A Characteristic Factor for the 3-Term IP Roth Theorem in $\mathbb{Z}_3^{\mathbb{N}}$

Randall McCutcheon Alistair Windsor

Department of Mathematical Sciences
The University of Memphis
Tennessee, U.S.A.

rmcctchn@memphis.edu awindsor@memphis.edu

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Abstract

Let $\Omega = \bigoplus_{i=1}^{\infty} \mathbb{Z}_3$ and $e_i = (0, \dots, 0, 1, 0, \dots)$ where the 1 occurs in the i-th coordinate. Let $\mathscr{F} = \{\alpha \subset \mathbb{N} : \varnothing \neq \alpha, \alpha \text{ is finite}\}$. There is a natural inclusion of $\mathscr{F}$ into Ω where $\alpha \in \mathscr{F}$ is mapped to $e_\alpha = \sum_{i\in\alpha} e_i$. We give a new proof that if $E \subset \Omega$ with $d^*(E) > 0$ then there exist $\omega \in \Omega$ and $\alpha \in \mathscr{F}$ such that

$$\{\omega, \omega + e_\alpha, \omega + 2e_\alpha\} \subset E.$$

Our proof establishes that for the ergodic reformulation of the problem there is a characteristic factor that is a one step compact extension of the Kronecker factor.

1 Introduction

Let $\Omega = \bigoplus_{i=1}^{\infty} \mathbb{Z}_3$. Ω is an abelian group and hence amenable. Let $e_i = (0, \dots, 0, 1, 0, \dots)$ where the 1 occurs in the i-th coordinate. For a set S let

$$\mathscr{F}(S) = \{\gamma \subset S : \gamma \text{ is non-empty and finite}\}.$$

We will denote $\mathscr{F}(\mathbb{N})$ by simply $\mathscr{F}$ and endow it with the discrete topology. There is a natural inclusion of $\mathscr{F}$ into Ω where $\alpha \in \mathscr{F}$ is mapped to $e_\alpha = \sum_{i\in\alpha} e_i$.

The upper Banach density of a set $E \subset \Omega$, denoted $d^*(E)$, is defined as

$$d^*(E) = \sup_{(\Phi_n) \text{ Følner}} \limsup_{n\to\infty} \frac{|E \cap \Phi_n|}{|\Phi_n|}$$

where the supremum is taken over the set of Følner sequences, i.e. over the set of sequences of finite sets $(\Phi_n)_{n=1}^{\infty}$ in Ω such that for all $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{|(\omega + \Phi_n)\triangle\Phi_n|}{|\Phi_n|} = 0.$$

We give a new proof of the following theorem:

Theorem 1. *Let $E \subset \Omega$ with $d^*(E) > 0$. There exists $\omega \in \Omega$ and $\alpha \in \mathscr{F}$ such that*

$$\{\omega, \omega + e_\alpha, \omega + 2e_\alpha\} \subset E. \tag{1}$$

One can derive Theorem 1 from Furstenberg's correspondence principle and the following recurrence theorem.

Theorem 2. *Let $(T_\omega)_{\omega\in\Omega}$ be a measure-preserving action of Ω on a probability space $(X, \mathscr{A}, \mu)$. If $A \in \mathscr{A}$ with $\mu(A) > 0$ then there exists $\alpha \in \mathscr{F}$ such that*

$$\mu(A \cap T_{e_\alpha} A \cap T_{2e_\alpha} A) > 0.$$

Theorem 2 is not new; it follows from the Furstenberg-Katznelson IP-Szemerédi Theorem [FK85]. However, our proof identifies a characteristic factor that is a 1-step compact extension of the Kronecker factor of T_ω. Identifying a characteristic factor is suggestive of a first step in obtaining a decent quantitative result.

2 Ultrafilter Preliminaries

We will be working with the Stone-Čech compactification of $\mathscr{F}$, $\beta\mathscr{F}$. Since $\mathscr{F}$ is discrete we may identify points of $\beta\mathscr{F}$ with ultrafilters on $\mathscr{F}$. An ultrafilter $\mathfrak{p}$ on $\mathscr{F}$ is a subset $\mathfrak{p} \subset \mathscr{P}(\mathscr{F})$ that satisfies the following axioms

1. $\emptyset \notin \mathfrak{p}$,
2. If $A \subset B$ and $A \in \mathfrak{p}$ then $B \in \mathfrak{p}$.
3. If $A, B \in \mathfrak{p}$ then $A \cap B \in \mathfrak{p}$.
4. if $A \subset \mathscr{F}$ then either $A \in \mathfrak{p}$ or $A^c \in \mathfrak{p}$.

We identify $\alpha \in \mathscr{F}$ with the principal ultrafilter $\mathfrak{p}_\alpha = \{A \subset \mathscr{F} : \alpha \in A\}$. We can endow $\beta\mathscr{F}$ with the Stone topology, that is for $A \subset \mathscr{F}$, we define $\overline{A} = \{\mathfrak{p} \in \beta\mathscr{F} : A \in \mathfrak{p}\}$, and the set $\{\overline{A} : A \subset \mathscr{F}\}$ is a basis for the closed sets of $\beta\mathscr{F}$. Indeed, from the ultrafilter property $\overline{A}^c = \overline{A^c}$ and so this is also a basis for the open sets.

For $\alpha, \beta \in \mathscr{F}$ we write $\alpha < \beta$ if $\max\alpha < \min\beta$. When $\alpha < \beta$ we define $\alpha * \beta = \alpha \cup \beta$ and we leave $\alpha * \beta$ undefined otherwise. This makes $(\mathscr{F}, *)$ into an adequate partial semigroup in the sense of [BBH94] (see also [HM01]]). Briefly this means that $*$ maps a subset of

$\mathscr{F} \times \mathscr{F}$ to $\mathscr{F}$, is associative for all triples where defined, and for any $\alpha_1, \dots, \alpha_n \in \mathscr{F}$ there exists $\beta \in \mathscr{F}$ such that $\alpha_i * \beta$ is defined for all $1 \leqslant i \leqslant n$. Notice that if $\alpha < \beta$ then

$$e_{\alpha * \beta} = e_\alpha + e_\beta.$$

In the case of a semi-group $\mathscr{F}$ the operation extends to an operation on $\beta\mathscr{F}$ that makes $\beta\mathscr{F}$ a semi-group. In our case however $\mathscr{F}$ is only a partial semi-group and $*$ does not extend to all of $\beta\mathscr{F} \times \beta\mathscr{F}$. We extend the operation $*$ to a partial semi-group operation on $\beta\mathscr{F}$ using the same definition as for semi-groups

$$A \in \mathfrak{p} * \mathfrak{q} \iff \{p : \{q : p * q \in A\} \in \mathfrak{q}\} \in \mathfrak{p}$$

where $p * q \in A$ means both that $p * q$ is defined and $p * q \in A$. This extends the existing operation $*$ in the following sense: if $\alpha, \beta \in \mathscr{F}$ then $\mathfrak{p}_\alpha * \mathfrak{p}_\beta$ is defined if $\alpha * \beta$ is defined, and in this case $\mathfrak{p}_\alpha * \mathfrak{p}_\beta = \mathfrak{p}_{\alpha*\beta}$. If we let $\mathscr{F}_n = \mathscr{F}\big(\{n+1, n+2, \dots\}\big)$ then we can define $\delta\mathscr{F} = \cap_{n=1}^\infty \overline{\mathscr{F}_n} \subset \beta\mathscr{F}$. If we wish $\mathfrak{p} * \mathfrak{q}$ to be defined for all $\mathfrak{p} \in \beta\mathscr{F}$ then it turns out that we must have $\mathfrak{q} \in \delta\mathscr{F}$.

Given an ultrafilter $\mathfrak{p}$ on $\mathscr{F}$ we say that a sequence $(x_\alpha)_{\alpha\in\mathscr{F}}$ in a Banach space X $\mathfrak{p}$-converges to L in norm if for every $\epsilon > 0$ we have $\{\alpha \in \mathscr{F} : \|x_\alpha - L\| < \epsilon\} \in \mathfrak{p}$. Since the limit, if it exists, is unique we write $\mathfrak{p}\text{-}\lim x_\alpha = L$. It can be shown that every pre-compact sequence $\mathfrak{p}$-converges. Given an ultrafilter $\mathfrak{p}$ on $\mathscr{F}$ we say that a sequence $(x_\alpha)_{\alpha\in\mathscr{F}}$ in a Hilbert space H $\mathfrak{p}$-converges to L weakly if for every $x \in H$ we have $\mathfrak{p}\text{-}\lim\langle x_\alpha, x\rangle = \langle L, x\rangle$.

It may be shown that $(\delta\mathscr{F}, *)$ is a compact Hausdorff right topological semigroup. An idempotent ultrafilter $\mathfrak{p}$ is one that satisfies $\mathfrak{p} * \mathfrak{p} = \mathfrak{p}$. For $E \subset \mathscr{F}$ write

$$\overline{d}(E) = \limsup_{n\to\infty} \frac{|E \cap \mathscr{F}(\{1, \dots, n\})|}{2^n}.$$

An ultrafilter $\mathfrak{p} \in \delta\mathscr{F}$ is called essential if for every $A \in \mathfrak{p}$ we have $\bar{d}(A \cap \mathscr{F}_n) > 0$. By [BM12, Proposition 2.1] there exists an essential idempotent ultrafilter in $\delta\mathscr{F}$.

3 Factors and Joinings

Crucial to our proof will be the following theorem. It is a combination of [BM12, Theorem 3.3] and [BM12, Theorem 4.3]. In [BM12] Theorem 4.3 is derived from [BKMP88, Corollary 1]; in our case, where we need only consider $\bigoplus_{n=1}^\infty \mathbb{Z}_3$, one may derive the appropriate version of [BM12, Theorem 4.3] from [BM83, Proposition 2.7], which is a direct consequence of a result of Woodall in [Woo77]. This theorem is the only place where the fact that $\mathfrak{p}$ is an essential ultrafilter is used.

Theorem 3. *[Ber03, Corollary 4.6] Let $(T_\omega)_{\omega\in\Omega}$ be a measure-preserving action of Ω on a probability space $(X, \mathscr{A}, \mu)$. The action T extends to a unitary action on $L^2(X, \mathscr{A}, \mu)$. Let $\mathfrak{p}$ be an essential idempotent ultrafilter in $\delta\mathscr{F}$. Define an operator P on $L^2(X, \mathscr{A}, \mu)$ by*

$$\mathfrak{p}\text{-}\lim_{\alpha\in\mathscr{F}} T_{e_\alpha} f = Pf \ \textit{weakly}$$

for $f \in L^2(X, \mathscr{A}, \mu)$*. The operator* P *is the orthogonal projection onto the Kronecker factor*

$$K = \big\{ f \in L^2(X, \mathscr{A}, \mu) : \{T_\omega f : \omega \in \Omega\} \text{ is norm precompact}\big\}.$$

Now we observe that $K = L^2(X, \mathscr{K}, \mu)$ for some sub-σ-algebra $\mathscr{K} \subset \mathscr{A}$ with $\mathscr{K}$ being T_ω invariant (see [FK91, Lemma 3.1] or [McC00, Theorem 2.7]). Let $(\mu_x)_{x\in X}$ be the disintegration of μ over the Kronecker algebra $\mathscr{K}$, so that

$$\int f \, d\mu_x = E\big(f|\mathscr{K}\big)(x) = Pf(x) \text{ a.e.},$$

where $E\big(f|\mathscr{K}\big)$ is the conditional expectation of f over $\mathscr{K}$ and P is the orthogonal projection onto K. To see that these are equal we observe that if $f \in L^2(X, \mathscr{A}, \mu)$ with $E\big(f|\mathscr{K}\big) = 0$ then $\int f \, g \, d\mu = 0$ for all $g \in K$.

Using the measures μ_x on X we may define a family of norms on X indexed by $x \in X$ as follows

$$\|f\|_x = \Big(\int |f|^2 d\mu_x\Big)^{\frac{1}{2}}.$$

We will use $\|\cdot\|$ without any subscript to denote the appropriate L^2 norm.

Definition 4. A function $f \in L^2(X, \mathscr{A}, \mu)$ is called *almost periodic over* K, or AP over K for short, if for all $\epsilon > 0$ there exist $g_1, \ldots, g_k \in L^2(X, \mathscr{A}, \mu)$ such that for almost every $x \in X$ and every $\omega \in \Omega$ there exists $1 \leqslant i = i(x, \omega) \leqslant k$ such that

$$\|T_\omega f - g_i\|_x \leqslant \epsilon.$$

One may show that the bounded AP over K functions are dense in the AP over K functions, the constant functions are AP over K, sums and products of bounded AP over K functions are again bounded AP over K functions. Therefore, the closure of the set of AP over K functions is of the form $L^2(X, \mathscr{B}, \mu)$ for some σ-algebra $\mathscr{B} \subset \mathscr{A}$ with $\mathscr{B}$ being T_ω invariant (see [FK91, Lemma 3.1] or [McC00, Theorem 2.7]). One may also observe that any bounded $f \in K$ is AP over K and hence any $f \in K$ is $\mathscr{B}$-measurable, so that $\mathscr{K} \subset \mathscr{B}$. We observe that if $E(f|\mathscr{B}) = 0$ then $E(f|\mathscr{K}) = Pf = 0$.

Consider the relative product measure $\tilde{\mu}$ on $X \times X$ defined by

$$\tilde{\mu}(A) = \int (\mu_x \times \mu_x)(A) \, d\mu(x)$$

so that

$$\int f(x)g(y) \, d\tilde{\mu}(x, y) = \int P(f)(x)P(g)(x) \, d\mu(x).$$

This measure is invariant under $\tilde{T}_\omega = T_\omega \times T_\omega$ for every $\omega \in \Omega$. If $\mathfrak{p}$ is an essential idempotent ultrafilter in $\delta\mathscr{F}$ then by Theorem 3

$$\mathfrak{p}\text{-}\lim_\alpha \tilde{T}_{e_\alpha} h = \mathfrak{p}\text{-}\lim_\alpha \tilde{T}_{2e_\alpha} h = Qh,$$

where Q is the orthogonal projection onto the Kronecker factor of $\tilde{T}$.

Given $H \in L^2(X \times X, \mathscr{A} \otimes \mathscr{A}, \tilde{\mu})$, where $\mathscr{A} \otimes \mathscr{A}$ denotes the (completion of the) sigma algebra generated by the rectangles $\{A_1 \times A_2 : A_1, A_2 \in \mathscr{A}\}$, and $\phi \in L^\infty(X, \mathscr{A}, \mu)$ we may define

$$H \star \phi(x) = \int H(x,s)\phi(s)\, d\mu_x(s).$$

Lemma 5. *If $QH = H$ then for all $\phi \in L^\infty(X, \mathscr{A}, \mu)$ $H \star \phi$ is $\mathscr{B}$-measurable.*

Proof. Since Q is the projection onto the Kronecker factor of $(X \times X, \tilde{T}_\omega, \tilde{\mu})$ we have that H is expressible as a countable sum of eigenfunctions for $\tilde{T}_\omega$. Without loss of generality we may assume that H is an eigenfunction, i.e. that

$$\big(\tilde{T}_\omega H\big)(t,s) = H\big(T_\omega t, T_\omega s\big) = \lambda(\omega)\, H(t,s)$$

where $\lambda : \Omega \to S^1$ is a character. Notice that

$$\begin{aligned} \big(T_\omega(H \star \phi)\big)(x) &= (H \star \phi)(T_\omega x) \\ &= \int H(T_\omega x, s)\phi(s)\, d\mu_{T_\omega x}(s) \\ &= \int H(T_\omega x, T_\omega s)\phi(T_\omega s)\, d\mu_x(s) \\ &= \lambda(\omega) \int H(x,s) T_\omega \phi(s)\, d\mu_x(s) \\ &= \lambda(\omega)\, \big(H \star T_\omega \phi\big)(x). \end{aligned} \tag{2}$$

Now for almost every $y \in X$ we have $\phi \mapsto H \star \phi$ is a compact operator on $L^2(X, \mathscr{A}, \mu_y)$ and range(λ) is finite so (2) shows that $\{T_\omega(H \star \phi) : \omega \in \Omega\}$ is precompact in $L^2(X, \mathscr{A}, \mu_y)$. Hence for almost every $y \in X$ and every $\epsilon > 0$ there exists $M(y, \epsilon)$ such that $\big\{T_\omega(H \star \phi) : \omega \in \operatorname{span}\{e_1, \dots, e_{M(y,\epsilon)}\}\big\}$ is ϵ-dense in $\{T_\omega(H \star \phi) : \omega \in \Omega\}$ in $L^2(X, \mathscr{A}, \mu_y)$. Let $\epsilon > 0$ be arbitrary. Choose M_n sufficiently large that $M_n > M(y, \frac{1}{n})$ except for $y \in E_n$ where $E_n \in \mathscr{K}$ and $\mu(E_n) \leqslant \epsilon 2^{-n}$. Define

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n=1}^\infty E_n, \\ H \star \phi(x) & \text{otherwise.} \end{cases}$$

It can be easily shown that $\|f_\epsilon - H \star \phi\| < \|H\|_\infty \|\phi\|_\infty \epsilon$. Thus it suffices to show that f_ϵ is almost periodic over K. From the construction it is easy to observe that $\{0\} \cup \big\{T_\omega(H \star \phi) : \omega \in \operatorname{span}\{e_1, \dots, e_{M_n}\}\big\}$ is $\frac{1}{n}$-dense in $\{T_\omega f_\epsilon : \omega \in \Omega\}$ in $L^2(\mu_y)$ for almost every $y \in X$. Thus f_ϵ is almost periodic over K as required. Since $H \star \phi$ lies in the closure of the functions that are almost periodic over K, $H \star \phi$ is $\mathscr{B}$-measurable. □

4 Projection Results

We now give applications of our results on joinings to the projections of products.

Lemma 6. *Let* $\mathfrak{p}$ *be an essential idempotent in* $\delta\mathscr{F}$ *and let* $f \in L^\infty(X, \mathscr{A}, \mu)$. *If* $E(f|\mathscr{B}) = 0$ *then*

$$\mathfrak{p}\text{-}\lim_\alpha \|P(f\, T_{e_\alpha} f)\| = \mathfrak{p}\text{-}\lim_\alpha \|P(f\, T_{2e_\alpha} f)\| = 0.$$

Proof. Let $f, g \in L^\infty(X, \mathscr{A}, \mu)$. One has

$$\begin{aligned}
&\mathfrak{p}\text{-}\lim_\alpha \|P\big(g(T_{e_\alpha} f)\big)\|^2 \\
&\qquad = \mathfrak{p}\text{-}\lim_\alpha \int \Big(P\big(g(T_{e_\alpha} f)\big)(x)\Big)^2 \, d\mu(x) \\
&\qquad = \mathfrak{p}\text{-}\lim_\alpha \int \Big(\int g(t)\,(T_{e_\alpha} f)(t)\, d\mu_x(t)\Big)^2 \, d\mu(x) \\
&\qquad = \mathfrak{p}\text{-}\lim_\alpha \int \Big(\int g(t)\,(T_{e_\alpha} f)(t)\, d\mu_x(t)\Big) \\
&\qquad\qquad\qquad \Big(\int g(s)\,(T_{e_\alpha} f)(s)\, d\mu_x(s)\Big)\, d\mu(x) \\
&\qquad = \mathfrak{p}\text{-}\lim_\alpha \int (g \otimes g)\, \tilde{T}_{e_\alpha}(f \otimes f)\, d\tilde{\mu} \\
&\qquad = \int (g \otimes g)\, Q(f \otimes f)\, d\tilde{\mu},
\end{aligned} \tag{3}$$

and the same equality holds for $\mathfrak{p}\text{-}\lim_\alpha \|P\big(f(T_{2e_\alpha} f)\big)\|^2$. Clearly $Q(f \otimes f)(t,s) = Q(f \otimes f)(s,t)$ and $Q(f \otimes f)$ is (essentially) bounded. We thus have that $Q(f \otimes f)$ is a positive definite symmetric kernel in the sense of [FK91, Section 3.6]. By [FK91, equation (3.6)] $Q(f \otimes f)(t,s) = \sum_k \lambda_k(t)\phi_k(t)\phi_k(s)$ where λ_k is $\mathscr{K}$-measurable (so that $\lambda_k(s) = \lambda_k(t)$ for $\tilde{\mu}$-almost every (t,s)), and for almost every $y \in X$ $\{\phi_k\}$ is orthonormal in $L^2(\mu_y)$. One may then check that $Q(f \otimes f) \star \phi_k = \lambda_k \phi_k$, so by Lemma 5 $\lambda_k \phi_k$ is $\mathscr{B}$-measurable.

Applying (3) with $g = f$, we get

$$\begin{aligned}
\mathfrak{p}\text{-}\lim_\alpha \|P\big(f\,(T_{e_\alpha} f)\big)\|^2 &= \int (f \otimes f)\, Q(f \otimes f)\, d\tilde{\mu} \\
&= \int f(t)\, f(s)\big(\sum_k \lambda_k(t)\, \phi_k(t)\, \phi_k(s)\big)\, d\tilde{\mu} \\
&= \sum_k \int f(t)\, f(s)\, \lambda_k(t)\, \phi_k(t)\, \phi_k(s)\, d\tilde{\mu} \\
&= \sum_k \int P(f\, \lambda_k\, \phi_k) P(f\, \phi_k)\, d\mu.
\end{aligned}$$

However, since $\lambda_k \phi_k$ is $\mathscr{B}$-measurable we have

$$E(f\, \lambda_k\, \phi_k | \mathscr{B}) = \lambda_k\, \phi_k\, E(f|\mathscr{B}) = 0$$

and consequently $P(f\, \lambda_k\, \phi_k) = 0$. □

Lemma 7. *Let $\mathfrak{p}$ be an essential idempotent on $\mathscr{F}$. If $f, g \in L^\infty(X, \mathscr{A}, \mu)$ with either $E(f|\mathscr{B}) = 0$ or $E(g|\mathscr{B}) = 0$ then $\mathfrak{p}\text{-}\lim_\alpha T_{e_\alpha} f\, T_{2e_\alpha} g = 0$ weakly.*

As is common in proofs of this type, a version of the Van der Corput lemma is crucial. This version appears as [BM12, Theorem 3.5].

Lemma 8 (Van der Corput Lemma)**.** *Let $(x_\alpha)_{\alpha\in\mathscr{F}}$ be a bounded $\mathscr{F}$-sequence of vectors in a Hilbert space and let $\mathfrak{p} \in \delta\mathscr{F}$ be an idempotent. If $\mathfrak{p}\text{-}\lim_\alpha \mathfrak{p}\text{-}\lim_\beta \langle x_{\alpha*\beta}, x_\beta\rangle = 0$ then $\mathfrak{p}\text{-}\lim_\alpha x_\alpha = 0$ weakly.*

We now use this Van der Corput Lemma to prove Lemma 7.

Proof of Lemma 7. Let $x_\alpha = T_{e_\alpha} f T_{2e_\alpha} g$. Then

$$\begin{aligned}
&\mathfrak{p}\text{-}\lim_\beta \mathfrak{p}\text{-}\lim_\alpha \langle x_{\alpha*\beta}, x_\alpha\rangle \\
&\qquad = \mathfrak{p}\text{-}\lim_\beta \mathfrak{p}\text{-}\lim_\alpha \int T_{e_\alpha} T_{e_\beta} f\, T_{2e_\alpha} T_{2e_\beta} g\, T_{e_\alpha} f\, T_{2e_\alpha} g\, d\mu \\
&\qquad = \mathfrak{p}\text{-}\lim_\beta \mathfrak{p}\text{-}\lim_\alpha \int T_{e_\beta} f\, T_{e_\alpha} T_{2e_\beta} g\, f\, T_{e_\alpha} g\, d\mu \\
&\qquad = \mathfrak{p}\text{-}\lim_\beta \mathfrak{p}\text{-}\lim_\alpha \int \big(f\, T_{e_\beta} f\big)\, T_{e_\alpha}\big(g\, T_{2e_\beta} g\big)\, d\mu \\
&\qquad = \mathfrak{p}\text{-}\lim_\beta \int P\big(f\, T_{e_\beta} f\big)\, P\big(g\, T_{2e_\beta} g\big)\, d\mu \\
&\qquad \leqslant \mathfrak{p}\text{-}\lim_\beta \|P(f T_{e_\beta} f)\|\, \|P(g T_{2e_\beta} g)\| = 0
\end{aligned}$$

since either $\mathfrak{p}\text{-}\lim_\beta \|P(f T_{e_\beta} f)\| = 0$ or $\mathfrak{p}\text{-}\lim_\beta \|P(g T_{2e_\beta} g)\| = 0$ by Lemma 6. Hence by the conclusion of the Van der Corput lemma we obtain that

$$\mathfrak{p}\text{-}\lim_\alpha T_{e_\alpha} f\, T_{2e_\alpha} g = 0 \text{ weakly,}$$

as required. □

5 Proof of Theorem 2

Our goal is to show that for $A \in \mathscr{A}$ with $\mu(A) > 0$,

$$\mathfrak{p}\text{-}\lim_\alpha \mu(A \cap T_{e_\alpha} A \cap T_{2e_\alpha} A) > 0.$$

It is equivalent that for $A \in \mathscr{A}$ with $\mu(A) > 0$ the characteristic function $f = \mathbb{1}_A$ satistfies

$$\mathfrak{p}\text{-}\lim_\alpha \int f\; T_{e_\alpha} f\; T_{2e_\alpha} f\; d\mu > 0.$$

To show this we will decompose f into

$$f_1 = E(f|\mathscr{B}),$$
$$f_2 = f - E(f|\mathscr{B}).$$

Clearly $E(f_2|\mathscr{B}) = 0$. Expanding, we get

$$\begin{aligned}
&\int f\, T_{e_\alpha} f\, T_{2e_\alpha} f\, d\mu \\
&\quad = \int f\, T_{e_\alpha}\big(f_1 + f_2\big)\, T_{2e_\alpha}\big(f_1 + f_2\big)\, d\mu, \\
&\quad = \int f\, T_{e_\alpha} f_1\, T_{2e_\alpha} f_1\, d\mu + \int f\, T_{e_\alpha} f_1\, T_{2e_\alpha} f_2\, d\mu \\
&\qquad + \int f\, T_{e_\alpha} f_2\, T_{2e_\alpha} f_1\, d\mu + \int f\, T_{e_\alpha} f_2\, T_{2e_\alpha} f_2\, d\mu.
\end{aligned}$$

From Lemma 7 we have

$$\mathfrak{p}\text{-}\lim_\alpha \int f\, T_{e_\alpha} f_1\, T_{2e_\alpha} f_2\, d\mu = 0,$$
$$\mathfrak{p}\text{-}\lim_\alpha \int f\, T_{e_\alpha} f_2\, T_{2e_\alpha} f_1\, d\mu = 0, \text{ and}$$
$$\mathfrak{p}\text{-}\lim_\alpha \int f\, T_{e_\alpha} f_2\, T_{2e_\alpha} f_2\, d\mu = 0.$$

so the only term which contributes is

$$\mathfrak{p}\text{-}\lim_\alpha \int f\, T_{e_\alpha} f_1\, T_{2e_\alpha} f_1\, d\mu.$$

Proposition 9. *Let $f = \mathbb{1}_A$ for $A \in \mathscr{A}$ with $\mu(A) > 0$ and let $f_1 = E(f|\mathscr{B})$. Then*

$$\mathfrak{p}\text{-}\lim_\alpha \int f\, T_{e_\alpha} f_1\, T_{2e_\alpha} f_1\, d\mu > 0.$$

Proof. By the decomposition of measures $f_1(x) > 0$ for μ-a.e. $x \in A$. So for some $a > 0$, if we let

$$A' = \{x \in X : f_1(x)^2 > a\}$$

then there exist $b > 0$ and a set $B_1 \in \mathscr{K}$ with $\mu(B_1) = 5\xi > 0$ such that for all $y \in B_1$ we have $\mu_y(A') > b$.

Note that $\int f\, f_1\, f_1\, d\mu_y > ab$ for all $y \in B_1$.

Let $\epsilon = \frac{a}{36}$. Now f_1 is $\mathscr{B}$-measurable and thus is in the closure of the AP over K functions. Hence we may choose an almost periodic over K function ϕ_1 such that $\|f_1 - \phi_1\| < \epsilon\sqrt{\xi}$. This means that $\|f_1 - \phi_1\|_y < \epsilon$ for every $y \in X \setminus C_1$, where $\mu(C_1) < \xi$. Since $\|f_1 - \phi_1\|_y$ is a $\mathscr{K}$-measurable function of y we have that C_1 is $\mathscr{K}$-measurable. We let $B_2 = B_1 \setminus C_1$. We have $\mu(B_2) > 4\xi$ and $\|f_1 - \phi_1\|_y < \epsilon$ for $y \in B_2$.

Since ϕ_1 is AP over K there exist $g_1, \dots, g_M \in L^2(X, \mu)$ such that for a.e. $y \in X$ and all $\omega \in \Omega$ one has $i = i(\omega, y)$ such that $1 \leqslant i \leqslant M$ and

$$\|T_\omega \phi_1 - g_i\|_y < \epsilon.$$

We claim that $\mathfrak{p}\text{-}\lim_\alpha \int f\ T_{e_\alpha} f_1\ T_{2e_\alpha} f_1\ d\mu \geqslant \frac{ab\xi}{4M^2}$.

Let $E_1 \in \mathfrak{p}$ be arbitrary. It suffices to find a single $\alpha \in E_1$ for which

$$\int f T_{e_\alpha} f_1\ T_{2e_\alpha} f_1\ d\mu > \frac{ab\xi}{4M^2}.$$

Let $N = M^2 + 1$. Since $B_2 \in \mathscr{K}$ we have $\mathfrak{p}\text{-}\lim_\alpha T_{e_\alpha} \mathbb{1}_{B_2} = \mathbb{1}_{B_2}$ weakly. However since there is no loss of norm we must have $\mathfrak{p}\text{-}\lim_\alpha T_{e_\alpha} \mathbb{1}_{B_2} = \mathbb{1}_{B_2}$ in norm. Writing this in terms of measure we immediately obtain $\mathfrak{p}\text{-}\lim_\alpha \mu(B_2 \triangle T_{e_\alpha} B_2) = 0$, and hence there exists $E_2 \in \mathfrak{p}$ such that for all $\alpha \in E_2$,

$$\mu(B_2 \triangle T_{e_\alpha} B_2) < \frac{\xi}{2^N}. \tag{4}$$

The same holds true under T_{2e_α}, hence there exists $E_3 \in \mathfrak{p}$ such that for all $\alpha \in E_3$,

$$\mu(B_2 \triangle T_{2e_\alpha} B_2) < \frac{\xi}{2^N}. \tag{5}$$

Now let $E = E_1 \cap E_2 \cap E_3 \subset E_1$. Since $\mathfrak{p}\text{-}\lim_\alpha T_{e_\alpha} h = \mathfrak{p}\text{-}\lim_\alpha T_{e_\alpha} h = h$ in norm for all $h \in L^2(X, \mathscr{K}, \mu)$, we have

$$\mathfrak{p}\text{-}\lim_\alpha \|T_{e_\alpha} h - T_{2e_\alpha} h\| = 0.$$

Taking $h(y) = \|f_1 - T_{2e_\alpha} f_1\|_y$ we obtain that for all α,

$$\mathfrak{p}\text{-}\lim_\beta \int \left| \|f_1 - T_{2e_\alpha} f_1\|_{T_{e_\beta} y} - \|f_1 - T_{2e_\alpha} f_1\|_{T_{2e_\beta} y} \right|^2 d\mu(y) = 0.$$

Let

$$A_\alpha = \left\{ \beta \in E : \int \left| \|f_1 - T_{2e_\alpha} f_1\|_{T_{e_\beta} y} - \|f_1 - T_{2e_\alpha} f_1\|_{T_{2e_\beta} y} \right|^2 d\mu(y) < \frac{\epsilon^2 \xi}{2M^2} \right\}.$$

We have $A_\alpha \in \mathfrak{p}$ for all $\alpha \in \mathscr{F}$. We will choose $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ inductively such that for all $\alpha, \beta \in FU(\alpha_1, \dots, \alpha_N)$, where $FU(\alpha_1, \dots, \alpha_N)$ denotes all finite unions of sets from $\{\alpha_1, \dots, \alpha_N\}$, with $\alpha < \beta$ one has $\beta \in A_\alpha$ i.e.

$$\int \left| \|f_1 - T_{2e_\alpha} f_1\|_{T_{e_\beta} y} - \|f_1 - T_{2e_\alpha} f_1\|_{T_{2e_\beta} y} \right|^2 d\mu(y) < \frac{\epsilon^2 \xi}{2M^2}. \tag{6}$$

We let $\alpha_1 \in E_2$ be arbitrary. We have $A_{\alpha_1} \in \mathfrak{p}$. Since $\mathfrak{p}$ is idempotent we have $A_{\alpha_1} \in \mathfrak{p} * \mathfrak{p}$, so that $\{\gamma \in \mathscr{F} : \gamma^{-1} A_{\alpha_1} \in \mathfrak{p}\} \in \mathfrak{p}$. Intersecting this set with $A_{\alpha_1} \in \mathfrak{p}$ and $\mathscr{F}_{\max \alpha_1} \in \mathfrak{p}$ we get $\{\gamma \in \mathscr{F}_{\max\alpha_1} \cap A_{\alpha_1} : \gamma^{-1}A_{\alpha_1} \in \mathfrak{p}\} \in \mathfrak{p}$ and consequently $\{\gamma \in \mathscr{F}_{\max\alpha_1} \cap A_{\alpha_1} : \gamma^{-1}A_{\alpha_1} \in \mathfrak{p}\} \neq \varnothing$. We choose $\alpha_2 \in \{\gamma \in \mathscr{F}_{\max\alpha_1} \cap A_{\alpha_1} : \gamma^{-1}A_{\alpha_1} \in \mathfrak{p}\}$ so that $\alpha_1 < \alpha_2$, $\alpha_2 \in A_{\alpha_1}$, and $\alpha_2^{-1}A_{\alpha_1} \in \mathfrak{p}$. Our inductive hypothesis is that for all $\alpha, \beta \in FU(\alpha_1, \ldots, \alpha_n)$ with $\alpha < \beta$ we have $\beta \in A_\alpha$ and $\beta^{-1}A_\alpha \in \mathfrak{p}$. For conciseness we will write $F_n = FU(\alpha_1, \ldots, \alpha_n)$. By the inductive hypothesis

$$\bigcap_{\alpha \in F_n} A_\alpha \cap \bigcap_{\substack{\alpha,\beta \in F_n \\ \alpha < \beta}} \beta^{-1}A_\alpha \in \mathfrak{p}.$$

Since $\mathfrak{p}$ is an idempotent we have

$$\Big\{\gamma \in \mathscr{F} : \gamma^{-1}\big(\bigcap_{\alpha \in F_n} A_\alpha \cap \bigcap_{\substack{\alpha,\beta \in F_n \\ \alpha < \beta}} \beta^{-1}A_\alpha \in \mathfrak{p}\big)\Big\} \in \mathfrak{p}$$

intersecting we then have

$$\Big\{\gamma \in \mathscr{F}_{\max\alpha_n} \cap \bigcap_{\alpha \in F_n} A_\alpha \cap \bigcap_{\substack{\alpha,\beta \in F_n \\ \alpha < \beta}} \beta^{-1}A_\alpha \\ : \gamma^{-1}\big(\bigcap_{\alpha \in F_n} A_\alpha \cap \bigcap_{\substack{\alpha,\beta \in F_n \\ \alpha < \beta}} \beta^{-1}A_\alpha \in \mathfrak{p}\big)\Big\} \in \mathfrak{p}.$$

We choose α_{n+1} from this set. The reader should verify that all the conditions for the induction to continue are satisfied. We define $B_3 \in \mathscr{K}$ as

$$B_3 = B_2 \cap \bigcap_{\alpha \in F_N} \big(T_{e_\alpha}^{-1}B_2 \cap T_{2e_\alpha}^{-1}B_2\big).$$

Using (4) and (5) we have that $\mu(B_3) > \xi$. Also for all $y \in B_3$, $\alpha \in F_N$, $T_{e_\alpha}y \in B_2$ and $T_{2e_\alpha}y \in B_2$.

Since $N = M^2 + 1$ for all $y \in B_3$ there exists $\ell = \ell(y)$, $m = m(y)$ with $1 \leqslant \ell < m \leqslant N$ such that

$$i(e_{\alpha_\ell \cup \cdots \cup \alpha_N}, y) = i(e_{\alpha_m \cup \cdots \cup \alpha_N}, y) \tag{7}$$

and

$$i(2e_{\alpha_\ell \cup \cdots \cup \alpha_N}, y) = i(2e_{\alpha_m \cup \cdots \cup \alpha_N}, y). \tag{8}$$

We may divide B_3 into at most M^2 cells on which both ℓ and m are constant. At least one of these cells must have measure at least $\mu(B_3)/M^2$. Let $B_4 \subset B_3$ be such a cell. Now $\mu(B_4) > \frac{\xi}{M^2}$. For conciseness we write $\beta_j = \alpha_j \cup \cdots \cup \alpha_N$.

For $y \in B_4$ we have

$$\begin{aligned} &\|T_{e_{\beta_m}}\phi_1 - T_{e_{\beta_\ell}}\phi_1\|_y \\ &\qquad < \|T_{e_{\beta_m}}\phi_1 - g_{i(e_{\beta_m}, y)}\|_y + \|g_{i(e_{\beta_\ell}, y)} - T_{e_{\beta_\ell}}\phi_1\|_y < 2\epsilon. \end{aligned} \tag{9}$$

Similarly,

$$\|T_{2e_{\beta_m}}\phi_1 - T_{2e_{\beta_\ell}}\phi_1\|_y < 2\epsilon. \tag{10}$$

Since T_ω is measure-preserving, by (9) and (10) we have

$$\begin{aligned} \|\phi_1 - T_{e_{\alpha_\ell \cup \cdots \cup \alpha_{m-1}}}\phi_1\|_{T_{e_{\beta_m}}y} < 2\epsilon \text{ and} \\ \|\phi_1 - T_{2e_{\alpha_\ell \cup \cdots \cup \alpha_{m-1}}}\phi_1\|_{T_{2e_{\beta_m}}y} < 2\epsilon. \end{aligned} \tag{11}$$

Since $\beta_m, \beta_\ell \in F_N$ we have $T_{e_{\beta_m}}y, T_{e_{\beta_\ell}}y \in B_2$ and consequently

$$\|\phi_1 - f_1\|_{T_{e_{\beta_m}}y} < \epsilon \text{ and } \|\phi_1 - f_1\|_{T_{e_{\beta_\ell}}y} < \epsilon.$$

Now $\|T_{e_{\alpha_\ell \cup \cdots \cup \alpha_{m-1}}} f_1 - T_{e_{\alpha_\ell \cup \cdots \cup \alpha_{m-1}}}\phi_1\|_{T_{e_{\beta_m}}y} = \|f_1 - \phi_1\|_{T_{e_{\beta_\ell}}y} < \epsilon$. Write $\alpha = \alpha_\ell \cup \cdots \cup \alpha_{m-1}$ and $\beta = \beta_m$. Now by the triangle inequality

$$\|f_1 - T_{e_\alpha} f_1\|_{T_{e_\beta}y} < 4\epsilon. \tag{12}$$

Similarly we can conclude that

$$\|f_1 - T_{2e_\alpha} f_1\|_{T_{2e_\beta}y} < 4\epsilon. \tag{13}$$

Since $\alpha < \beta$, by (6) we have

$$\big| \|f_1 - T_{2e_\alpha} f_1\|_{T_{e_\beta}y} - \|f_1 - T_{2e_\alpha} f_1\|_{T_{2e_\beta}y} \big| < \epsilon \tag{14}$$

for all $y \in X \setminus C_2$, where $\mu(C_2) < \frac{\xi}{2M^2}$. Let $B_5 = B_4 \setminus C_2$. Clearly $\mu(B_5) > \frac{\xi}{2M^2}$. For all $y \in B_5$ we can combine (14) with (13) to get

$$\|f_1 - T_{2e_\alpha} f_1\|_{T_{e_\beta}y} < 5\epsilon. \tag{15}$$

Now

$$\begin{aligned} &\int f\, T_{e_\alpha} f_1 T_{2e_\alpha} f_1 \, d\mu_y \\ &\quad = \int f \big(f_1 + (T_{e_\alpha} f_1 - f_1)\big)\big(f_1 + (T_{2e_\alpha} f_1 - f_1)\big)\, d\mu_y \\ &\quad = \int f\, f_1\, f_1\, d\mu_y + \int f\, (T_{e_\alpha} f_1 - f_1)\, f_1\, d\mu_y + \int f\, f_1\, (T_{2e_\alpha} f_1 - f_1)\, d\mu_y \\ &\qquad\quad + \int f\, (T_{e_\alpha} f_1 - f_1)\, (T_{2e_\alpha} f_1 - f_1)\, d\mu_y. \end{aligned}$$

Using $|f| \leqslant 1$ and $|f_1| \leqslant 1$ together with (12) and (15) we obtain

$$\int f\, T_{e_\alpha} f_1 T_{2e_\alpha} f_1 \, d\mu_{T_{e_\beta}y} > (a - 18\epsilon)b > \frac{ab}{2}$$

for all $y \in B_5$. Since $\mu(B_5) > \frac{\xi}{2M^2}$ we have

$$\int f\, T_{e_\alpha} f_1\, T_{2e_\alpha} f_1 \, d\mu > \frac{ab\xi}{4M^2}$$

as claimed. □

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