A note on a Ramsey-type problem for sequences

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Abstract

Two sequences $\{x_i\}_{i=1}^t$ and $\{y_i\}_{i=1}^t$ of distinct integers are *similar* if their entries are order-isomorphic. Let $f(r,X)$ be the length of the shortest sequence Y such that any r-coloring of the entries of Y yields a monochromatic subsequence that is also similar to X. In this note we show that for any fixed non-monotone sequence X, $f(r,X) = \Theta(r^2)$, otherwise, for a monotone X, $f(r,X) = \Theta(r)$.

Keywords: Permutations; Sequences; Ramsey problems

1 Introduction

We consider the following Ramsey-type question. We say that two sequences $\{x_i\}_{i=1}^t$ and $\{y_i\}_{i=1}^t$ of distinct integers are *similar* if their entries are order-isomorphic, *i.e.*, $x_i < x_j$ if and only if $y_i < y_j$ for all $1 \leqslant i < j \leqslant t$. For a given sequence X and a positive integer r a sequence Y is *Ramsey* for X if for every r-coloring of the entries of Y there is a subsequence of Y which is both monochromatic and similar to X. Denote by $f(r,X)$ the length of the shortest sequence Y that is Ramsey for X, *i.e.*,

$$f(r,X) = \min_Y |Y|,$$

where the minimum is taken over all Ramsey sequences for X. Moreover, let

$$f(r,t) = \max_X f(r,X),$$

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where the maximum is taken over all sequences X with $|X| = t$.

Frankl, Rödl and the author [3] asked to determine for a fixed t the order of magnitude of $f(r,t)$ as a function of r. Here we show that $f(r,t) = \Theta(r^2)$. Indeed, we give a stronger result identifying the asymptotic behavior of $f(r,X)$ for every X.

Theorem 1.

(i) Let X be a monotone sequence, i.e., X is similar to $(1,2,\ldots,|X|)$ or $(|X|,\ldots,2,1)$. Then

$$f(r,X) = \Theta(r).$$

(ii) Let X be a non-monotone sequence. Then

$$f(r,X) = \Theta(r^2).$$

(The hidden constants depend only on X.)

It is also worth mentioning that the proof shows that for each t there is a (universal) sequence Y of length $O(r^2)$ which is Ramsey for every sequence X of length t and any number of colors r. Furthermore, the entries of such Y colored by the majority color contain a subsequence similar to X.

2 Proof of Theorem 1

For (i) it is enough to observe that $(1,2,\ldots,r|X|)$ is Ramsey for $(1,2,\ldots,|X|)$, and similarly, $(r|X|,\ldots,2,1)$ is Ramsey for $(|X|,\ldots,2,1)$.

Now we prove (ii). First we show the lower bound. The proof is based on the Erdős-Szekeres [4] theorem which says that any sequence S of length m contains a monotone subsequence of length $\lceil\sqrt{m}\,\rceil$. It is not difficult to observe (see, *e.g.*, [1, 8]) that the repetitive application of this result shows that S can be partitioned into at most $\lfloor 2\sqrt{m}\rfloor$ monotone subsequences. For the sake of completeness we prove a similar result here.

Let X be any sequence of length t which is non-monotone. Assume that Y is Ramsey for X. We show that $|Y| > \left(\frac{r}{2}\right)^2$. Suppose not, *i.e.*, $|Y| \leqslant \left(\frac{r}{2}\right)^2$. We will repeatedly apply the Erdős-Szekeres theorem. We start with Y of length $a_0 = |Y| \leqslant \left(\frac{r}{2}\right)^2$ and find a monotone subsequence of length $\lceil\sqrt{a_0}\,\rceil$. Then we remove it from Y obtaining a sequence of length $a_1 = a_0 - \lceil\sqrt{a_0}\,\rceil$ and repeat the whole process again. After the i-th step the length of the remaining sequence is given by the recursive formula

$$a_{i+1} = a_i - \lceil\sqrt{a_i}\,\rceil.$$

Let N be the least integer for which $a_N = 0$. We show that $N \leqslant r$. First observe that for each $i < N$, we have $a_i \geqslant 1$ and

$$a_{i+1} = a_i - \lceil\sqrt{a_i}\,\rceil \leqslant a_i - \sqrt{a_i} \leqslant \left(\sqrt{a_i} - \frac{1}{2}\right)^2$$

implying

$$\sqrt{a_{i+1}} \leqslant \sqrt{a_i} - \frac{1}{2},$$

and consequently,

$$\sum_{i=0}^{N-1} \sqrt{a_{i+1}} \leqslant \sum_{i=0}^{N-1} \left(\sqrt{a_i} - \frac{1}{2} \right).$$

Thus,

$$\sqrt{a_N} \leqslant \sqrt{a_0} - \frac{N}{2} \leqslant \frac{r}{2} - \frac{N}{2}$$

and after at most r steps we end up with an empty sequence. Summarizing, we just found a decomposition of Y into at most r monotone subsequences. Now we color each monotone subsequence with a different color. Since X is non-monotone, there is no monochromatic subsequence similar to X, a contradiction.

Next we show the upper bound. First we need some notation. Let A and P be 0-1 matrices. We say that A *contains* the $t \times t$ matrix $P = (p_{i,j})$ if there exists a $t \times t$ submatrix $B = (b_{i,j})$ of A with $b_{i,j} = 1$ whenever $p_{i,j} = 1$. Otherwise we say that A *avoids* P. Notice that we can delete rows and columns of A to obtain the submatrix B but we cannot permute the remaining rows and columns. Given a permutation π of t elements its *permutation matrix* is the $t \times t$ matrix $P_\pi = (p_{i,j})$ whose entries are all 0 except that in column i, the entry $\pi(i)$ equals 1, *i.e.*, the only non-zero entries are $p_{\pi(i),i}$.

We will use the following result conjectured by Füredi and Hajnal [6] and proved by Marcus and Tardos [7]. Let P be a permutation matrix. Denote by $g(P, m)$ the maximum number of ones in a 0-1 matrix of size $m \times m$ avoiding P. Then, due to Marcus and Tardos [7], there exists a positive constant $c = c(P)$ such that

$$g(P, m) \leqslant cm. \tag{1}$$

Let X be a given sequence of t different integers. (Here non-monotonicity is not required.) Without loss of generality we may assume that X is a permutation of $\{1, \ldots, t\}$. Let P_X be the corresponding permutation matrix and let $c = c(P_X)$ be as in (1) yielding

$$g(P_X, m) \leqslant cm. \tag{2}$$

Set

$$m = \lfloor cr \rfloor + 1. \tag{3}$$

Now we define a sequence Y which is a permutation of $\{1, \ldots, m^2\}$. Let

$$\begin{aligned} Y = (\ & 1, && m+1, && 2m+1, && 3m+1, && \ldots, && (m-1)m+1, \\ & 2, && m+2, && 2m+2, && 3m+2, && \ldots, && (m-1)m+2, \\ & 3, && m+3, && 2m+3, && 3m+3, && \ldots, && (m-1)m+3, \\ & \cdots && \cdots && \cdots && \cdots && \cdots && \cdots \\ & m, && 2m, && 3m, && 4m, && \ldots, && m^2\ ). \end{aligned}$$

Clearly, $|Y| = \Theta(r^2)$. It remains to show that Y is Ramsey for X.

Let A_Y be the following matrix of size $m \times m$ based on Y. The first m elements of Y form the first column in A_Y in reverse order. The next m elements of Y form the second column in A_Y in reverse order, etc. Thus,

$$A_Y = \begin{pmatrix} (m-1)m+1 & (m-1)m+2 & (m-1)m+3 & \dots & m^2 \\ (m-2)m+1 & (m-2)m+2 & (m-2)m+3 & \dots & (m-1)m \\ (m-3)m+1 & (m-3)m+2 & (m-3)m+3 & \dots & (m-2)m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m+1 & m+2 & m+3 & \dots & 2m \\ 1 & 2 & 3 & \dots & m \end{pmatrix}.$$

Now let us arbitrarily color the elements of Y with r colors. We need to show that there is a monochromatic subsequence in Y that is similar to X.

Clearly, every coloring of Y uniquely induces a coloring of the entries of A_Y. Choose the most frequent color, say red, and let $A = (a_{ij})$ be the 0-1 matrix of size $m \times m$ whose entries correspond to it. That means $a_{ij} = 1$ if and only if the ij-entry in A_Y is colored red. The key observation is the following: if A does not avoid P_X, then Y contains a monochromatic subsequence similar to X. By (3) and (2), we get that the number of ones in A is at least

$$\frac{m^2}{r} > cm \geqslant g(P_X, m).$$

Hence, A does not avoid P_X. This completes the proof of (ii).

3 Concluding remarks

It may be of some interest to study $f(r,t)$ in more detail. Theorem 1 implies that

$$c_1 r^2 \leqslant f(r,t) \leqslant c_2 r^2,$$

for some positive constants $c_1 = c_1(t)$ and $c_2 = c_2(t)$. For the sake of simplicity we did not attempt to optimize theses constants. The proof gives $c_1 = \frac{1}{4}$ and this constant can be improved to $\frac{1}{2}$ by using a result of Brandstädt and Kratsch [2]. On the other hand, c_2 is entirely based on the result of Marcus and Tardos [7] and so is exponential in t (see also a result of Fox [5]).

It would be also interesting to consider a similar question and study the growth of $f(r,t)$ for a fixed r and large t.

For only two colors it is not difficult to see that

$$f(2,t) = \Theta(t^2). \tag{4}$$

Indeed, let $X = \{x_i\}_{i=1}^t$ be any sequence. Without loss of generality we may assume that X is a permutation of $\{0,\dots,t-1\}$. For the upper bound let us define $Y = Y^{(1)}Y^{(2)}\dots Y^{(t)}$, where $Y^{(i)} = (tx_i + x_1,\ tx_i + x_2,\ \dots,\ tx_i + x_t)$ for $1 \leqslant i \leqslant t$. Now let us arbitrarily color the entries of Y with two colors. Since each $Y^{(i)}$ is similar to X, we may assume that there is no monochromatic $Y^{(i)}$. Thus, there is a monochromatic subsequence $(y_1, y_2, \ldots, y_t)$ such that $y_i \in Y^{(i)}$ for $1 \leqslant i \leqslant t$. It is easy to see that such $(y_1, y_2, \ldots, y_t)$ is similar to X. Consequently, Y is Ramsey for X and $f(2, X) \leqslant |Y| = t^2$.

To see the lower bound of (4) consider $X = \big(1, 2, \ldots, \lfloor \frac{t}{2} \rfloor, t, t-1, \ldots, \lfloor \frac{t}{2} \rfloor + 1\big)$. Let Y be any Ramsey sequence for X. Clearly, Y must contain many subsequences similar to $Z = \big(1, 2, \ldots, \lfloor \frac{t}{2} \rfloor\big)$. Starting with $Y_0 = Y$, we find a subsequence similar to Z and remove it obtaining Y_1 (of length $|Y| - \lfloor \frac{t}{2} \rfloor$). We repeatedly continue the process of removing subsequences similar to Z until we cannot longer find a subsequence similar to Z. Let m denote the number of steps and Y_m be the remaining sequence. Now we color Y_m blue and $Y \setminus Y_m$ red. Since Y_m contains no subsequence similar to Z, there is no blue subsequence similar to X in Y. Therefore, there must be a red subsequence in Y which is similar to X. In particular, there is a red subsequence similar to $X \setminus Z$. Since $Y \setminus Y_m$ is a disjoint union of m (increasing) sequences similar to Z, each of these m subsequences can contain at most one element of the (decreasing) sequence $X \setminus Z$. Thus, $m \geqslant t - \lfloor \frac{t}{2} \rfloor = \lceil \frac{t}{2} \rceil$ and so

$$|Y| \geqslant |Y \setminus Y_m| \geqslant m \left\lfloor \frac{t}{2} \right\rfloor \geqslant \left\lceil \frac{t}{2} \right\rceil \left\lfloor \frac{t}{2} \right\rfloor \geqslant \frac{t^2 - 1}{4}.$$

By recursively extending the above construction one can get an upper bound for any $r \geqslant 2$ and show that

$$f(r, t) \leqslant t^r. \tag{5}$$

For example, for $r = 3$ and a permutation $X = \{x_i\}_{i=1}^t$ of $\{0, \ldots, t-1\}$ it is enough to take $Y = Y^{(1)} Y^{(2)} \ldots Y^{(t)}$, where

$$\begin{aligned} Y^{(i)} = (\ & t^2 x_i + t x_1 + x_1, && t^2 x_i + t x_1 + x_2, && \ldots, && t^2 x_i + t x_1 + x_t, \\ & t^2 x_i + t x_2 + x_1, && t^2 x_i + t x_2 + x_2, && \ldots, && t^2 x_i + t x_2 + x_t, \\ & \ldots && \ldots && \ldots && \ldots \\ & t^2 x_i + t x_t + x_1, && t^2 x_i + t x_t + x_2, && \ldots, && t^2 x_i + t x_t + x_t \ ), \end{aligned}$$

for $1 \leqslant i \leqslant t$. Observe that Y is Ramsey for X. Hence, $f(3, X) \leqslant |Y| = t^3$.

The lower bound in (4) and the upper bound in (5) imply that for a fixed $r \geqslant 2$

$$\Omega(t^2) = f(r, t) = O(t^r).$$

Determining the right order of magnitude of $f(r, t)$ as a function of t remains open. [1]

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[1] Very recently, together with Klimošová and Král' we showed that $f(r, t) = \Omega\big(\frac{t^{(r+1)/2}}{\mathrm{polylog}(t)}\big)$. The proof uses some ideas from [5].

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