/\gamma}\\&<\frac{1}{A}\cdot\frac{\gamma\,(p-1)}{2p}A^2<\frac{\gamma\,(p-1)}{2p}.\end{aligned}$$

Therefore, (25) is a contradiction and (23) holds.

Hence the graph H has the required properties and Theorem 7 is proved. □

4.5 Proof of Theorem 5

The proof of Theorem 5 is based on the following nonspectral result, proved in [3].

Lemma 12. *Let $r\geqslant 2$ and G be graph a of order n. If G contains a K_{r+1} and $\delta(G)>(1-1/r-1/r^4)\,n$, then $\mathrm{js}_{r+1}(G)>n^{r-1}/r^{r+3}$.* □

The idea of Lemma 12 can be traced back to Erdős; its main advantage is that the bound on the jointsize can be deduced from two simpler conditions: presence of K_{r+1} and sufficient minimum degree. Although, these conditions may not hold in G, Theorem 7 guarantees that there is a large subgraph H of G for which the conditions do hold. Now, applying Lemma 12 to H, we obtain the desired bound on $\mathrm{js}_{r+1}(G)$.

Proof of Theorem 5. Let G be a graph of order n such that $\lambda^{(p)}(G) \geqslant \lambda^{(p)}(T_r(n))$ and assume that $G \neq T_r(n)$. Theorem 1 implies that G contains a K_{r+1}. Now, if

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{26}$$

then Lemma 12 implies that

$$\mathrm{js}_{r+1}(G) > \frac{n^{r-1}}{r^{r+3}} > \frac{n^{r-1}}{r^{r^6 p/(p-1)}},$$

completing the proof. Thus we shall assume that (26) fails. Then, letting

$$\gamma := 1/r^4, \quad A := 1 - 1/r, \quad R := r/4,$$

we see that $\delta(G) \leqslant (A - \gamma) n$, and, in view of (5), we also see that

$$\lambda^{(p)}(G) n^{2/p-1} \geqslant 2e(T_r(n))/n \geqslant (1 - 1/r) n - r/4n = An - R/n.$$

We want to apply Theorem 7, but to do so we have to ensure that $1 < p \leqslant 2$ and that

$$n > \frac{4(R+1)p}{\gamma(p-1)} A^{-p/(\gamma p - \gamma)} = \frac{(r+4) r^4 p}{p-1} \left(\frac{r}{r-1}\right)^{r^4 p/(p-1)}. \tag{27}$$

First we shall show that if (27) fails, then the proof is trivially completed. Assume that (27) fails. Since $K_{r+1} \subset G$, we have $\mathrm{js}_{r+1}(G) \geqslant 1$; hence the proof would be completed, if we can show that

$$1 > \frac{n^{r-1}}{r^{r^6 p/(p-1)}}. \tag{28}$$

Assume for a contradiction that (28) fails. Then

$$r^{r^6 p/(p-1)} < n^{r-1} < \left(\frac{(r+4) r^4 p}{p-1}\right)^{r-1} \left(\frac{r}{r-1}\right)^{r^4 (r-1) p/(p-1)}.$$

To simplify the right side, we use the obvious inequalities $r/(r-1) < r$ and

$$\left(\frac{(r+4) r^4 p}{p-1}\right)^{r-1} < \left(\frac{4r^5 p}{p-1}\right)^{r} < r^{5r} \left(\frac{4p}{p-1}\right)^{r},$$

thus getting

$$r^{r^6 p/(p-1)} < \left(\frac{4p}{p-1}\right)^{r} r^{r^4 (r-1) p/(p-1) + 5r}.$$

Since $r \geqslant 2$, and $4^x > e^x > x$ for $x > 0$, we see that

$$r^{8rp/(p-1)} > 4^{4rp/(p-1)} > (4p/(p-1))^r .$$

Hence,

$$\begin{aligned} r^6 p/(p-1) &< r^5 (r-1) p/(p-1) + 5r + 8rp/(p-1) \\ &= \left(r^6 - r^5 + 5r(p-1)/p + 8r\right) p/(p-1) \\ &< \left(r^6 - r^5 + 13r\right) p/(p-1), \end{aligned}$$

and after obvious cancellations, we find that $r^5 < 13r$, which is the desired contradiction. Therefore, we can assume that (27) holds.

Now, suppose that $1 < p \leqslant 2$. All parameter conditions of Theorem 7 are met, and so there is an induced subgraph $H \subset G$ of order

$$k > A^{p/(\gamma p - \gamma)} n = (1 - 1/r)^{r^4 p/(p-1)} n > \frac{n}{r^{r^4 p/(p-1)}}$$

such that $\lambda^{(p)}(H) > (1 - 1/r) k$ and $\delta(H) > (1 - 1/r - 1/r^4) k$. By Theorem 1, $K_{r+1} \subset H$; hence, Lemma 12 implies that

$$\mathrm{js}_{r+1}(H) > \frac{k^{r-1}}{r^{r+3}} > \left(\frac{n}{r^{r^4 p/(p-1)}}\right)^{r-1} \frac{1}{r^{r+3}} > \frac{n^{r-1}}{r^{r^4(r-1)p/(p-1)+r+3}} > \frac{n^{r-1}}{r^{r^6 p/(p-1)}}.$$

Since $\mathrm{js}_{r+1}(G) \geqslant \mathrm{js}_{r+1}(H)$, the proof of Theorem 5 is completed whenever $1 < p \leqslant 2$.

To finish the proof for all p, assume that $p > 2$. Then in view of (10) and (5) we have

$$\lambda^{(2)}(G) \geqslant \lambda^{(p)}(G) n^{2/p-1} \geqslant \lambda^{(p)}(T_r(n)) n^{2/p-1} > \left(1 - \frac{1}{r}\right) n - \frac{r}{4n}.$$

Applying Theorem 7 with $p = 2$, we get a subgraph $H \subset G$ of order

$$k > A^{2r^4} n = (1 - 1/r)^{2r^4} n > \frac{n}{r^{2r^4}}$$

such that $\lambda^{(p)}(H) > (1 - 1/r) k$ and $\delta(H) > (1 - 1/r - 1/r^4) k$. By Theorem 1, $K_{r+1} \subset H$; hence, Lemma 12 implies that

$$\mathrm{js}_{r+1}(H) > \frac{k^{r-1}}{r^{r+3}} > \left(\frac{n}{r^{2r^4}}\right)^{r-1} \frac{1}{r^{r+3}} > \frac{n^{r-1}}{r^{2r^4(r-1)+r+3}} > \frac{n^{r-1}}{r^{2r^5}} > \frac{n^{r-1}}{r^{r^6 p/(p-1)}}.$$

Since $\mathrm{js}_{r+1}(G) \geqslant \mathrm{js}_{r+1}(H)$, Theorem 5 is proved completely. $\square$

4.6 Proof of Theorem 6

For the proof of Theorem 6 we rely on Theorem 7 and on a non-spectral result, proved in [16], Theorem 6. To state it we first extend the definition of $K_r^+(t)$ as follows: given the integers $r \geqslant 2$, $s \geqslant 2$, and $t \geqslant 1$, let $K_r^+(s;t)$ be the complete r-partite graph with first $r-1$ parts of size s and the last part of size t, and with an edge added to its first part.

Theorem 13. *Let r, c, and n satisfy*

$$r \geqslant 2, \qquad 0 < c \leqslant r^{-(r+8)r}, \qquad \textit{and} \qquad \log n \geqslant 2/c.$$

If G is a graph of order n, with $K_{r+1} \subset G$ and $\delta(G) > (1-1/r-1/r^4)\, n$, then G contains a $K_r^+\left(\lfloor c\log n\rfloor ; \left\lceil n^{1-cr^3}\right\rceil\right)$. $\square$.

For the proof of Theorem 6 we shall need a corollary of this statement.

Corollary 14. *Let r, a, and n satisfy*

$$r \geqslant 2, \qquad 0 < a \leqslant r^{-(r+8)r}, \qquad \textit{and} \qquad \log n \geqslant 2/a.$$

If G is a graph of order n, with $K_{r+1} \subset G$ and $\delta(G) > (1-1/r-1/r^4)\, n$, then G contains a $K_r^+(\lfloor a\log n\rfloor)$.

Proof. There is not much to prove here. Indeed assume that r, a, and n are as required and let G be a graph of order n, with $K_{r+1} \subset G$ and $\delta(G) > (1-1/r-1/r^4)\, n$. By Theorem 13, G contains a $K_r^+\left(\lfloor a\log n\rfloor ; \left\lceil n^{1-ar^3}\right\rceil\right)$. First, note that

$$\log n \geqslant \frac{2}{a} \geqslant 2r^{(r+8)r} > 2;$$

hence $n^{1/2} > e$ and so, $n^{1/2} > \log n^{1/2}$. Now, with a lot to spare, we see that

$$n^{1-ar^3} > n^{1/2} > \frac{1}{2}\log n > r^{-(r+8)r}\log n \geqslant a\log n.$$

So

$$K_r^+(\lfloor a\log n\rfloor) \subset K_r^+\left(\lfloor a\log n\rfloor ; \left\lceil n^{1-ar^3}\right\rceil\right) \subset G,$$

completing the proof of Corollary 14. $\square$

Proof of Theorem 6. Let r, p, c, and n be as required, and let G be a graph of order n with $\lambda^{(p)}(G) > \lambda^{(p)}(T_r(n))$; thus, by Theorem 1, G contains a K_{r+1}. If

$$\delta(G) > \left(1-1/r-1/r^4\right) n, \tag{29}$$

then Corollary 14 implies that G contains a $K_r^+(\lfloor c\log n\rfloor)$, completing the proof. Thus, we shall assume that (29) fails. Then, letting

$$\gamma := 1/r^4, \quad A := 1 - 1/r, \quad R := r/4,$$

we see that $\delta(G) \leqslant (A-\gamma)n$, and, in view of (5), we also see that

$$\lambda^{(p)}(G)\, n^{2/p-1} \geqslant 2e(T_r(n))/n \geqslant (1-1/r)n - r/4n = An - R/n.$$

To apply Theorem 7, we have to ensure that $1 < p \leqslant 2$ and that

$$n > \frac{4(R+1)p}{\gamma(p-1)} A^{-p/(\gamma p-\gamma)} = \frac{r^4(r+4)p}{p-1}\left(\frac{r}{r-1}\right)^{r^4p/(p-1)}. \tag{30}$$

First from $\log n > 2p/(cp-c)$ we obtain

$$\log n \geqslant \frac{2p}{c(p-1)} \geqslant 2r^{(r+8)r}\frac{p}{p-1} \geqslant 2r^{(r+8)r}\frac{p}{p-1}\frac{\log r}{r^2} > r^5\frac{p}{p-1}\log r.$$

On the other hand,

$$\frac{r^4(r+4)p}{p-1}\left(\frac{r}{r-1}\right)^{r^4p/(p-1)} < r^5\left(\frac{4p}{p-1}\right)r^{r^4p/(p-1)} < r^{r^4p/(p-1)+5+8p/(p-1)} < r^{r^5p/(p-1)}.$$

Thus, we see that (30) holds.

Now, suppose that $1 < p \leqslant 2$. All parameter conditions of Theorem 7 are met, and so there is an induced subgraph $H \subset G$ of order

$$k > A^{p/(\gamma p-\gamma)}n = (1-1/r)^{r^4p/(p-1)}\, n > \frac{n}{r^{r^4p/(p-1)}},$$

with $\lambda^{(p)}(H) > (1-1/r)k$ and $\delta(H) > (1-1/r-1/r^4)k$. We shall prove that $\log k > 2r^{(r+8)r}$. Indeed,

$$\begin{aligned}\log k &> \log n - \log r^{r^4p/(p-1)} > \frac{1}{2}\log n + 2r^{(r+8)r}\frac{p}{p-1} - r^4\frac{p}{p-1}\log r \\ &> \frac{1}{2}\log n + \left(2r^{(r+8)r-2} - r^4\right)\frac{p}{p-1}\log r > \frac{1}{2}\log n > \frac{1}{c} > 2r^{(r+8)r}.\end{aligned}$$

Now, setting $a := 2c$, Corollary 14 implies that $K_r^+(\lfloor 2c\log k\rfloor) \subset H$. Since

$$2c\log k > c\log n,$$

Theorem 6 is proved if $1 < p \leqslant 2$. Now, assume that $p > 2$. Then, in view of (10) and (5), we have

$$\lambda^{(2)}(G) \geqslant \lambda^{(p)}(G)\, n^{2/p-1} \geqslant \lambda^{(p)}(T_r(n)) > \left(1-\frac{1}{r}\right)n - \frac{r}{4n},$$

and applying Theorem 7 with $p = 2$, we get a subgraph $H \subset G$ of order

$$k > A^{p/(\gamma p - \gamma)} n = (1 - 1/r)^{2r^4} n > \frac{n}{r^{2r^4}},$$

with $\lambda^{(p)}(H) > (1 - 1/r)k$ and $\delta(H) > (1 - 1/r - 1/r^4)k$. Again we see that $\log k > 2r^{(r+8)r}$, due to

$$\begin{aligned}\log k > \log n - \log r^{2r^4} &> \frac{1}{2}\log n + 2r^{(r+8)r}\frac{p}{p-1} - 2r^4 \log r \\ &> \frac{1}{2}\log n + \left(2r^{(r+8)r-2} - 2r^4\right)\log r > \frac{1}{2}\log n > \frac{1}{c} > 2r^{(r+8)r}.\end{aligned}$$

Hence, setting $a := 2c$, Corollary 14 implies that $K_r^+(\lfloor 2c \log k \rfloor) \subset H$. Since

$$2c \log k > c \log n,$$

Theorem 6 is proved for $p > 2$, as well. $\square$

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References

[1] B. Bollobás, *Extremal Graph Theory,* Academic Press Inc., London-New York, 1978, xx+488 pp.

[2] B. Bollobás. *Modern Graph Theory.* Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[3] B. Bollobás and V. Nikiforov. Joints in graphs. *Discrete Math.* , 308:9–19, 2008.

[4] P. Erdős. On the structure of linear graphs. *Israel J. Math.*, 1:156–160, 1963.

[5] P. Erdős. On the number of complete subgraphs and circuits contained in graphs. *Časopis Pěst. Mat.*, 94:290–296, 1969.

[6] P. Erdős and M. Simonovits. On a valence problem in extremal graph theory. *DisArete Math.*, 5:323-334, 1973.

[7] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities, 2nd edition*, Cambridge University Press, 1988, vi+324 pp.

[8] B. He, Y.L. Jin and X.D. Zhang. Sharp bounds for the signless Laplacian spectral radius in terms of clique number. *Linear Algebra Appl.*, 438:3851–3861, 2013.

[9] L. Kang, V. Nikiforov, and X.Yuan. The p-spectral radius of k-partite and k-chromatic uniform hypergraphs. arXiv:1402.0442.

[10] P. Keevash, J. Lenz, and D. Mubayi. Spectral extremal problems for hypergraphs. arXiv:1304.0050.

[11] T. Motzkin and E. Straus. Maxima for graphs and a new proof of a theorem of Turán. *Canad. J. Math.*, 17:533-540, 1965.

[12] V. Nikiforov. Some inequalities for the largest eigenvalue of a graph. *Combin. Probab. Comput.*, 11:179-189, 2002.

[13] V. Nikiforov. Bounds on graph eigenvalues II. *Linear Algebra Appl.*, 427:183-189, 2007.

[14] V. Nikiforov. A spectral condition for odd cycles. *Linear Algebra Appl.*, 428:1492-1498, 2008.

[15] V. Nikiforov. Spectral saturation: inverting the spectral Turán theorem. *Electronic J. Combin.*, 15:R33, 2009.

[16] V. Nikiforov. Turán's theorem inverted. *Discrete Math.*, 310:125-131,2010.

[17] V. Nikiforov. Some new results in extremal graph theory. In *Surveys in Combinatorics,* Cambridge University Press, 2011, pp. 141–181.

[18] V. Nikiforov. Some extremal problems for hereditary properties of graphs. *Electron. J. Combin.*, 21:P1.17, 2014.

[19] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra Appl.*, 457:455-535, 2014.

[20] V. Nikiforov. An analytic theory of extremal hypergraph problems. arXiv:1305.1073v2.

[21] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966),* pp. 279–319, Academic Press, New York, 1968.

[22] P. Turán. On an extremal problem in graph theory (in Hungarian). *Mat. és Fiz. Lapok*, 48:436-452, 1941.

[23] H. Wilf. Spectral bounds for the clique and independence numbers of graphs. *J. Combin. Theory Ser. B*, 40:113-117, 1986.

[24] A. A. Zykov. On some properties of linear complexes (in Russian). *Mat. Sbornik N.S.*, 24(66):163–188, 1949.