On Fence Patrolling by Mobile Agents*

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Abstract

Suppose that a fence needs to be protected (perpetually) by k mobile agents with maximum speeds $v_1, \ldots, v_k$ so that no point on the fence is left unattended for more than a given amount of time. The problem is to determine if this requirement can be met, and if so, to design a suitable patrolling schedule for the agents. Alternatively, one would like to find a schedule that minimizes the *idle time*, that is, the longest time interval during which some point is not visited by any agent. We revisit this problem, introduced by Czyzowicz et al. (2011), and discuss several strategies for the cases where the fence is an open and a closed curve, respectively.

In particular: (i) we disprove a conjecture by Czyzowicz et al. regarding the optimality of their algorithm $\mathcal{A}_2$ for unidirectional patrolling of a closed fence; (ii) we present a schedule with a lower idle time for patrolling an open fence, improving an earlier result of Kawamura and Kobayashi.

Keywords: Multi-agent patrolling, idle time, approximation algorithm.

1 Introduction

A set of k mobile agents with (possibly distinct) maximum speeds v_i $(i = 1, \ldots, k)$ are in charge of *patrolling* a given region of interest. Patrolling problems find applications

in the field of robotics where surveillance of a region is necessary. An interesting one-dimensional variant has been introduced by Czyzowicz et al. [7], where the agents move along a rectifiable Jordan curve representing a *fence*. The fence is either a *closed* curve (the boundary of a compact region in the plane), or an *open* curve (the boundary between two regions). For simplicity (and without loss of generality) it can be assumed that the open curve is a line segment and the closed curve is a circle. The movement of the agents over the time interval $[0, \infty)$ is described by a *patrolling schedule* (or *guarding schedule*), where the speed of the ith agent, a_i ($i = 1, \ldots, k$), may vary between zero and its maximum value v_i in any of the two moving directions (left or right).

Given a closed or open fence of length ℓ and maximum speeds $v_1, \ldots, v_k > 0$ of k agents, the goal is to find a patrolling schedule that minimizes the *idle time I*, defined as the longest time interval in $[0, \infty)$ during which a point on the fence remains unvisited, taken over all points. A straightforward volume argument [7] yields the lower bound $I \geqslant \ell / \sum_{i=1}^{k} v_i$ for an (open or closed) fence of length ℓ. A *patrolling algorithm* computes a patrolling schedule for a given fence and set of speeds $v_1, \ldots, v_k > 0$.

For an open fence (line segment), Czyzowicz et al. [7] proposed a simple partitioning strategy, algorithm $\mathcal{A}_1$, where each agent moves back and forth perpetually in a segment whose length is proportional with its speed. Specifically, for a segment of length ℓ and k agents with maximum speeds $v_1, \ldots, v_k$, algorithm $\mathcal{A}_1$ partitions the segment into k pieces of lengths $\ell v_i / \sum_{j=1}^{k} v_j$, and schedules the ith agent to patrol the ith interval with speed v_i. Algorithm $\mathcal{A}_1$ has been proved to be optimal for uniform speeds [7], i.e., when all maximum speeds are equal. Algorithm $\mathcal{A}_1$ achieves an idle time $2\ell / \sum_{i=1}^{k} v_i$ on a segment of length ℓ, and so $\mathcal{A}_1$ is a 2-approximation algorithm for the shortest idle time. It has been conjectured [7, Conjecture 1] that $\mathcal{A}_1$ is optimal for arbitrary speeds, however this was disproved by Kawamura and Kobayashi [10]: they selected speeds $v_1, \ldots, v_6$ and constructed a schedule for 6 agents that achieves an idle time of $\frac{41}{42} \left(2\ell / \sum_{i=1}^{k} v_i \right)$.

A patrolling algorithm $\mathcal{A}$ is *universal* if it can be executed with any number of agents k and any speed setting $v_1, \ldots, v_k > 0$ for the agents. For example, $\mathcal{A}_1$ above is universal, however certain algorithms (e.g., algorithm $\mathcal{A}_3$ in Section 3 or the algorithm in Section 4) can only be executed with certain speed settings or number of agents, i.e., they are not universal.

For the closed fence (circle), no universal algorithm has been proposed to be optimal. For uniform speeds (i.e., $v_1 = \ldots = v_k = v$), it is not difficult to see that placing the agents uniformly around the circle and letting them move in the same direction yields the shortest idle time. Indeed, the idle time in this case is $\ell/(kv) = \ell / \sum_{i=1}^{k} v_i$, matching the lower bound mentioned earlier.

For the variant in which all agents are *required* to move in the same direction along a circle of unit length (say clockwise), Czyzowicz et al. [7, Conjecture 2] conjectured that the following algorithm $\mathcal{A}_2$ always yields an optimal schedule. Label the agents so that $v_1 \geqslant v_2 \geqslant \ldots \geqslant v_k > 0$. Let r, $1 \leqslant r \leqslant k$, be an index such that $\max_{1 \leqslant i \leqslant k} iv_i = rv_r$. Place the agents at equal distances of $1/r$ around the circle, so that each moves clockwise at the same speed v_r. Discard the remaining agents, if any. Since all agents move in the same direction, we also refer to $\mathcal{A}_2$ as the "runners" algorithm. It achieves an idle time of $1/\max_{1 \leqslant i \leqslant k} iv_i$ [7, Theorem 2]. Observe that $\mathcal{A}_2$ is also universal.

Related problems. Multi-agent patrolling is a variation of the problem of multi-robot coverage [4, 5], studied extensively in the robotics community. A variety of models has been considered for patrolling, including deterministic and randomized, as well as centralized and distributed strategies, under various objectives [1, 9]. *Idleness*, as a measure of efficiency for a patrolling strategy, was introduced by Machado et al. [11] in a graph setting; see also the article by Chevaleyre [4].

The closed fence patrolling problem is reminiscent of the classical *lonely runners conjecture*, introduced by Wills [12] and Cusick [6], independently, in number theory and discrete geometry. Assume that k agents run clockwise along a circle of length 1, starting from the same point at time $t = 0$. They have distinct but constant speeds (the speeds cannot vary, unlike in the model considered in this paper). A runner is called *lonely* when he/she is at distance of at least $\frac{1}{k}$ from any other runner (along the circle). The conjecture asserts that each runner a_i is lonely at some time $t_i \in (0, \infty)$. The conjecture has only been confirmed for up to $k = 7$ runners [2, 3]. A recent survey [8] lists a few other related problems.

Notation and terminology. A *unit* circle is a circle of unit length. We parameterize a line segment and a circle of length ℓ by the interval $[0, \ell]$. A *schedule* of k agents consists of k functions $f_i : [0, \infty] \to [0, \ell]$, for $i = 1, \ldots, k$, where $f_i(t)$ is the position of agent i at time t. Each function f_i is continuous (for a closed fence, the endpoints of the interval $[0, \ell]$ are identified), it is piecewise differentiable, and its derivative (speed) is bounded by $|f_i'| \leqslant v_i$.

A schedule is called *periodic* with period $T > 0$ if $f_i(t) = f_i(t + T)$ for all $i = 1, \ldots, k$ and $t \geqslant 0$. The *idle time* I of a schedule is the maximum length of an open time interval (t_1, t_2) such that there is a point $x \in [0, \ell]$ where $f_i(t) \neq x$ for all $i = 1, \ldots, k$ and $t \in (t_1, t_2)$. Given a fence length ℓ, a fence type (closed or open), and maximum speeds $v_1, \ldots, v_k$, $\texttt{idle}(\mathcal{A})$ denotes the idle time of a schedule produced by algorithm $\mathcal{A}$ for these parameters.

We use *position-time diagrams* to plot the agent trajectories with respect to time. One axis represents the position $f_i(t)$ of the agents along the fence and the other axis represents time. In Fig. 1, for instance, the horizontal axis represents the position of the agents along the fence and the vertical axis represents time. In Fig. 2, however, the vertical axis represents the position and the vertical axis represents time. A schedule with idle time I is equivalent to a covering problem in such a diagram (see Fig. 1). For a straight-line (i.e., constant speed) trajectory between points (x_1, y_1) and (x_2, y_2) in the diagram, construct a shaded parallelogram with vertices, (x_1, y_1), $(x_1, y_1 + I)$, (x_2, y_2), $(x_2, y_2 + I)$, where I denotes the desired idle time and the shaded region represents the covered region. In particular, if an agent stays put in a time-interval, the parallelogram degenerates to a vertical segment. A schedule for the agents ensures idle time I if and only if the entire area of the diagram in the time interval $[I, \infty)$ is covered.

The efficiency of a patrolling algorithm $\mathcal{A}$ is measured by the ratio

$$\rho = \texttt{idle}(\mathcal{A})/\texttt{idle}(\mathcal{A}_1)$$

between the idle times of $\mathcal{A}$ and the partition-based algorithm $\mathcal{A}_1$. Lower values of ρ indicate better (more efficient) algorithms. Recall however that certain algorithms can only be executed with certain speed settings or number of agents.
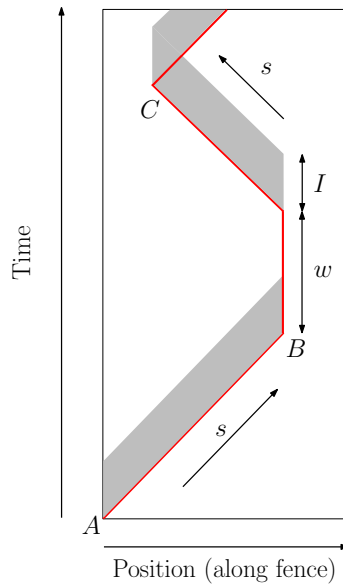
Figure 1: Agent moving with speed s from A to B, waiting at B for time w and then moving from B to C with speed s.

Our results.

1. Consider the unidirectional unit circle (where all agents are required to move in the same direction).

   (i) We disprove a conjecture by Czyzowicz et al. [7, Conjecture 2] regarding the optimality of algorithm $\mathcal{A}_2$. Specifically, we construct a schedule for 32 agents with harmonic speeds $v_i = 1/i$, $i = 1, \ldots, 32$, that has an idle time strictly less than 1. In contrast, algorithm $\mathcal{A}_2$ yields a unit idle time for harmonic speeds ($\texttt{idle}(\mathcal{A}_2) = 1$), hence it is suboptimal. See Theorem 1, Section 2.

   (ii) For every $\tau \in (0,1]$ and $t \geqslant \tau$, there exists a positive integer $k = k(t) \leqslant e^{4t/\tau^2}$ and a schedule for the system of k agents with harmonic speeds $v_i = 1/i$, $i = 1, \ldots, k$, that ensures an idle time at most τ during the time interval $[0, t]$. See Theorem 2, Section 2.

2. Consider the open fence patrolling. For every integer $x \geqslant 2$, there exist $k = 4x + 1$ agents with $\sum_{i=1}^{k} v_i = 16x + 1$ and a guarding schedule for a segment of length $25x/3$. Alternatively, for every integer $x \geqslant 2$ there exist $k = 4x + 1$ agents with suitable speeds $v_1, \ldots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\frac{48x+3}{50x} \frac{2}{\sum_{i=1}^{k} v_i}$. In particular, for every $\varepsilon > 0$, there exist k agents with suitable speeds $v_1, \ldots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\left(\frac{24}{25} + \varepsilon\right) \frac{2}{\sum_{i=1}^{k} v_i}$. This improves the previous bound of $\frac{41}{42} \frac{2}{\sum_{i=1}^{k} v_i}$ by Kawamura and Kobayashi [10]. See Theorem 3, Section 4.

3. Consider the bidirectional unit circle.

   (i) For every $k \geqslant 4$, there exist maximum speeds $v_1, \ldots, v_k > 0$ and a patrolling algorithm $\mathcal{A}_3$ with a shorter idle time than that achieved by both $\mathcal{A}_1$ and $\mathcal{A}_2$. In

particular, for large k, the idle time of $\mathcal{A}_3$ with these speeds is about $2/3$ of that achieved by $\mathcal{A}_1$ and $\mathcal{A}_2$. See Proposition 1, Section 3.

(ii) For every $k \geqslant 2$, there exist maximum speeds $v_1, \ldots, v_k > 0$ and an optimal schedule for patrolling the circle that does not use up to $k - 1$ of the agents $a_2, \ldots, a_k$. In contrast, for a segment, any optimal schedule must use all agents. See Proposition 2, Section 3.

(iii) There exist settings in which if all k agents are used by a patrolling algorithm, then some agent(s) need overtake (pass) other agent(s). This partially answers a question left open by Czyzowicz et al. [7, Section 3]. See the remark at the end of Section 3.

2 Unidirectional Circle Patrolling

A counterexample for the optimality of algorithm $\mathcal{A}_2$. We show that algorithm $\mathcal{A}_2$ by Czyzowicz et al. [7] for unidirectional circle patrolling is not always optimal. We consider agents with *harmonic speeds* $v_i = 1/i$, $i \in \mathbb{N}$. Obviously, for this setting we have $\mathtt{idle}(\mathcal{A}_2) = 1$, which is already achieved by the agent a_1 with the highest (here unit) speed. We design a periodic schedule (patrolling algorithm) for $k = 32$ agents with idle time $I < 1$. In this schedule, agent a_1 moves continuously with unit speed, and it remains to schedule agents $a_2, \ldots, a_{32}$ such that every point is visited at least one more time in the unit length open time interval between two consecutive visits of a_1. We start with a weaker claim, for *closed* intervals but using only 6 agents.

Lemma 1. *Consider the unit circle, where all agents are required to move in the same direction. For $k = 6$ agents of harmonic speeds $v_i = 1/i$, $i = 1, \ldots, 6$, there is a schedule where agent a_1 moves continuously with speed 1, and every point on the circle is visited by some other agent in every closed unit time interval between two consecutive visits of a_1.*
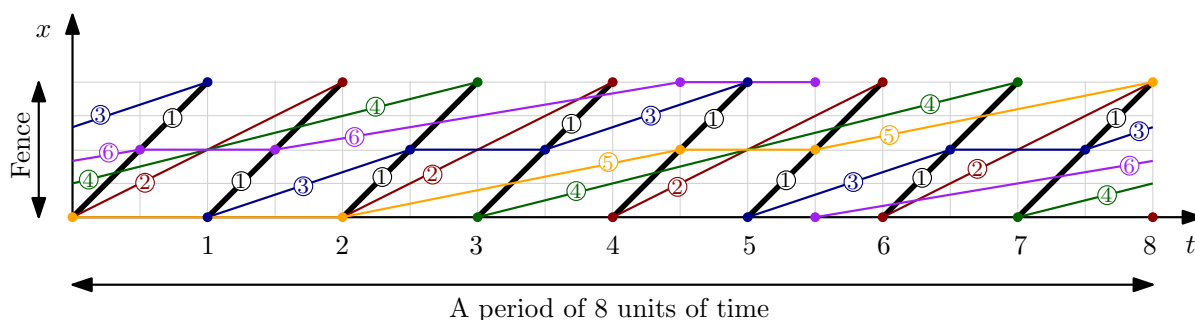

Figure 2: A periodic schedule of 6 agents of speeds $1/i$, $i = 1, \ldots, 6$, on a unit circle with period 8. Agent a_1 moves continuously with speed 1. Each point is visited by one of the agents a_2, a_3, a_4, a_5, a_6 between any two consecutive visits of agent a_1.

Proof. Our proof is constructive. We construct a periodic schedule for the 6 agents with period 8; refer to Fig. 2. Agents a_1, a_2 and a_4 continuously move with maximum speed,

while agents a_3, a_5 and a_6 each stop at certain times in their movements. Their schedule in one period $t \in [0, 8]$ is given by the following piecewise linear functions.

$$f_1(t) = t \mod 1, \qquad f_2(t) = t/2 \mod 1, \qquad f_4(t) = (t-3)/4 \mod 1.$$

$$f_3(t) = \begin{cases} (t-1)/3 \mod 1 & \text{for } t \in [0, 2.5] \cup [7.5, 8] \\ 0.5 & \text{for } t \in [2.5, 3.5] \cup [6.5, 7.5] \\ (t-2)/3 \mod 1 & \text{for } t \in [3.5, 6.5] \\ t/3 \mod 1 & \text{for } t \in [7.5, 8]. \end{cases}$$

$$f_5(t) = \begin{cases} 0 & \text{for } t \in [0, 2] \\ (t-2)/5 \mod 1 & \text{for } t \in [2, 4.5] \\ 0.5 & \text{for } t \in [4.5, 5.5] \\ (t-3)/5 \mod 1 & \text{for } t \in [5.5, 8]. \end{cases}$$

$$f_6(t) = \begin{cases} (t-3.5)/6 \mod 1 & \text{for } t \in [0, 0.5] \\ 0.5 & \text{for } t \in [0.5, 1.5] \\ (t-4.5)/6 \mod 1 & \text{for } t \in [1.5, 4.5] \\ 1 & \text{for } t \in [4.5, 5.5] \\ (t-5.5)/6 \mod 1 & \text{for } t \in [5.5, 8]. \end{cases}$$

$\square$

Theorem 1. *Consider the unit circle, where all agents are required to move in the same direction. For 32 agents of harmonic speeds $v_i = 1/i$, $i = 1, \ldots, 32$, there is a periodic schedule with idle time strictly less than 1.*

Proof. Agents $a_1, \ldots, a_6$ follow the periodic schedule described in Lemma 1. A time-position pair $(t, x) \in [0, 8) \times [0, 1)$ is a *critical point* in the time-position diagram if (1) point x on the fence is *not* visited by any agent in the open time interval $(t, t+1)$, or (2) point x is only visited by an agent stationary at x during the time interval $(t, t+1)$. To establish the theorem, we need to take care of the critical points. The problem with critical points of type (2) is that some other points in the neighborhood of x will not be visited for some time interval in $(t, t+1)$ whose duration can be arbitrarily close to 1, even though the point x itself is covered by the stationary agent. There are exactly 12 critical points in the schedule in Fig. 2. Specifically, these points are $(j, 0)$ for $j = 0, 1, \ldots, 7$; and $(j + \frac{1}{2}, \frac{1}{2})$ for $j = 1, 3, 5, 7$.

For each critical point (t, x), we assign one, two, or four agents such that they jointly traverse a small neighborhood of the critical point in each period in the periodic schedule.

We schedule agents a_7 and a_8 to move continuously with speed $1/8$, as follows.

$$f_7(t) = \frac{1}{8}\left(t - \frac{1}{3}\right) \mod 1, \qquad f_8(t) = \frac{1}{8}\left(t - \frac{7}{3}\right) \mod 1.$$

Agent a_7 traverses the unit intervals of the critical points $(0, 0)$ and $(3 + \frac{1}{2}, \frac{1}{2})$; and agent a_8 traverses the unit intervals of the critical points $(2, 0)$ and $(5 + \frac{1}{2}, \frac{1}{2})$. We are left with 8 critical points, which will be taken care of by agents $a_9, \ldots, a_{32}$.

Agents $a_9, \ldots, a_{16}$ are scheduled to move with constant speed $1/16$. These 8 agents form 4 pairs, where each pair is responsible to visit the neighborhood of a critical point

in each period of length 8 (each agent in a pair returns to the same critical point after 16 units of time). Finally, agents $a_{17}, \ldots, a_{32}$ move with constant speed $1/32$. These 16 agents form 4 quadruples, where each quadruple is responsible to visit the neighborhood of a critical point in each period of 8 (each agent in a quadruple returns after 32 units of time).

This schedule ensures that every point on the fence within a small neighborhood of the 12 critical points is visited by some agent within every time interval of length $1 - \varepsilon$, where $\varepsilon > 0$ is a sufficiently small constant. Apart from these neighborhoods, the first 6 agents already visit every point within every time interval of length $1 - \varepsilon$ provided that $\varepsilon > 0$ is sufficiently small. $\qquad\square$

Remark. In Theorem 1, we required that all agents move in the same direction (clockwise) along the unit circle, but we allowed agents to stop (i.e., have zero speed). If all agents are required to maintain a *strictly* positive speed, the proof of Theorem 1 would still go through: in this case, agents a_3, a_5 and a_6 could move at an extremely slow but positive speed instead of stopping. As a result, some points at the neighborhoods of the 12 critical points would remain unvisited for 1 unit of time (this frequency is maintained by agent a_1 alone). However, agents $a_7, \ldots, a_{32}$ would still ensure that every point in these neighborhoods is also visited within every time interval of length $1 - \varepsilon$.

Finite time patrolling. Interestingly enough, we can achieve any prescribed idle time below 1 for an arbitrarily long time in this setting, provided we choose the number of agents k large enough.

Theorem 2. *Consider the unit circle, where all agents are required to move in the same direction. For every $0 < \tau \leqslant 1$ and $t \geqslant \tau$, there exists $k = k(t) \leqslant e^{4t/\tau^2}$ and a schedule for the system of k agents with maximum speeds $v_i = 1/i$, $i = 1, \ldots, k$, that ensures an idle time at most τ during the time interval $[0, t]$.*

Proof. We construct a schedule with an idle time at most τ. Let agent a_1 start at time 0 and move clockwise at maximum (unit) speed, i.e., $f_1(t) = t \mod 1$ denotes the position on the unit circle of agent a_1 at time t. Assume without loss of generality that t is a multiple of τ, i.e., $t = m\tau$, where m is a natural number. Divide the time interval $[0, t]$ into $2m$ subintervals of length $\tau/2$. For $j = 1, \ldots, 2m$, $[(j-1)\tau/2, j\tau/2]$ is the jth interval.

For each j, cover the unit circle C so that every point of C is visited at least once by some agent. This ensures that each point of the circle is visited at least once in the time interval $[0, \tau/2]$ and no two consecutive visits to any one point are separated in time by more than τ thereafter until time t, as required.

To achieve the covering condition in each interval j, we use the first agent (a_1, of unit speed), and as many other unused agents as needed. The 'origin' on C is reset to the current position of a_1 at time $(j-1)\tau/2$, i.e., the beginning of the current time interval. So the fastest agent is used (continuously) in all $2m$ time intervals. Agent a_1 can cover a distance of $\tau/2$ during one interval. From its endpoint, at time $(j-1)\tau/2$, start the unused agent with the smallest index, say $i_1(j)$; this agent can cover a distance of $\frac{\tau}{2} \frac{1}{i_1(j)}$ during the interval. Continue in the same way using new agents, all starting at time

$(j-1)\tau/2$, until the entire circle C is covered; let the index of the last agent used be $i_2(j)$. The covering condition can be written as:

$$\frac{\tau}{2}\left(1+\sum_{i=i_1(j)}^{i_2(j)}\frac{1}{i}\right)\geqslant 1,\ \text{or equivalently,}\ 1+\sum_{i=i_1(j)}^{i_2(j)}\frac{1}{i}\geqslant \frac{2}{\tau}. \tag{1}$$

For example[1], if $\tau=2/3$: $j=1$ requires agents a_1 through a_{11}, since $H_{11}\geqslant 3$, but $H_{10}<3$; $j=2$ requires agents a_1 and agents a_{12} through a_{85}, since $1+(H_{85}-H_{11})\geqslant 3$, but $1+(H_{84}-H_{11})<3$.

We now bound from above the total number k of distinct agents used. Observe that the covering condition (1) may lead to overshooting the target. Because the harmonic series has decreasing terms, the overshooting error cannot exceed the term $\frac{1}{i_2(1)+1}$ for $\tau=1$, namely $1/5$ (the overshooting for $\tau=1$ is only $\frac{1}{3}-\frac{1}{4}=\frac{1}{12}<\frac{1}{5}$). So inequality (1) becomes

$$\frac{2}{\tau}\leqslant 1+\sum_{i=i_1(j)}^{i_2(j)}\frac{1}{i}\leqslant \frac{2}{\tau}+\frac{1}{5}. \tag{2}$$

Recall that $t=m\tau$. By adding inequality (2) over all $2m$ time intervals yields (in equivalent forms)

$$H_k-1+\frac{8m}{5}\leqslant \frac{4m}{\tau},\ \text{or}\ H_k\leqslant \frac{4t}{\tau^2}+1-\frac{8t}{5\tau}. \tag{3}$$

For $t\geqslant \tau$ we have $1\leqslant \frac{8t}{5\tau}$. Since $\ln k\leqslant H_k$, it follows from (3) that

$$\ln k\leqslant \frac{4t}{\tau^2},\ \text{or}\ k\leqslant e^{4t/\tau^2},$$

as required. $\square$

3 Bidirectional Circle Patrolling

The "train" algorithm for closed fence patrolling. Czyzowicz et al. [7, Theorem 5] showed that for $k=3$ there exist maximum speeds v_1,v_2,v_3 and a schedule that achieves a shorter idle time than both algorithm $\mathcal{A}_1$ and $\mathcal{A}_2$, namely $35/36$ versus $12/11$ and 1. We extend their result for all $k\geqslant 4$. We propose a new patrolling algorithm, $\mathcal{A}_3$, for maximum speeds $v_1>v_2\geqslant\ldots\geqslant v_k>0$. That is, $\mathcal{A}_3$ assumes that one of the agents is faster than all others; we then show that for all $k\geqslant 4$ there exist k maximum speeds for which $\mathcal{A}_3$ outperforms both $\mathcal{A}_1$ and $\mathcal{A}_2$.

Place the $k-1$ agents $a_2,\ldots,a_k$ at equal distances, x on the unit circle, and let them move all clockwise perpetually at the same speed v_k; we say that $a_2,\ldots,a_k$ make a "train". Let a_1 move back and forth (i.e., clockwise and counterclockwise) perpetually on the moving arc of length $1-(k-2)x$, i.e., between the start and the end of the train. Refer to Fig. 3.

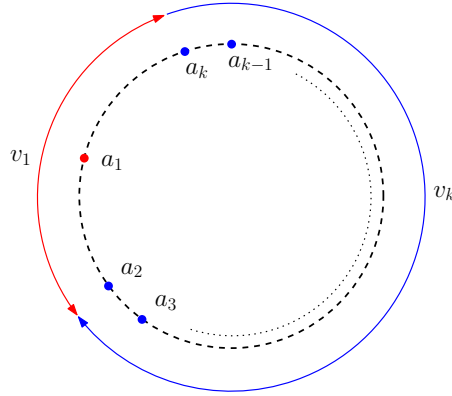[1]$H_n=\sum_{i=1}^n 1/i$ denotes the nth *harmonic number*.

Figure 3: Train algorithm: the train $a_2, \ldots, a_k$ moving unidirectionally with speed v_k and the bidirectional agent a_1 with speed v_1.

Proposition 1. *For every $k \geqslant 4$, there exist maximum speeds $v_1 > v_2 \geqslant \ldots \geqslant v_k$ such that algorithm $\mathcal{A}_3$ achieves a shorter idle time than both $\mathcal{A}_1$ and $\mathcal{A}_2$. In particular, for large k, the idle time achieved by the train algorithm is about $2/3$ of the idle times achieved by $\mathcal{A}_1$ and $\mathcal{A}_2$.*

Proof. Consider the speed setting $v_1 = a$, $v_2 = \ldots = v_k = b$, where $a > b > 0$, and $\max_{1 \leqslant i \leqslant k} i v_i = kb$ (i.e., $a \leqslant kb$). Put $y = 1 - (k-2)x$. To determine the idle time, x/b, write:

$$[1 - (k-2)x]\left(\frac{1}{a-b} + \frac{1}{a+b}\right) = \frac{x}{b}, \text{ or equivalently, } \frac{2ay}{a^2 - b^2} = \frac{1-y}{(k-2)b}.$$

Solving for x/b yields

$$\texttt{idle}(\mathcal{A}_3) = \frac{2a}{a^2 - b^2 + 2(k-2)ab}.$$

For our speed setting, we also have

$$\texttt{idle}(\mathcal{A}_1) = \frac{2}{a + (k-1)b}, \text{ and } \texttt{idle}(\mathcal{A}_2) = \frac{1}{kb}.$$

Write $t = a/b$. It can be checked that for $k \geqslant 4$, $\texttt{idle}(\mathcal{A}_3) \leqslant \texttt{idle}(\mathcal{A}_1)$ and $\texttt{idle}(\mathcal{A}_3) \leqslant \texttt{idle}(\mathcal{A}_2)$ when $a^2 - b^2 - 4ab \geqslant 0$, i.e., $t \geqslant 2 + \sqrt{5}$. In particular, for $a = 1$, and $b = 1/k$ (note that $a \leqslant kb$), we have

$$\texttt{idle}(\mathcal{A}_3) = \frac{2}{1 - 1/k^2 + 2(k-2)/k} \xrightarrow[k\to\infty]{} \frac{2}{3},$$

while

$$\texttt{idle}(\mathcal{A}_1) = \frac{2}{1 + (k-1)/k} \xrightarrow[k\to\infty]{} 1, \text{ and } \texttt{idle}(\mathcal{A}_2) = \frac{1}{k(1/k)} = 1. \qquad \square$$

Useless agents for circle patrolling. Czyzowicz et al. [7] showed that for $k = 2$ there are maximum speeds for which an optimal schedule does not use one of the agents. Here we extend this result for all $k \geqslant 2$:

Proposition 2. (i) *For every $k \geqslant 2$, there exist maximum speeds $v_1, \ldots, v_k > 0$ and an optimal schedule for patrolling the circle with these speeds that does not use up to $k - 1$ of the agents $a_2, \ldots, a_k$.*
(ii) *In contrast, for a segment, any optimal schedule must use all agents.*

Proof. (i) Let $v_1 = 1$ and $v_2 = \ldots = v_k = \varepsilon/k$, for a small positive $\varepsilon \leqslant 1/300$, and C be a unit circle. Obviously by using agent a_1 alone (moving perpetually clockwise) we can achieve unit idle time. Assume for contradiction that there exists a schedule achieving an idle time less than 1. Let $f_1(t)$ denote the position of agent a_1 at time t. Assume without loss of generality that $f_1(0) = 0$ and consider the time interval $[0, 2]$. For $2 \leqslant i \leqslant k$, let J_i be the interval of points visited by agent a_i during the time interval $[0, 2]$, and put $J = \cup_{i=2}^{k} J_i$. We have $|J_i| \leqslant 2\varepsilon/k$, thus $|J| \leqslant 2\varepsilon$. We make the following observations:

1. $f_1(1) \in [-2\varepsilon, 2\varepsilon]$. Indeed, if $f_1(1) \notin [-2\varepsilon, 2\varepsilon]$, then either some point in $[-2\varepsilon, 2\varepsilon]$ is not visited by any agent during the time interval $[0, 1]$, or some point in $C \setminus [-2\varepsilon, 2\varepsilon]$ is not visited by any agent during the time interval $[0, 1]$.

2. a_1 has done almost a complete (say, clockwise) rotation along C during the time interval $[0, 1]$, i.e., it starts at $0 \in [-2\varepsilon, 2\varepsilon]$ and ends in $[-2\varepsilon, 2\varepsilon]$, otherwise some point in $C \setminus [-2\varepsilon, 2\varepsilon]$ is not visited during the time interval $[0, 1]$.

3. $f_1(2) \in [-4\varepsilon, 4\varepsilon]$, by a similar argument.

4. a_1 has done almost a complete rotation along C during the time interval $[1, 2]$, i.e., it starts in $[-2\varepsilon, 2\varepsilon]$ and ends in $[-4\varepsilon, 4\varepsilon]$. Moreover this rotation must be in the same clockwise sense as the previous one, since otherwise there would exist points not visited for at least one unit of time.

Pick three points $x_1, x_2, x_3 \in C \setminus J$ close to 1/4, 2/4, and 3/4, respectively, i.e., $|x_i - i/4| \leqslant 1/100$, for $i = 1, 2, 3$. By Observations 2 and 4, these three points must be visited by a_1 in the first two rotations during the time interval $[0, 2]$ in the order $x_1, x_2, x_3, x_1, x_2, x_3$. Since a_1 has unit speed, successive visits to x_1 are separated in time by at least one time unit, contradicting the assumption that the idle time of the schedule is less than 1.

(ii) Given $v_1 \geqslant v_2 \geqslant \ldots \geqslant v_k > 0$, assume for contradiction that there is an optimal guarding schedule with unit idle time for a segment s of maximum length that does not use agent a_j (with maximum speed v_j), for some $1 \leqslant j \leqslant k$. Extend s at one end by a subsegment of length $v_j/2$ and assign a_j to this subsegment to move back and forth from one end to the other, perpetually. We now have a guarding schedule with unit idle time for a segment longer than s, which is a contradiction. $\qquad \square$

Overtaking other agents. Consider an optimal schedule for circle patrolling (with unit idle time) for the agents in the proof of Proposition 2, with $v_1 = 1$ and $v_2 = \ldots = v_k = \varepsilon/k$, in which all agents move clockwise at their maximum speeds. Obviously a_1 will overtake all other agents during the time interval $[0, 2]$. Thus there exist settings in which if all k agents are used by a patrolling algorithm, then some agent(s) need to overtake (pass) other agent(s). Observe however that overtaking can be easily avoided in this setting by not making use of any of the agents $a_2, \ldots, a_k$.

4   An Improved Idle Time for Open Fence Patrolling

Kawamura and Kobayashi [10] showed that algorithm $\mathcal{A}_1$ by Czyzowicz et al. [7] does not always produce an optimal schedule for open fence patrolling. They presented two counterexamples: their first example uses 6 agents and achieves an idle time of $\frac{41}{42}$ $\texttt{idle}(\mathcal{A}_1)$; their second example uses 9 agents and achieves an idle time of $\frac{99}{100}$ $\texttt{idle}(\mathcal{A}_1)$. By replicating the strategy from the second example with a number of agents larger than 9, i.e., iteratively using blocks of agents, we improve the ratio to $24/25 + \varepsilon$ for any $\varepsilon > 0$. We need two technical lemmas to verify this claim.
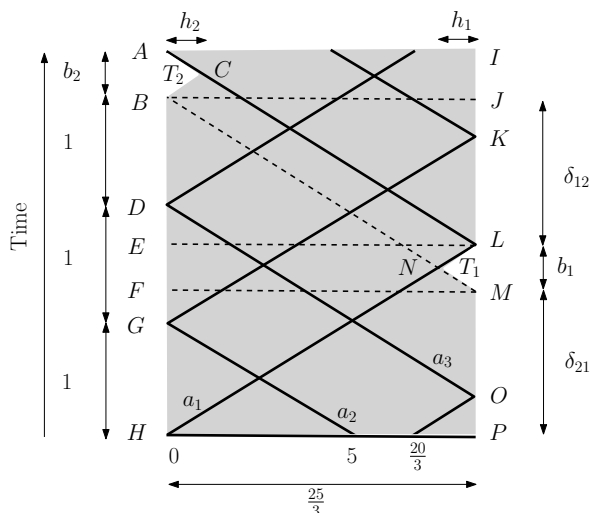

Figure 4: Three agents each with a speed of 5 patrolling a fence of length $25/3$; their start positions are 0, 5, and $20/3$, respectively. Figure is *not* to scale.

Lemma 2. *Consider a segment of length $\ell = \frac{25}{3}$ such that three agents a_1, a_2, a_3 are patrolling perpetually each with speed of 5 and generating an alternating sequence of uncovered triangles $T_2, T_1, T_2, T_1, \ldots$, as shown in the position-time diagram in Fig. 4. Denote the vertical distances between consecutive occurrences of T_1 and T_2 by δ_{12} and between consecutive occurrences of T_2 and T_1 by δ_{21}. Denote the bases of T_1 and T_2 by b_1 and b_2 respectively, and the heights of T_1 and T_2 by h_1 and h_2 respectively. Then*

(i) $\frac{10}{3}$ *is a period of the schedule.*

(ii) *T_1 and T_2 are congruent; further, $b_1 = b_2 = \frac{1}{3}$, $\delta_{12} = \delta_{21} = \frac{4}{3}$, and $h_1 = h_2 = \frac{5}{6}$.*

Proof. (i) Observe that a_1, a_2 and a_3 reach the left endpoint of the segment at times $2(25/3)/5 = 10/3$, $5/5 = 1$, and $(25/3 + 5/3)/5 = 2$, respectively. During the time interval $[0, 10/3]$, each agent traverses the distance 2ℓ and the positions and directions of the agents at time $t = 10/3$ are the same as those at time $t = 0$. Hence $10/3$ is a period for their schedule.

(ii) Since $AL \parallel BM$ and $AB \parallel LM$, we have $b_1 = b_2$. Since L is the midpoint of IP, we have $\delta_{12} + b_2 = \delta_{21} + b_1$, thus $\delta_{12} = \delta_{21}$. Since all the agents have same speed, 5, all the trajectory line segments in the position-time diagram have the same slope, $1/5$. Hence
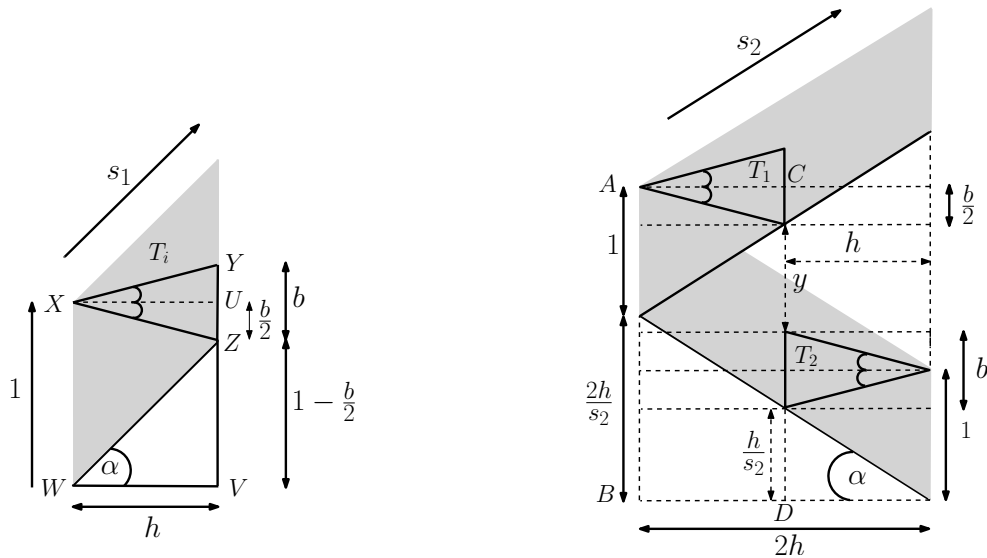
Figure 5: Left: agent covering an uncovered triangle T_i. Right: agent covering an alternate sequence of congruent triangles T_1, T_2, with collinear bases.

$\angle BAC = \angle ABC = \angle MLN = \angle LMN$. Thus, T_1 is similar to T_2. Since $b_1 = b_2$, T_1 is congruent to T_2, and consequently $h_1 = h_2$.

Put $b = b_1$, $h = h_1$, and $\delta = \delta_{12}$. Recall from (i) that $|AH| = 10/3$. By construction, we have $|BD| = 1$, thus $|BH| = |BD| + |DG| + |GH| = 1 + 1 + 1 = 3$. We also have $|AH| = b + |BH|$, thus $b = 10/3 - 3 = 1/3$. Since L is the midpoint of IP, we have $\delta + b = 5/3$, thus $\delta = 5/3 - b = 4/3$.

Let $x(N)$ denote the x-coordinate of point N; then $x(N) + h = 25/3$. To compute $x(N)$ we compute the intersection of the two segments HL and BM. We have $H = (0,0)$, $L = (25/3, 5/3)$, $B = (0,3)$, and $M = (25/3, 4/3)$. The equations of HL and BM are $HL : x = 5y$ and $BM : x + 5y = 15$, and solving for x yields $x = 15/2$, and consequently $h = 25/3 - 15/2 = 5/6$. $\qquad\square$

Lemma 3. (i) *Let s_1 be the speed of an agent needed to cover an uncovered isosceles triangle T_i; refer to Fig. 5 (left). Then $s_1 = \frac{h}{1-b/2}$, where $b < 1$ and h are the base and height of T_i, respectively.*

(ii) *Let s_2 be the speed of an agent needed to cover an alternate sequence of congruent isosceles triangles T_1, T_2 with bases on same vertical line; refer to Fig. 5 (right). Then $s_2 = \frac{h}{3b/2+y-1}$ where y is the vertical distance between the triangles, $b < 1$ is the base and h is the height of the congruent triangles.*

Proof. (i) In Fig. 5 (left), $\tan \alpha = 1/s_1$, $|UZ| = b/2$, hence $|VZ| = 1 - b/2$. Also, $\frac{|VZ|}{|WV|} = \tan \alpha = \frac{1-b/2}{h} = \frac{1}{s_1}$, which yields $s_1 = \frac{h}{1-b/2}$.

(ii) In Fig. 5 (right), $|AB| = 1 + \frac{2h}{s_2}$. Also, $|CD| = \frac{b}{2} + y + b + \frac{h}{s_2}$. Equating $1 + \frac{2h}{s_2} = \frac{3b}{2} + y + \frac{h}{s_2}$ and solving for s_2, we get $s_2 = \frac{h}{3b/2+y-1}$. $\qquad\square$

Theorem 3. *For every integer $x \geqslant 2$, there exist $k = 4x+1$ agents with $\sum_{i=1}^{k} v_i = 16x+1$ and a guarding schedule for a segment of length $25x/3$. Alternatively, for every integer*
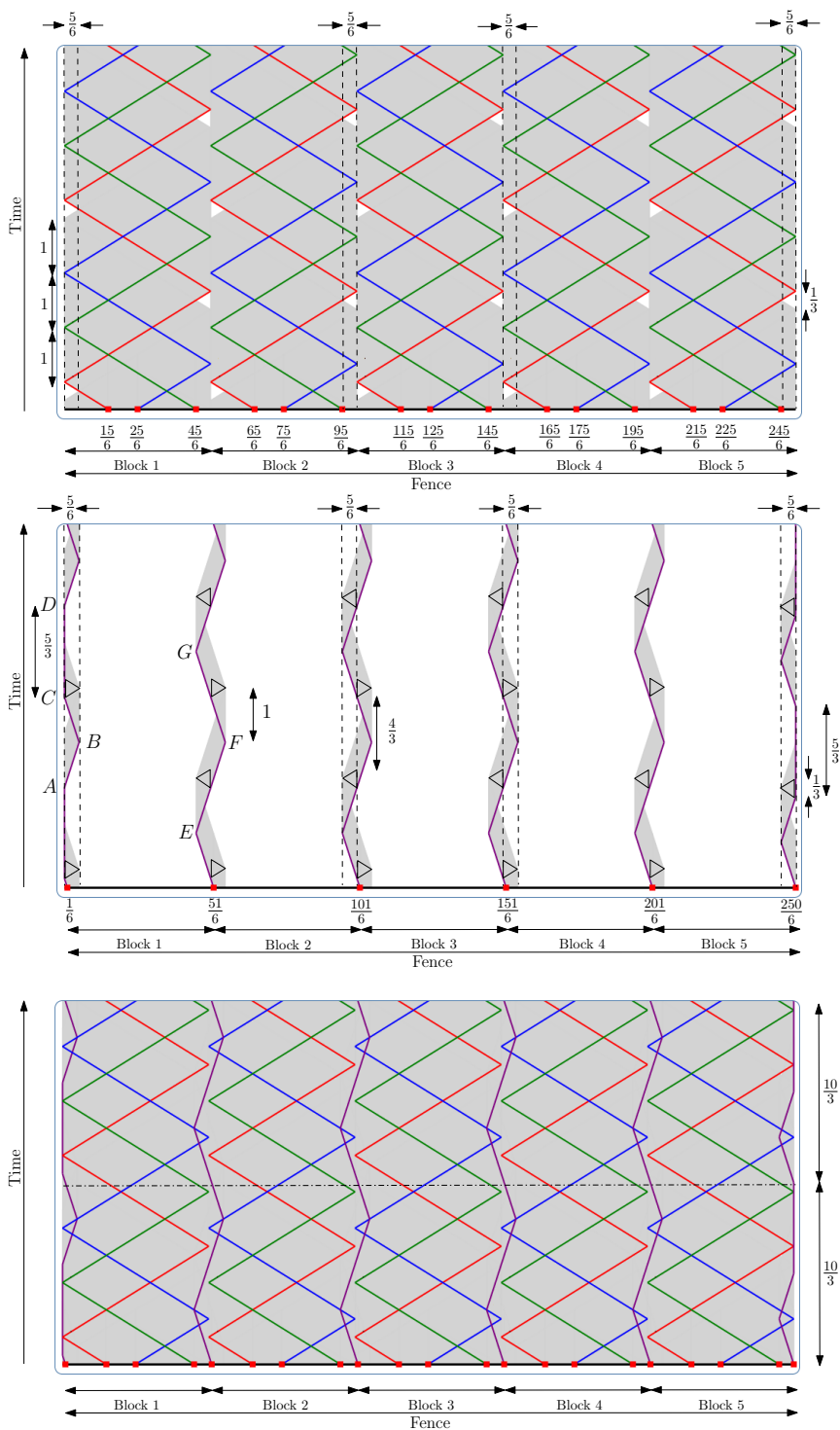
Figure 6: Top: iterative construction with 5 blocks; each block has three agents with speed 5. Middle: 6 agents with speed 1. Bottom: patrolling strategy for 5 blocks using 21 agents for two time periods (starting at $t = 1/3$ relative to Fig. 4); the block length is 25/3 and the period is 10/3.

$x \geqslant 2$ there exist $k = 4x+1$ agents with suitable speeds $v_1, \ldots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\frac{48x+3}{50x} \frac{2}{\sum_{i=1}^{k} v_i}$. In particular, for every $\varepsilon > 0$, there exist k agents with suitable speeds $v_1, \ldots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\left(\frac{24}{25} + \varepsilon\right) \frac{2}{\sum_{i=1}^{k} v_i}$.

Proof. Refer to Fig. 6. We use a long fence divided into x blocks; each block is of length $25/3$. Each block has 3 agents each of speed 5 running in zig-zag fashion. Consecutive blocks share one agent of speed 1 which covers the uncovered triangles from the trajectories of the zig-zag agents in the position-time diagram. The first and the last block use two agents of speed 1 not shared by any other block. The setting of these speeds is explained below.

From Lemma 2(ii), we conclude that all the uncovered triangles generated by the agents of speed 5 are congruent and their base is $b = 1/3$ and their height is $h = 5/6$. By Lemma 3(i), we can set the speeds of the agents not shared by consecutive blocks to $s_1 = \frac{5/6}{1-1/6} = 1$. Also, in our strategy, Lemma 2(ii) yields $y = \delta = 4/3$. Hence, by Lemma 3(ii), we can set the speeds of the agents shared by consecutive blocks to $s_2 = \frac{5/6}{1/2+4/3-1} = 1$.

In our strategy, we have 3 types of agents: agents running with speed 5 as in Fig. 6 (top), unit speed agents not shared by 2 consecutive blocks and unit speed agents shared by two consecutive blocks as in Fig. 6 (middle). By Lemma 2(i), the agents of first type have period $10/3$. In Fig. 6 (middle), there are two agents of second type and both have a similar trajectory. Thus, it is enough to verify for the leftmost unit speed agent. It takes $5/6$ time from A to B and again $5/6$ time from B to C. Next, it waits for $5/3$ time at C. Hence after $5/6 + 5/6 + 5/3 = 10/3$ time, its position and direction at D is same as that at A. Hence, its time period is $10/3$. For the agents of third type, refer to Fig. 6 (middle): it takes $10/6$ time from E to F and $10/6$ time from F to G. Thus, arguing as above, its time period is $10/3$. Hence, overall, the time period of the strategy is $10/3$.

For x blocks, we use $3x + (x+1) = 4x+1$ agents. The sum of all speeds is $5(3x) + 1(x+1) = 16x+1$ and the total fence length is $\frac{25x}{3}$. The resulting ratio is $\rho = \frac{16x+1}{2} / \frac{25x}{3} = \frac{48x+3}{50x}$. For example, when $x = 2$ we reobtain the bound of Kawamura and Kobayashi [10] (from their 2nd example), when $x = 39$, $\rho = \frac{100}{104}$ and further on, $\rho \xrightarrow[x \to \infty]{} \frac{24}{25}$. Thus an idle time of at most $\left(\frac{24}{25} + \varepsilon\right) \frac{2}{\sum_{i=1}^{k} v_i}$ can be achieved for every given $\varepsilon > 0$, as required. $\qquad\square$

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