Arithmetic properties of overcubic partition pairs

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Abstract

Let $\overline{b}(n)$ denote the number of overcubic partition pairs of n. In this paper, we establish two Ramanujan type congruences and several infinite families of congruences modulo 3 satisfied by $\overline{b}(n)$. For modulus 5, we obtain one Ramanujan type congruence and two congruence relations for $\overline{b}(n)$, from which some strange congruences are derived.

Keywords: overcubic partition pairs; theta function; congruence

1 Introduction

In a series of papers [4, 5, 6], Chan investigated congruence properties of cubic partition function $a(n)$ which is defined by

$$\frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \sum_{n=0}^{\infty} a(n)q^n. \tag{1}$$

Throughout this paper, we assume $|q| < 1$ and adopt the following customary notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}).$$

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After Chan's work, many analogous partition functions have been studied. Kim [10] studied its overpartiton analog in which the overcubic partition function $\overline{a}(n)$ was given by

$$\frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} \overline{a}(n)q^n. \tag{2}$$

It is worth mentioning that Hirschhorn [8] has given an elementary proof of the results satisfied by $\overline{a}(n)$ appeared in [10]. Later, Zhao and Zhong [14] established congruences modulo 5, 7 and 9 for the following partition function

$$\frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \sum_{n=0}^{\infty} b(n)q^n. \tag{3}$$

Kim [11] introduced two partition statistics to explain the congruences modulo 5 and 7 for $b(n)$. Since $b(n)$ counts a pair of cubic partitions, Kim [11] christened $b(n)$ the number of cubic partition pairs. Zhou [15] also found combinatorial interpretations for the congruences modulo 5 and 7 satisfied by $b(n)$. Recently, Kim [12] focused on studying congruence properties of $\overline{b}(n)$ whose generating function is

$$\frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \sum_{n=0}^{\infty} \overline{b}(n)q^n. \tag{4}$$

Similarly, Kim named $\overline{b}(n)$ as the number of overcubic partition pairs of n. Using arithmetic properties of quadratic forms and modular forms, Kim [12] derived the following two congruences

$$\overline{b}(8n+7) \equiv 0 \pmod{64}, \tag{5}$$

$$\overline{b}(9n+3) \equiv 0 \pmod{3}. \tag{6}$$

The paper is organized as follows. In Section 2 we introduce necessary notation and some preliminary results. In Section 3 we aim to establish two Ramanujan type congruences and several infinite families of congruences modulo 3 satisfied by $\overline{b}(n)$. We obtain some unexpected congruence results for $\overline{b}(n)$ with modulus 5 in Section 4.

2 Preliminaries

We first recall that Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n-1)/2}b^{n(n+1)/2}, \quad |ab|<1. \tag{7}$$

In Ramanujan's notation, the Jacobi triple product identity takes the shape

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{8}$$

Three special cases of $f(a,b)$ are defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{9}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{10}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{11}$$

where the above three product representations follows from (8).

We now list the necessary preliminary results in the following lemmas, which will be used in our later proofs.

Lemma 1.

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \tag{12}$$

$$\varphi^2(-q^2) = \varphi(q)\varphi(-q). \tag{13}$$

Proof. Applying (8), we have

$$\begin{aligned}\varphi(-q) &= (q;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty \\ &= \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.\end{aligned}$$

Multiplying (9) by (12), we immediately get (13).

□

Lemma 2.

$$\varphi(q) = \varphi(q^9) + 2qf(q^3,q^{15}), \tag{14}$$

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{15}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{16}$$

Proof. The detailed proofs can be found in [2, p. 49] and [2, p. 40] respectively. □

In Section 4, we will involve Ramanujan's congruence modulo 5 for partition function $p(n)$. It is well known that the generating function of $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

The congruence modulo 5 for $p(n)$ which we require is stated as follows.

Lemma 3. *For all $n \geqslant 0$,*

$$p(5n+4) \equiv 0 \pmod 5. \tag{17}$$

Proof. See [3, p. 31] for a proof. $\square$

An overpartition of n is a partition of n for which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. For convenience, define $\overline{p}(0) = 1$. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, the generating function for overpartitions satisfies

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{1}{\varphi(-q)}. \tag{18}$$

Recently, Chen and Xia [7] confirmed the following congruence first conjectured by Hirschhorn and Sellers [9]

$$\overline{p}(40n+35) \equiv 0 \pmod 5. \tag{19}$$

In [13], the author presented an alternative proof of (19) by firstly establishing the following congruence relation.

Lemma 4.

$$\sum_{n=0}^{\infty} \overline{p}(5n) q^n \equiv \varphi(-q)\varphi^2(-q^5) + q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\psi(q^2)\varphi(q) \pmod 5. \tag{20}$$

At the end of this section, we introduce the following congruence relations which will be frequently adopted throughout the paper without explicitly mentioning it.

Lemma 5. *For any prime p, we have*

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod p,$$
$$\varphi(-q)^p \equiv \varphi(-q^p) \pmod p.$$

Proof. By the binomial theorem, we have

$$(1-q)^p \equiv 1 - q^p \pmod p,$$

which yields the first congruence relation. The second congruence relation follows from the first congruence relation and the product representation for $\varphi(-q)$. $\square$

3 Congruences modulo 3 for $\overline{b}(n)$

In this section, we aim to establish two Ramanujan type congruences and several infinite families of congruences modulo 3 for $\overline{b}(n)$. We begin by rewriting the generating function for $\overline{b}(n)$ in the following form

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(n)q^n &= \frac{(q^4;q^4)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^2}\\ &= \varphi^2(q)\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^{12}}. \qquad (21)\end{aligned}$$

First we introduce two Ramanujan type congruences modulo 3 for $\overline{b}(n)$.

Theorem 6. *For all $n \geqslant 0$,*

$$\overline{b}(12n+10) \equiv 0 \pmod 3, \qquad (22)$$

$$\overline{b}(24n+16) \equiv 0 \pmod 3. \qquad (23)$$

Proof. Invoking (16), we can reformulate (21) as

$$\sum_{n=0}^{\infty}\overline{b}(n)q^n = \left(\varphi^2(q^2)+4q\psi^2(q^4)\right)\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^{12}}.$$

Choosing the terms for which the power of q is a multiple of 2, replacing q^2 by q yields

$$\sum_{n=0}^{\infty}\overline{b}(2n)q^n = \varphi^2(q)\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^{12}}, \qquad (24)$$

and

$$\sum_{n=0}^{\infty}\overline{b}(2n)q^n \equiv \left(\varphi(q^9)+2qf(q^3,q^{15})\right)^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^4} \pmod 3.$$

If we extract those terms whose power of q is congruent to 2 modulo 3, divide by q^2, and replace q^3 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{b}(6n+4)q^n \equiv f^2(q,q^5)\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} \pmod 3.$$

It is straightforward to check that

$$f(q,q^5) \equiv \frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} \pmod 3.$$

Combining the above two identities together, we have

$$\sum_{n=0}^{\infty}\overline{b}(6n+4)q^n \equiv (q^4;q^4)_\infty(q^{12};q^{12})_\infty \pmod 3. \qquad (25)$$

Equating the coefficients of q^{2n+1} and q^{4n+2} respectively, we deduce that for all $n \geqslant 0$,

$$\overline{b}(12n+10) \equiv \overline{b}(24n+16) \equiv 0 \pmod 3.$$

This completes the proof. □

As a consequence of (25), we obtain the following corollary.

Corollary 7. *We have*

$$\sum_{n=0}^{\infty} \overline{b}(24n+4)q^n \equiv (q;q)_\infty (q^3;q^3)_\infty \pmod{3}. \tag{26}$$

With the aid of (26), we get the following result.

Theorem 8. *For any prime* $p \geqslant 5$, $\left(\frac{-3}{p}\right) = -1$, *we have*

$$\sum_{n=0}^{\infty} \overline{b}(24pn+4p^2)q^n \equiv (q^p;q^p)_\infty (q^{3p};q^{3p})_\infty \pmod{3}. \tag{27}$$

Proof. Substituting (11) into (26), we obtain

$$\sum_{n=0}^{\infty} \overline{b}(24n+4)q^n \equiv \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m(3m+1)/2+3n(3n+1)/2} \pmod{3}. \tag{28}$$

We claim that if

$$\frac{m(3m+1)}{2} + \frac{3n(3n+1)}{2} \equiv \frac{p^2-1}{6} \pmod{p}, \tag{29}$$

there exist some k and l such that

$$\frac{m(3m+1)}{2} + \frac{3n(3n+1)}{2} = \frac{p^2-1}{6} + \frac{p^2(3k^2+k)}{2} + \frac{3p^2(3l^2+l)}{2}$$

and $(-1)^{m+n} = (-1)^{k+l}$.

First it follows from (29) that

$$(6m+1)^2 + 3(6n+1)^2 \equiv 0 \pmod{p},$$

which implies that $6m+1 \equiv 6n+1 \equiv 0 \pmod{p}$ since $\left(\frac{-3}{p}\right) = -1$.

Case 1. If $p \equiv 1 \pmod{6}$, then

$$m \equiv n \equiv \frac{p-1}{6} \pmod{p}.$$

Let $m = kp + (p-1)/6$ and $n = lp + (p-1)/6$, we have

$$\frac{m(3m+1)}{2} + \frac{3n(3n+1)}{2} = \frac{p^2-1}{6} + \frac{p^2(3k^2+k)}{2} + \frac{3p^2(3l^2+l)}{2}$$

and

$$(-1)^{m+n} = (-1)^{(k+l)p+(p-1)/3} = (-1)^{(k+l)p} = (-1)^{k+l}.$$

Case 2. If $p \equiv -1 \pmod 6$, then

$$m \equiv n \equiv \frac{-p-1}{6} \pmod p.$$

Let $m = -kp - \frac{p+1}{6}$ and $n = -lp - (p+1)/6$, we also have

$$\frac{m(3m+1)}{2} + \frac{3n(3n+1)}{2} = \frac{p^2-1}{6} + \frac{p^2(3k^2+k)}{2} + \frac{3p^2(3l^2+l)}{2}$$

and

$$(-1)^{m+n} = (-1)^{-(k+l)p-(p+1)/3} = (-1)^{(k+l)p} = (-1)^{k+l}.$$

Hence our claim holds.

If we extract those terms whose power of q is congruent to $(p^2-1)/6$ modulo p from (28), and employ the above analysis, we obtain

$$\sum_{n=0}^{\infty} \overline{b}(24(pn+(p^2-1)/6)+4)q^{pn+\frac{p^2-1}{6}} \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{\frac{p^2-1}{6}+\frac{p^2(3k^2+k)}{2}+\frac{3p^2(3l^2+l)}{2}} \pmod 3,$$

which can be simplified as

$$\sum_{n=0}^{\infty} \overline{b}(24pn+4p^2)q^n \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{p(3k^2+k)/2+3p(3l^2+l)/2} \pmod 3.$$

Applying (11), we have

$$\sum_{n=0}^{\infty} \overline{b}(24pn+4p^2)q^n \equiv (q^p;q^p)_\infty (q^{3p};q^{3p})_\infty \pmod 3,$$

which finishes the proof. □

From Theorem 8 and by induction, we obtain the following theorem.

Theorem 9. *For any prime $p \geqslant 5$, $\left(\frac{-3}{p}\right) = -1$, $\alpha \geqslant 1$, we have*

$$\sum_{n=0}^{\infty} \overline{b}(24p^{2\alpha-1}n+4p^{2\alpha})q^n \equiv (q^p;q^p)_\infty (q^{3p};q^{3p})_\infty \pmod 3. \tag{30}$$

Note that the right-hand side of (30), when expanded as a power series, contains only terms of the form q^{pm} for some m. Based on this fact, we deduce the following corollary.

Corollary 10. *For any prime $p \geqslant 5$, $\left(\frac{-3}{p}\right) = -1$, $\alpha \geqslant 1$, and all $n \geqslant 0$, we have*

$$\overline{b}(24p^{2\alpha}n + 24p^{2\alpha-1}i + 4p^{2\alpha}) \equiv 0 \pmod 3, \tag{31}$$

where $i = 1, 2, \ldots, p-1$.

To conclude this section, we present another two infinite families of congruences modulo 3 for $\overline{b}(n)$.

Theorem 11. *For $\alpha \geqslant 2$ and all $n \geqslant 0$,*

$$\overline{b}(3^{\alpha}(3n+2)) \equiv 0 \pmod 3, \tag{32}$$

$$\overline{b}(3^{\alpha}(4n+2)) \equiv 0 \pmod 3. \tag{33}$$

Proof. Putting (14) into (21), we find that

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n \equiv \left(\varphi(q^9) + 2qf(q^3, q^{15})\right)^2 \frac{(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^4} \pmod 3.$$

Collecting the terms whose power of q is a multiple of 3, replacing q^3 by q, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{b}(3n)q^n &\equiv \varphi^2(q^3)\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^4} \\
&\equiv \varphi^2(q^3)\frac{1}{\varphi(-q^2)^2} \\
&\equiv \varphi^2(q^3)\frac{\varphi(-q^2)}{\varphi(-q^6)} \\
&\equiv \varphi^2(q^3)\frac{\varphi(-q^{18}) - 2q^2 f(-q^6, -q^{30})}{\varphi(-q^6)} \pmod 3. \qquad (34)
\end{aligned}$$

It can be readily seen that no terms on the right-hand of (34) can have its power of q to be congruent to 1 modulo 3. Thus, equating the coefficients of q^{3n+1} yields

$$\overline{b}(9n+3) \equiv 0 \pmod 3,$$

which is due to Kim [12].

Extracting those terms with power of q being a multiple of 3 from (34), then replacing q^3 by q, we conclude that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{b}(9n)q^n &\equiv \frac{\varphi^2(q)\varphi(-q^6)}{\varphi(-q^2)} \\
&\equiv \varphi^2(q)\varphi^2(-q^2) \\
&\equiv \varphi(q^3)\varphi(-q) \qquad (35) \\
&\equiv \varphi(q^3)\left(\varphi(-q^9) - 2qf(-q^3, -q^{15})\right) \pmod 3. \qquad (36)
\end{aligned}$$

Furthermore, if we choose those terms in which the power of q is a multiple of 3 from (36), and replace q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{b}(27n)q^n \equiv \varphi(q)\varphi(-q^3) \pmod 3.$$

Combing together (35) with the above congruence relation, we find that

$$\sum_{n=0}^{\infty} \overline{b}(9n)q^n \equiv \sum_{n=0}^{\infty} \overline{b}(27n)(-q)^n \pmod 3,$$

from which we deduce that for $n \geqslant 0$,

$$\overline{b}(9n) \equiv (-1)^n \overline{b}(27n) \pmod 3. \tag{37}$$

For each term on the right-hand of (36), the power of q can not be congruent to 2 modulo 3, thus we immediately derive that for $n \geqslant 0$,

$$\overline{b}(27n + 18) \equiv 0 \pmod 3. \tag{38}$$

On the other hand, substituting (15) into (35), we see that

$$\sum_{n=0}^{\infty} \overline{b}(9n)q^n \equiv \big(\varphi(q^{12}) + 2q^3\psi(q^{24})\big) \times \big(\varphi(q^4) - 2q\psi(q^8)\big) \pmod 3.$$

It follows from the fact there exist no terms of the form q^{4n+2} in the above identity that

$$\overline{b}(36n + 18) \equiv 0 \pmod 3. \tag{39}$$

Based on (37), (38) and (39), and proceeding by induction on α, the desired results (32) and (33) follows immediately. □

4 Congruences modulo 5 for $\overline{b}(n)$

The goal of this section is devoted to proving the following unexpected results, from which we obtain some strange congruences modulo 5 for $\overline{b}(n)$.

Theorem 12. *For all $n \geqslant 0$, we have $\overline{b}(20n + 10) \equiv 0 \pmod 5$ and*

$$\overline{b}(40n) \equiv \overline{b}(80n) \pmod 5, \tag{40}$$

$$\overline{b}(20n) \equiv \overline{b}(100n) \pmod 5. \tag{41}$$

Proof. From (24), we have

$$\sum_{n=0}^{\infty} \overline{b}(2n)q^n = \frac{\varphi^2(q)}{\varphi^6(-q)} \equiv \frac{\varphi^2(q)\varphi^4(-q)}{\varphi^2(-q^5)} \pmod 5.$$

Thus,

$$\sum_{n=0}^{\infty} \overline{b}(2n)(-q)^n \equiv \frac{\varphi^2(-q)\varphi^4(q)}{\varphi^2(q^5)} \pmod 5. \tag{42}$$

Recall Ramanujan's identity [1, p. 28, Entry 1.6.2]

$$16q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 = (\varphi^2(q)-\varphi^2(q^5))(5\varphi^2(q^5)-\varphi^2(q)).$$

It follows that

$$-q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 \equiv \varphi^4(q)-\varphi^2(q)\varphi^2(q^5) \pmod 5, \tag{43}$$

and

$$\begin{aligned}
\varphi^2(-q)\varphi^4(q) &\equiv \varphi^2(-q)\left(\varphi^2(q)\varphi^2(q^5)-q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2\right)\\
&= \varphi^4(-q^2)\varphi^2(q^5)-q(q;q)_\infty^4(q^{10};q^{10})_\infty^2\\
&\equiv \frac{\varphi^2(q^5)\varphi(-q^{10})}{\varphi(-q^2)} - q\frac{(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q;q)_\infty} \pmod 5.
\end{aligned}$$

Invoking the above identity, we can rewrite (42) in the following form

$$\sum_{n=0}^\infty \overline{b}(2n)(-q)^n \equiv \frac{\varphi(-q^{10})}{\varphi(-q^2)} - \frac{(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{\varphi^2(q^5)}\sum_{n=0}^\infty p(n)q^{n+1} \pmod 5.$$

If we extract the terms whose power of q is divisible by 5, replace q^5 by q, we find that

$$\sum_{n=0}^\infty \overline{b}(10n)(-q)^n \equiv \varphi(-q^2)\sum_{n=0}^\infty \overline{p}(5n)q^{2n} - \frac{(q;q)_\infty(q^2;q^2)_\infty^2}{\varphi^2(q)}\sum_{n=0}^\infty p(5n+4)q^n.$$

Applying (17), we have

$$\sum_{n=0}^\infty \overline{b}(10n)(-q)^n \equiv \varphi(-q^2)\sum_{n=0}^\infty \overline{p}(5n)q^{2n} \pmod 5. \tag{44}$$

Note that the right-hand side of (44), when expanded as a power series, only contains terms of the form q^{2m} for some m. Thus, we get

$$\overline{b}(20n+10) \equiv 0 \pmod 5.$$

Selecting the terms for which the power of q is even from (44), replacing q^2 by q, we obtain

$$\sum_{n=0}^\infty \overline{b}(20n)q^n \equiv \varphi(-q)\sum_{n=0}^\infty \overline{p}(5n)q^n \pmod 5. \tag{45}$$

Employing (20) and (43), we deduce that

$$\begin{aligned}
\sum_{n=0}^\infty \overline{b}(20n)q^n &\equiv \varphi^2(-q)\varphi^2(-q^5) + q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\psi(q^2)\varphi^2(-q^2) &\quad (46)\\
&\equiv \varphi^4(-q) - q(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2 + q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\cdot\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\cdot\frac{(q^2;q^2)_\infty^4}{(q^4;q^4)_\infty^2}\\
&\equiv \varphi(-q^5)\sum_{n=0}^\infty \overline{p}(n)q^n \pmod 5. &\quad (47)
\end{aligned}$$

Applying (16) to (46), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(20n)q^n &\equiv \left(\varphi^2(q^2)-4q\psi^2(q^4)\right)\times\left(\varphi^2(q^{10})-4q^5\psi^2(q^{20})\right)\\ &+ q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\psi(q^2)\varphi^2(-q^2) \pmod 5.\end{aligned}$$

If we extract the terms with even power of q from the above identity and then replace q^2 by q, we find that

$$\sum_{n=0}^{\infty}\overline{b}(40n)q^n \equiv \varphi^2(q)\varphi^2(q^5)+q^3\psi^2(q^2)\psi^2(q^{10}) \pmod 5. \tag{48}$$

Using the same argument to (48) yields

$$\sum_{n=0}^{\infty}\overline{b}(80n)q^n \equiv \varphi^2(q)\varphi^2(q^5)+q^3\psi^2(q^2)\psi^2(q^{10}) \pmod 5. \tag{49}$$

Combining (48) with (49) and equating the coefficients of q^n, we obtain

$$\overline{b}(40n)\equiv\overline{b}(80n) \pmod 5.$$

It follows from (47) that

$$\sum_{n=0}^{\infty}\overline{b}(100n)q^n \equiv \varphi(-q)\sum_{n=0}^{\infty}\overline{p}(5n)q^n \pmod 5. \tag{50}$$

Combining the above identity and (45) together, we conclude that

$$\overline{b}(20n)\equiv\overline{b}(100n) \pmod 5.$$

This completes the proof. □

Employing (40) and (41), and by induction, we have the following corollary.

Corollary 13. *For all $k \geqslant 0$ and $n \geqslant 0$,*

$$\overline{b}(40\times 2^k n) \equiv \overline{b}(40n) \pmod 5, \tag{51}$$

$$\overline{b}(20\times 5^k n) \equiv \overline{b}(20n) \pmod 5. \tag{52}$$

Utilizing the above result and the known fact that

$$\overline{b}(20)\equiv 2 \pmod 5,\ \overline{b}(40)\equiv 4 \pmod 5,\ \overline{b}(380)\equiv 0 \pmod 5,$$

we obtain the following strange congruences modulo 5 for $\overline{b}(n)$.

Corollary 14. *For all $k \geqslant 0$,*

$$\overline{b}(20\cdot 5^k) \equiv 2 \pmod 5, \tag{53}$$

$$\overline{b}(40\cdot 2^k) \equiv 4 \pmod 5, \tag{54}$$

$$\overline{b}(380\cdot 5^k) \equiv 0 \pmod 5. \tag{55}$$

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