Note on the subgraph component polynomial

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Abstract

Tittmann, Averbouch and Makowsky [The enumeration of vertex induced subgraphs with respect to the number of components, *European J. Combin.* 32 (2011) 954–974] introduced the subgraph component polynomial $Q(G;x,y)$ of a graph G, which counts the number of connected components in vertex induced subgraphs. This polynomial encodes a large amount of combinatorial information about the underlying graph, such as the order, the size, and the independence number. We show that several other graph invariants, such as the connectivity and the number of cycles of length four in a regular bipartite graph are also determined by the subgraph component polynomial. Then, we prove that several well-known families of graphs are determined by the polynomial $Q(G;x,y)$. Moreover, we study the distinguishing power and find simple graphs which are not distinguished by the subgraph component polynomial but distinguished by the characteristic polynomial, the matching polynomial and the Tutte polynomial. These are partial answers to three open problems proposed by Tittmann et al.

1 Introduction

All graphs in this paper are simple and finite. Let $G = (V(G), E(G))$ be a graph. The *order* and the *size* of G are the number of vertices and the number of edges of G, respectively. As usual, the *complete graph*, the *cycle*, and the *path* on n vertices are denoted by K_n, C_n and P_n, respectively; the *complete bipartite graph* with part sizes m and n is $K_{m,n}$, and $K_{1,n}$ is the *star*. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{w : w \in V(G) \quad \textit{and} \quad \{v,w\} \in E(G)\}$, and the *closed neighborhood*

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$N[v] = N_G(v) \cup \{v\}$; if there is no ambiguity on G, we use $N(v)$ and $N[v]$, respectively. The degree $d(v)$ of vertex v is the number of edges incident with v. A vertex v is called a *pendant vertex* if $d(v) = 1$. The *minimum degree* of G is denoted by $\delta(G)$. Given a subset $U \subseteq V(G)$, we write $G[U]$ for the vertex induced subgraph of G by U. An *independent set* in G is a set of vertices no two of which are adjacent. The *independence number* $\alpha(G)$ is defined as the cardinality of a maximum independent set in graph G.

A graph G is *connected* if any two of its vertices are linked by a path. A *separating set* of a connected graph G is a set of vertices whose removal renders G disconnected. The *vertex connectivity* $c(G)$ (where G is not a complete graph) is the order of a minimal separating set. Obviously, $c(G) \leqslant \delta(G)$. A graph is called *k-connected* if its vertex connectivity is not less than k. This means that if a graph G is k-connected, then $G[V \backslash U]$ is connected for every subset $U \subseteq V(G)$ with $|U| < k$. The complete graph K_n has no separating set, but by convention $c(K_n) = n - 1$. A connected graph G is said to be *unicycle* if $|V(G)| = |E(G)|$. Observe that a unicycle can be regarded as a cycle with trees attached to its vertices. For more standard definitions, we refer the reader to the text of Diestel [6].

A number of different graph polynomials have been introduced and widely studied: such as the chromatic polynomial [19], the Tutte polynomial [4, 8, 22], the matching polynomial [7, 10, 25], the domination polynomial [1, 16] and the edge elimination polynomial [3, 21]. Recently, Tittmann, Averbouch and Makowsky [20] defined the *subgraph component polynomial* $Q(G; x, y)$, which counts the number of connected components in induced subgraphs. This polynomial $Q(G; x, y)$ has many interesting properties; for example, it is universal with respect to vertex elimination, and it determines the order, the size, the number of components, and the independence number of G. In addition, Tittmann et al. [20] found that the star $K_{1,n}$ is determined by $Q(G; x, y)$, and they posed a number of problems concerning this polynomial. In this paper, we are mainly concerned with three of those problems:

Problem 1. *Are there simple graphs distinguished by the characteristic polynomial $p(G; x)$, the matching polynomial $m(G; x)$, the bivariate chromatic polynomial $P(G; x, y)$ or the Tutte polynomial $T(G; x, y)$, which are not distinguished by $Q(G; x, y)$?*

Problem 2. *Find more graph invariants which are determined by $Q(G; x, y)$.*

Problem 3. *Find more classes of graphs which are determined by $Q(G; x, y)$.*

Our main findings are:

- We discover much more information contained in the polynomial $Q(G; x, y)$, e.g. the vertex connectivity $c(G)$ (Theorem 3.2), the regularity (Proposition 3.4), and the number of cycles of length four when G is a regular bipartite graph (Theorem 3.5). It is a well-known fact that the latter parameter is also determined by the Tutte polynomial [17].

- We find several classes of graphs which are determined by $Q(G; x, y)$, e.g. the path P_n, the cycle C_n, the tadpole graph $T_{m,n}$, the complete bipartite graph $K_{m,n}$, the friendship graph C_3^n, the book graph B_n, and the n-cube Q^n (Section 4).

- We find two simple graphs are distinguished by the characteristic polynomial, the matching polynomial and the Tutte polynomial, but are not distinguished by $Q(G;x,y)$ (Proposition 5.2).

2 The subgraph component polynomial

The polynomial $Q(G;x,y)$ was aroses from analyzing community structures in social networks by Tittmann et al., and has been further studied by Garijo et al. in [11, 12]. Its formal definition is the following.

Let $k\left(G\right)$ be the number of components of G, and let $q_{i,j}\left(G\right)$ be the number of vertex subsets $X \subseteq V$ with i vertices such that $G\left[X\right]$ has exactly j components, that is

$$q_{i,j}\left(G\right) = \left|\left\{X \subseteq V : |X| = i \wedge k\left(G\left[X\right]\right) = j\right\}\right|.$$

The *subgraph component polynomial* $Q(G;x,y)$ of G is defined as an ordinary generating function for these numbers:

$$Q\left(G;x,y\right) = \sum_{i=0}^{n}\sum_{j=0}^{n} q_{i,j}\left(G\right) x^i y^j.$$

If we sum over all the possible subsets of vertices, the definition can be rewritten in a slightly different way:

$$Q\left(G;x,y\right) = \sum_{X \subseteq V} x^{|X|} y^{k(G[X])}.$$

Tittmann et al. defined three types of vertex elimination operations on graphs:

- *Deletion*. $G - v$ denote the graph obtained by simply removing the vertex v.
- *Extraction*. $G - N[v]$ denote the graph obtained from G by removal of all vertices adjacent to v including v itself.
- *Contraction*. G/v denote the graph obtained from G by removal of v and insertion of edges between all pairs of non-adjacent neighbor vertices of v.

They showed that $Q(G;x,y)$ satisfies the following linear recurrence relation with respect to the preceding operations and is universal in this respect.

Proposition 2.1. **[20]** *Let $G = (V,E)$ be a graph and $v \in V$. Then the subgraph component polynomial satisfies the decomposition formula*

$$Q(G;x,y) = Q(G-v;x,y) + x(y-1)Q(G-N[v];x,y) + xQ(G/v;x,y).$$

The previous definition of $q_{i,j}(G)$ yields the following observation. If H is subgraph of G, then $q_{i,j}(H) \leqslant q_{i,j}(G)$.

Two graphs G and H are said to be *subgraph component equivalent*, or simply *Q-equivalent*, if $Q(G;x,y) = Q(H;x,y)$. A graph G is *Q-unique* if $Q(H;x,y) = Q(G;,x,y)$ implies that H is isomorphic to G for any graph H. Let $[x^iy^j]Q(G;x,y)$ denote the coefficient of x^iy^j in $Q(G;x,y)$, and let $deg_xQ(G;x,y)$ be the degree of $Q(G;x,y)$ with respect to the variable x.

3 Graph invariants determined by the subgraph component polynomial

In general, various aspects of combinatorial invariants/properties and numbers of a graph is stored in the coefficients of a specific graph polynomial, such as polynomials studied in [2, 18, 21]. In this section, we will study the coefficients of $Q(G; x, y)$.

Proposition 3.1. **[20]** *The following graph properties can be easily obtained from the subgraph polynomial:*

(1) The number of vertices:

$$|G| = \deg_x Q(G; x, y) = \log_2 Q(G; 1, 1) = [xy]\, Q(G; x, y)$$

(2) The number of edges:

$$|E(G)| = \left[x^2 y\right] Q(G; x, y)$$

(3) The number of components:

$$k(G) = \deg_y \left(\left[x^{|G|}\right] Q(G; x, y)\right)$$

(4) The number of independent sets of each size; in particular, the independence number:

$$\alpha(G) = \deg_y Q(G; x, y)$$

Since the order, the size and the number of components of a graph G is determined by its subgraph component polynomial $Q(G; x, y)$, it is clear that if G is a tree and H is Q-equivalent to G, then H is also a tree.

Theorem 3.2. *The vertex connectivity of a graph G is determined by its subgraph component polynomial.*

Proof. As it was shown by Tittmann et al. [20] that complete graphs were determined by the subgraph component polynomial, we just need to consider non-complete graphs here. Let $G = (V, E)$ be a graph of order n. Given a vertex subset S with cardinality s, then $k(G[V \backslash S]) \geqslant 2$ when S is a separating set of G. From the definition of the vertex connectivity $c(G)$, we have the following relation:

$$\begin{aligned} c(G) &= min\left\{s : \left[x^{n-s} y^j\right] Q(G; x, y) > 0 \wedge j \geqslant 2\right\} \\ &= n - max\left\{deg_x \left(\left[y^j\right] Q(G; x, y)\right) : j \geqslant 2\right\}. \end{aligned}$$

□

Remark 3.3. Theorem 3.2 can also be obtained as an application of Theorem 1 of [20].

Since the vertex connectivity is a lower bound for the minimum degree $\delta(G)$, and the order and the size of a graph G are determined by $Q(G;x,y)$ as well, then we can conclude that the regularity of graph G is also determined by $Q(G;x,y)$.

Proposition 3.4. *Let G be a k-regular graph. If H is Q-equivalent to G, then H is k-regular.*

Theorem 3.5. *Let G be a k-regular bipartite graph of order n. Then the number of subgraphs isomorphic to C_4 and the number of subgraphs isomorphic to P_4 are determined by $Q(G;x,y)$.*

Proof. There are three non-isomorphic subgraphs of G (paths of order 4, cycles of order 4 and complete bipartite subgraphs $K_{1,3}$), which contribute to $[x^4y]Q(G;x,y)$ (see Fig. 1).

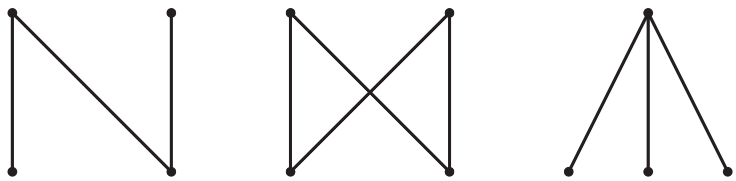

Figure 1: All possible non-isomorphic subgraphs of G with 4 vertices and 1 component.

Since G is k-regular, the contribution of $K_{1,3}$ is $n\binom{k}{3}$. It follows that

$$\left[x^4y\right] Q\left(G;x,y\right) = p + c + n\binom{k}{3} \tag{1}$$

where p is the number of paths of order 4 and c is the number of cycles of order 4.

For a bipartite graph G, a subgraph $A \subseteq G$ contribute to $[x^3y]Q(G;x,y)$ if and only if A is isomorphic to $K_{1,2}$. Let $K_{1,2}^+$ be a subgraph obtained from $K_{1,2}$ by adding a new vertex u which is adjacent to at least one of the two vertices x_1, x_2 in the partite set of cardinality two. Then $K_{1,2}^+$ is the P_4 if the new vertex u is adjacent to only one vertex of x_1, x_2, and $K_{1,2}^+$ is the C_4 if u is adjacent to both x_1 and x_2. We can count the number of $K_{1,2}^+$ (counted with multiplicity) in two ways. As G is k-regular, for every subgraph $K_{1,2}$, x_1 and x_2 have $2k-2$ neighbor vertices (counted with multiplicity) other than the vertex in the partite set of cardinality one. On the other hand, every P_4 is counted twice and every C_4 is counted 8 times in the counting process. Then the following relation holds:

$$(2k-2)\left[x^3y\right] Q\left(G;x,y\right) = 2p + 8c. \tag{2}$$

Equations (1) and (2) imply that p and c can be obtained from the coefficients of $Q(G;x,y)$. □

4 Graphs determined by their subgraph component polynomial

The Tutte polynomial is a well-studied graph polynomial. In [17], Mier and Noy found that wheels, squares of cycles, ladders, Möbius Ladders, complete multipartite graphs, and hypercubes are determined by their Tutte polynomials. In [20], it was shown that the star, complete graphs, and the class of empty graphs E_n are Q-unique. To find more classes of graphs which are determined by $Q(G;x,y)$ is an open problem posed in [20]. In this section we will show that paths, cycles, tadpole graphs, complete bipartite graphs, friendship graphs, book graphs and hypercubes are Q-unique.

4.1 Paths, cycles, tadpole graphs and complete bipartite graphs

The cycle C_n is a 2-connected unicycle with n vertices.

Proposition 4.1. *The cycle C_n is Q-unique.*

Proof. Let H be Q-equivalent to the cycle C_n. Then, by Proposition 3.1 and Theorem 3.2, H is a 2-connected graph, and $|V(H)| = |E(H)| = n$. Thus, it follows that $H \cong C_n$. $\square$

Theorem 4.2. *The path P_n is Q-unique.*

Proof. Let H be Q-equivalent to the path P_n. By Proposition 3.1, H is a tree of order n. Since removing a vertex in a path could increase the number of connected components by at most 1, $deg_y[x^{n-1}]Q(H;x,y) = deg_y[x^{n-1}]Q(P_n;x,y) \leqslant 2$.

Claim. There are at most two leaves in H.

Proof. Suppose that H has at least three leaves, say x, y, z. As H is a tree, there is a unique path P_y from x to y, and we denote the unique path from x to z by P_z. Let $meet_{yz}$ be the last vertex in the intersection of P_y and P_z. That is, the path from $meet_{yz}$ to y and the path from $meet_{yz}$ to z are disjoint except at $meet_{yz}$. Let $X = V(H)\backslash meet_{yz}$, then $|X| = n-1$ and $k(G[X]) \geqslant 3$ which leads to $deg_y[x^{n-1}]Q(H;x,y) \geqslant 3$, a contradiction. $\square$

As a tree has at least two leaves, then H has exactly two leaves. Therefore, H is the path P_n. $\square$

The (m,n)-tadpole graph $T_{m,n}$ is the graph obtained by joining a path P_m to a cycle C_n with a bridge.

Theorem 4.3. *The tadpole graph $T_{m,n}$ is Q-unique.*

Proof. Let H be Q-equivalent to $T_{m,n}$. Then, by Proposition 3.1 and Theorem 3.2, H is a 1-connected graph, and $|V(H)| = |E(H)| = m+n$. It follows that H is unicyclic. Similar to the proof of the Claim in Theorem 4.2, we can prove that H has at most one pendant vertex. Since H is connected but not 2-connected, then H can be constructed by joining one path to a cycle by a bridge. In addition, the order of the path is counted by $[x^{m+n-1}y^2]Q(H;x,y)$. This proves the theorem. $\square$

Theorem 4.4. *The complete bipartite graph $K_{m,n}$ is Q-unique.*

Proof. Let H be a graph with the same subgraph component polynomial as $K_{m,n}$, for some $m \geqslant n$. Then, by Proposition 3.1 and Theorem 3.2, H is a n-connected graph, $|H| = m + n$, $|E(H)| = mn$, and $\alpha(H) = m$. For every vertex subset A of cardinality $m+1$, $K_{m,n}[A]$ is connected and so $deg_y[x^{m+1}]Q(H;x,y) = deg_y[x^{m+1}]Q(K_{m,n};x,y) = 1$. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a maximum independent set of H, and $Y = V(H)\backslash X = \{y_1, y_2, \ldots, y_n\}$. We claim that every vertex in X is adjacent to every vertex in Y. If not, we can assume that there are two vertices $x_i \in X$, $y_j \in Y$ such that $\{x_i, y_j\} \notin E(H)$. Let $Z = X \cup \{y_j\}$. Then $|Z| = m+1$ and $k(H[Z]) \geqslant 2$ which leads to $deg_y[x^{m+1}]Q(H;x,y) \geqslant 2$, a contradiction. Therefore, $|E(K_{m,n})| = mn = |E(H)|$. Hence Y is an independent set of H, so $H \cong K_{m,n}$. $\square$

4.2 Friendship graphs and book graphs

The *friendship graph* C_3^n can be constructed by joining n copies of the cycle C_3 with a common vertex u, see Fig. 2. Wang et al. [23] found that the friendship graph C_3^n can be determined by the signless Laplacian spectrum.

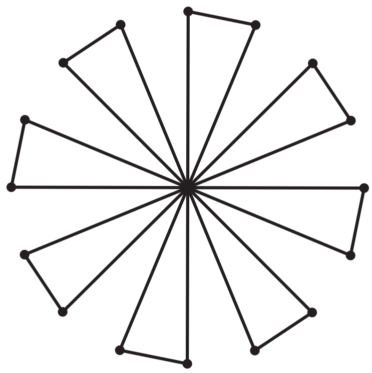

Figure 2: The friendship graph C_3^8.

Theorem 4.5. *The friendship graph C_3^n is Q-unique.*

Proof. Let H be Q-equivalent to C_3^n. Then, by Theorem 3.2, H is a 1-connected graph, $|V(H)| = 2n+1$, $|E(H)| = 3n$, and $\alpha(H) = n$. Since $[x^{2n}y^n]Q(H;x,y) = [x^{2n}y^n]Q(C_3^n;x,y) = 1$, then there is a subgraph of H with $2n$ vertices and n components. We denote these components by $H_1, H_2, \cdots, H_n$. As H is connected, there must be a vertex u in H such that u is connected to each of the above n components.

Claim. Each component H_i has exactly two vertices.

Proof. We first claim that $|H_i| \geqslant 2$ for each i. If there exists j such that $|H_j| = 1$, let $H_j = \{x\}$. Then $V(H)\backslash\{u, x\}$ induces a subgraph in H with $2n-1$ vertices and $n-1$

components; but such a subgraph does not exist in C_3^n. Since the total number of vertices in H is $2n+1$, we have $|H_i| = 2$ for each i. □

For each i, let $V(H_i) = \{x_i, y_i\}$. We first prove that neither x_i nor y_i is a pendant vertex. We suppose x_i is a pendant vertex in H, then y_i is the unique neighbor vertex of x_i since x_i, y_i constitute a component. Then $V(H)\backslash\{y_i\}$ induces a subgraph in H that has $2n$ vertices and 2 components; but such a subgraph does not exist in C_3^n. Therefore $\{x_i, y_i, u\}$ induces a cycle C_3 in H and we complete the proof. □

The *n-book graph* B_n can be constructed by joining n copies of the cycle graph C_4 with a common edge $\{u, v\}$, see Fig. 3.

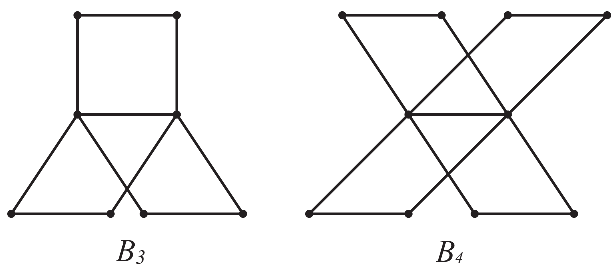

Figure 3: The book graphs B_3 and B_4.

Theorem 4.6. *The book graph B_n is Q-unique.*

Proof. Let H be a graph Q-equivalent to B_n. Then H is a 2-connected graph with $2n+2$ vertices, $3n+1$ edges, and $\alpha(H) = n+1$. According to the special structure of B_n, the following two equations hold:

$$[x^{2n}y^n]Q(H; x, y) = [x^{2n}y^n]Q(B_n; x, y) = 1, \tag{3}$$

$$[x^{n+1}y^{n+1}]Q(H; x, y) = [x^{n+1}y^{n+1}]Q(B_n; x, y) = 2. \tag{4}$$

It follows from Equation (3) that there are two vertices u and v in $V(H)$ such that $V(H) \setminus \{u, v\}$ induces a subgraph of H with $2n$ vertices and n components. We denote these components by $H_1, \cdots, H_n$. Similar to the proof of the Claim in Theorem 4.5 we can show H_i has exactly two vertices x_i, y_i, for each i. H is a 2-connected graph implies that for each i, there is one vertex in H_i adjacent to u and another vertex in H_i adjacent to v. Without loss of generality we let x_i adjacent to u and y_i adjacent to v. We suppose there is a vertex y_k in the component H_k such that $\{u, y_k\} \in H$. Since H has exactly $3n+1$ edges, then $\{u, v\} \notin H$ and $\{v, x_i\} \notin H$ for each i. Therefore, $[x^{n+1}y^{n+1}]Q(H; x, y) = 1$ which is a contradiction with Equation (4). Hence, $\{u, y_i\} \notin H_i$ for each i. Analogously, $\{v, x_i\} \notin H_i$ for each i. So $\{x_i, y_i, u, v\}$ constitute a cycle C_4 for each i and we have finished the proof. □

4.3 Hypercubes

The *n-cube* Q^n is the product graph of n copies of K_2. n-cubes have been extensively studied in computer science [5, 13, 15], and the following two lemmas for the case of vertex fault were shown in [9, 24].

Lemma 4.7. *[9, 24] Let F be a set of at most $2n-3$ vertices in Q^n. If $N(u) \subsetneq F$ for each vertex u in Q^n, then $Q^n - F$ is connected.*

Lemma 4.8. *[24] Let u be a vertex in Q^n. Then $c(Q^n - N[u]) = n-2$.*

Lemmas 4.7 and 4.8 imply that in order to get a subgraph of Q^n with more than two components, at least $2n-2$ vertices should be deleted.

Lemma 4.9. *Let F be a set of at most $2n-3$ vertices in Q^n. Then $k(Q^n - F) \leqslant 2$.*

From the proof of Theorem 6.2 in [17], we can conclude the following characterization of the n-cube.

Lemma 4.10. *A connected n-regular graph is isomorphic to the n-cube if and only if it has 2^n vertices, $2^{n-2}\binom{n}{2}$ subgraphs isomorphic to C_4 and no subgraph isomorphic to $K_{2,3}$.*

We are now in a position to prove the hypercube Q^n is Q-unique.

Theorem 4.11. *The n-cube Q^n is Q-unique for every $n \geqslant 2$.*

Proof. Let H be Q-equivalent to the n-cube Q^n. Then H is n-connected, n-regular, has 2^n vertices and $n2^{n-1}$ edges, $\alpha(H) = 2^{n-1}$.

Claim 1. For every $s \geqslant 1$, $deg_y[x^{2^n-s}]Q(H;x,y) \leqslant s$.

Proof. If graph G is Hamiltonian, then $k(G-A) \leqslant |A|$ for every subset $A \subseteq V(G)$. As well and long known, Q^n is Hamiltonian. Therefore, for every vertex subset $A \subseteq V(Q^n)$ of cardinality s, $k(Q^n[V\backslash A]) \leqslant s$. Then $deg_y[x^{2^n-s}]Q(H;x,y) = deg_y[x^{2^n-s}]Q(Q^n;x,y) \leqslant s$. ☐

It is evident that $[x^{2^{n-1}}y^{2^{n-1}}]Q(H;x,y) = [x^{2^{n-1}}y^{2^{n-1}}]Q(Q^n);x,y) = 2$. Let X, Y be two separating sets of H and $|X| = |Y| = 2^{n-1}$.

Claim 2. X and Y are disjoint.

Proof. If not, let us suppose $Z = X \cap Y$ and $|Z| = s \geqslant 1$. Let $X = \{x_1, \ldots, x_{2^{n-1}-s}, z_1, \ldots, z_s\}$, $Y = \{y_1, \ldots, y_{2^{n-1}-s}, z_1, \ldots, z_s\}$, $Z = \{z_1, \ldots, z_s\}$ and $U = X \cup Y$. Then $|U| = 2^n - s$ and $k(H[U]) \geqslant s+1$, which leads to $deg_y[x^{2^n-s}]Q(H;x,y) \geqslant s+1$, a contradiction to Claim 1. ☐

Claim 2 implies that H is a regular bipartite graph. Consequently, H has $2^{n-2}\binom{n}{2}$ subgraphs isomorphic to C_4. In view of Lemma 4.10, we just need to prove that H has no subgraph isomorphic to $K_{2,3}$. Let $H = (V_1, V_2, E)$ with $|V_1| = |V_2| = 2^{n-1}$ and $V_1 \cap V_2 = \varnothing$. For every pair of vertices a and b at distance 2 in H, a and b belong to the same partite set. Let $n(a, b)$ be the number of common neighbor vertices they have.

Claim 3. $n(a, b) \leqslant 2$.

Proof. We suppose on the contrary there is a pair of vertices a and b at distance 2 in, say V_1 such that $n(a, b) \geqslant 3$. Let c_1, c_2, c_3 be three vertices which are adjacent to both a and b. Then $c_1, c_2, c_3 \in V_2$. Since H is n-regular, we can let

$$\begin{aligned} N(a) &= \{c_1, c_2, c_3, a_1, a_2, \ldots, a_{n-3}\}, \\ N(b) &= \{c_1, c_2, c_3, b_1, b_2, \ldots, b_{n-3}\}, \end{aligned}$$

where a_i and b_i are neighbor vertices of a and b, respectively. Let $N(a, b) = N(a) \cup N(b)$. Then $|N(a, b)| \leqslant 2n - 3$ and $k(H[V \backslash N(a, b)]) \geqslant 3$, which is a contradiction with Lemma 4.9. □

It follows from Claim 3 that H has no subgraph isomorphism to $K_{2,3}$. □

5 Distinguishing power

The Tutte polynomial does not distinguish 1-connected graphs and the subgraph component polynomial does not distinguish graphs which differ only by the multiplicity of their edges. In this section we shall give a family of 2-connected simple graphs with the same Tutte polynomials but different subgraph component polynomials. Moreover, we find two Q-equivalent simple graphs which can be distinguished by the characteristic polynomial $p(G; x)$, the matching polynomial $m(G; x)$ and the Tutte polynomial $T(G; x, y)$. This gives an answer of a problem (Problem 1 in introduction) of Tittmann et al.

The join $G \vee H$ of two graphs $G = (V, E)$ and $H = (W, F)$ with $V \cap W = \varnothing$ is the graph obtained from $G \cup H$ by introducing edges from each vertex of G to each vertex of H [14].

The graph $K_1 \vee P_n$ is called the *fan graph* F_n. In the fan F_n, the vertices corresponding to the path P_n are labeled from v_1 to v_n, and the central vertex corresponding to K_1 is labeled as v_0. F^+_{n-1} arises from F_{n-1} by adding a new vertex v_n and two new edges $\{v_{n-2}, v_n\}, \{v_{n-1}, v_n\}$, see Fig. 4.

Proposition 5.1. *For* $n \geqslant 5$, $T(F_n; x, y) = T(F^+_{n-1}; x, y)$ *but* $Q(F_n; x, y) \neq Q(F^+_{n-1}; x, y)$.

Proof. Observe that F_n and F^+_{n-1} have the same dual graphs. Since $T(G; x, y) = T(G^*; y, x)$ for a planar graph G and its dual graph G^* [8], we have $T(F_n; x, y) = T(F^+_{n-1}; x, y)$. We can check that $F_n - \{v_n\} = F^+_{n-1} - \{v_n\} = F_{n-1}$, $F_n/v_n = F^+_{n-1}/v_n = F_{n-1}$, $F_n - N[v_n] = P_{n-2}$ and $F_{n-1} - N[v_n] = F_{n-3}$. Since paths are Q-unique, then $Q(P_{n-2}; x, y) \neq Q(F_{n-3}; x, y)$. Proposition 2.1 implies that $Q(F_n; x, y) \neq Q(F_{n-1}; x, y)$. □

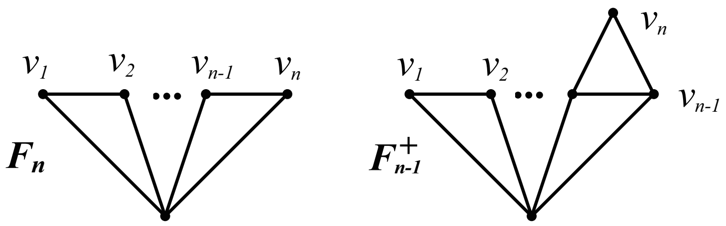

Figure 4: The graphs F_n, F^+_{n-1}.

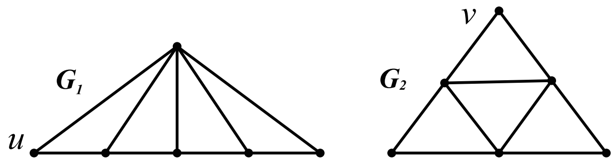

Figure 5: The graphs G_1, G_2.

Proposition 5.2. *For the graphs G_1 and G_2 illustrated in Fig. 5, we have*
(1) $Q(G_1; x, y) = Q(G_2; x, y)$.
(2) $p(G_1; x) \neq p(G_2; x)$.
(3) $m(G_1; x) \neq m(G_2; x)$.
(4) $T(G_1; x, y) \neq T(G_2; x, y)$.

Proof. We eliminate vertices u and v in graphs G_1 and G_2, respectively. It is not difficult to see that $G_1 - u = G_2 - v = F_4$, $G_1 - N[u] = G_2 - N[v] = P_3$ and $G_1/u = G_2/v = F_4$. Then $Q(G_1; x, y) = Q(G_2; x, y)$. Using the graph package for Maple we can obtain the characteristic polynomials, the matching polynomials, and the Tutte polynomials of G_1 and G_2 as following:

$$\begin{aligned}
p(G_1; x) &= x^6 - 9x^4 - 8x^3 + 9x^2 + 8x - 1, \\
p(G_2; x) &= x^6 - 9x^4 - 8x^3 + 9x^2 + 6x - 4, \\
m(G_1; x) &= x^6 - 9x^4 + 15x^2 - 2, \\
m(G_2; x) &= x^6 - 9x^4 + 15x^2 - 3,
\end{aligned}$$

$$\begin{aligned}
T(G_1; x, y) &= x^5 + 4x^4 + 4x^3y + 3x^2y^2 + 2xy^3 + y^4 + 6x^3 \\
&\quad + 9x^2y + 7xy^2 + 3y^3 + 4x^2 + 6xy + 3y^2 + x + y, \\
T(G_2; x, y) &= x^5 + 4x^4 + 4x^3y + 3x^2y^2 + 3xy^3 + y^4 + 6x^3 \\
&\quad + 9x^2y + 6xy^2 + 2y^3 + 4x^2 + 6xy + 3y^2 + x + y.
\end{aligned}$$

□

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