Schur-Positivity in a Square

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Abstract

Determining if a symmetric function is Schur-positive is a prevalent and, in general, a notoriously difficult problem. In this paper we study the Schur-positivity of a family of symmetric functions. Given a partition ν, we denote by ν^c its complement in a square partition (m^m). We conjecture a Schur-positivity criterion for symmetric functions of the form $s_{\mu'}s_{\mu^c} - s_{\nu'}s_{\nu^c}$, where ν is a partition of weight $|\mu| - 1$ contained in μ and the complement of μ is taken in the same square partition as the complement of ν. We prove the conjecture in many cases.

1 Introduction

The ring of symmetric functions has as a basis the Schur functions, s_λ, indexed by partitions λ. This basis is of particular importance in representation theory because its elements occur as characters of the general linear group, GL_n, and they correspond to characters of the symmetric group, S_n, via the Frobenius map. In addition, the Schur functions are representatives of Schubert classes in the cohomology of the Grassmanian. Often, given a symmetric function, we are interested in writing it in the Schur basis. If the coefficients in the Schur basis expansion are all non-negative integers, then the symmetric

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function corresponds to the character of a representation of GL_n or S_n. In this case, the coefficients are simply giving the decomposition of the character in terms of irreducible characters. We will call a symmetric function *Schur-positive* if it is a linear combination of Schur functions with non-negative coefficients.

In recent years there has been increased interest in studying the Schur-positivity of expressions of the form

$$s_\lambda s_\mu - s_\alpha s_\beta. \tag{1}$$

See for example [1, 2, 3, 4, 5, 7, 13]. These expressions can also be interpreted as differences of skew Schur functions which have been studied in [11, 12]. The Schur-positivity of such expressions is equivalent to inequalities between Littlewood-Richardson coefficients. In this paper, we study the Schur-positivity of a family of expressions of this form. Let ν and μ be partitions such that $|\mu| = |\nu| + 1$ and $\nu \subseteq \mu$, and let m be an integer such that $m \geqslant \mu_1 + \ell(\mu)$. We denote by ν^c (respectively μ^c) the complement of ν (respectively μ) in the square partition (m^m) and by ν' (respectively μ') the conjugate of ν (respectively μ). We are interested in the Schur-positivity of expressions of the form

$$s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}. \tag{2}$$

As we explain in section 2.3, these expressions arise in the study of the Kronecker product of a square shape and a hook shape. Determining the Schur decomposition of this Kronecker product is of particular interest in a paper by Scharf, Thibon and Wybourne [14] on the powers of the Vandermonde determinant and its application to the quantum Hall effect. In their paper, the q-discriminant is expanded as a linear combination of Schur functions where the coefficients are specializations of the Kronecker products mentioned above.

Central to our paper is a conjectural criterion for the Schur-positivity of (2).

Conjecture I: Let $\nu \vdash n$ and $\mu \vdash n+1$ be partitions such that $\nu \subseteq \mu$. Then, if complements are taken in a large enough square, $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$ is Schur-positive if and only if the following conditions are satisfied.

(C1') The partition ν is such that, if δ_k is the smallest staircase partition which is not contained in ν, then there is a single box $(a+1, b+1)$ which is in δ_k but not in ν, and ν contains the box $(a, a+b)$ or the box $(a+b, b)$. (By convention, ν contains the boxes $(0, a+b)$ and $(a+b, 0)$.)

(C2') If ν is as in (C1'), then μ is the partition obtained from ν by adding the box $(a+1, b+1)$.

The conjecture above can be restated in the following equivalent form.

Conjecture II: Let $\nu \vdash n$ and $\mu \vdash n+1$ be partitions such that $\nu \subseteq \mu$. Then, if complements are taken in a large enough square, $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$ is Schur-positive if and only if the following conditions are satisfied.

(C1) ν or ν' is of the form $\beta + (s^s) + \alpha$, where β and α are partitions such that $\ell(\alpha) \leqslant s$, and

(i) if $\beta_1 = i$, then β contains $(i^{s+2}, i-1, i-2, \ldots, 1)$,

(ii) $s \neq 0$ if and only if $\alpha \neq \emptyset$.

(C2) If ν (respectively ν') is a partition $\beta + (s^s) + \alpha$ as in (C1), then μ (respectively μ') is the partition $\beta + (s^s, 1) + \alpha$.

If ν satisfies (C1) in Conjecture II, we say that ν is of ***type* 1** if $s = 0$ and of ***type* 2** if $s \geqslant 1$.

In [10] McNamara gives several necessary conditions for Schur-positivity. However, most articles considering special cases of (1) focus on sufficient conditions. The strength of our conjecture lies in the fact that is it a criterion. In this article, we prove the conjecture in many cases.

All results of this paper are valid if all complements are taken in a large enough rectangle instead of a square. Since the proofs for complements in rectangles do not add any new insight, to simplify the exposition, we present the results with complements taken in a large enough square.

The paper is organized as follows. In Section 1, we review the notation and basic facts about partitions and Schur functions and discuss products of the form $s_{\mu'}s_{\mu^c}$. In Section 2, we discuss symmetry and stability properties of expressions of the form (2), introduce the main conjecture and prove that the two formulations are equivalent, and discuss type possibilities for a partition and its conjugate. In Section 3, we study partitions of type 1 and show that the conjecture holds if the partition contains a large rectangle of the same width as the partition. Moreover, we show that the conjecture holds for all partitions of type 1 of width at most four. In this section we also state the Lex-minimality conjecture that would imply that condition (C2) on μ is necessary for the Schur-positivity of $s_{\mu'}s_{\mu^c} - s_{\nu'}s_{\nu^c}$, if ν satisfies (C1). In Section 4, we establish several properties of partitions of type 2 and prove the conjecture for partitions of type 2 with $\beta_1 = 0$ or 1. In Section 5, we consider the failure of Schur-positivity for the symmetric function (2) if ν does not satisfy (C1) (*i.e.*, ν is neither of type 1, nor of type 2). In Section 6, we offer some final remarks, including unsuccessful strategies we considered in our attempt to prove the conjecture.

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2 Preliminaries

2.1 Partitions and Schur functions

For details and proofs of the contents presented here see [9] or [15, Chapter 7]. Let n be a non-negative integer. A *partition* of n is a weakly decreasing sequence of non-negative integers, $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, such that $|\lambda| = \sum \lambda_i = n$. We write $\lambda \vdash n$ to mean λ is a partition of n. The nonzero integers λ_i are called the *parts* of λ. We identify a partition with its *Young diagram*, *i.e.*, the array of left-justified squares (boxes) with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. The rows are arranged in matrix form from top to bottom. By the box in position (i, j) we mean the box in the i-th row and j-th column of λ. The *length* of λ, $\ell(\lambda)$, is the number of rows in the Young diagram.

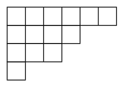


Figure 1: $\lambda = (6, 4, 3, 1)$, $\ell(\lambda) = 4$, $|\lambda| = 14$

Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if and only if $\ell(\mu) \leqslant \ell(\lambda)$ and $\mu_i \leqslant \lambda_i$ for all $1 \leqslant i \leqslant \ell(\mu)$. If $\mu \subseteq \lambda$, we denote by λ/μ the skew shape obtained by removing the boxes corresponding to μ from λ.

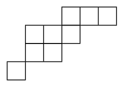


Figure 2: λ/μ with $\lambda = (6, 4, 3, 1)$ and $\mu = (3, 1, 1)$

For any two Young diagrams (or skew shapes) λ and μ, we denote by $\lambda \times \mu$ any skew diagram consisting only of the diagram λ followed by the diagram μ such that λ and μ have no common edges. That is, the rows (respectively columns) of λ are above the rows (respectively to the right of the columns) of μ. Note that λ and μ could have one common corner, *i.e*, the highest northeast corner of μ can connect with the lowest southwest corner of λ.

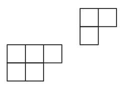


Figure 3: $\lambda \times \mu$ with $\lambda = (2, 1)$ and $\mu = (3, 2)$.

For any two partitions λ and μ, we define the *sum* $\lambda + \mu$ to be the partition obtained by adding corresponding parts. That is, $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. For example, if $\lambda = (3, 2, 1)$ and $\mu = (2, 2)$, then $\lambda + \mu = (5, 4, 1)$.

A *semi-standard Young tableau* (SSYT) *of shape* λ/μ is a filling of the boxes in the Young diagram of the skew shape λ/μ with positive integers so that the numbers weakly increase in each row from left to right and strictly increase in each column from top to bottom. The *type* of a SSYT T is the sequence of non-negative integers $(t_1, t_2, \ldots)$, where t_i is the number of labels i in T. The *superstandard tableau* of a partition λ is the SSYT of shape λ and type λ.

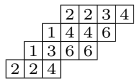


Figure 4: SSYT of shape $\lambda/\mu = (7,6,5,3)/(3,2,1)$ and type $(2,4,2,4,0,3)$.

Given a SSYT T of shape λ/μ and type $(t_1, t_2, \ldots)$, we define its *weight*, $w(T)$, to be the monomial obtained by replacing each i in T by x_i and taking the product over all boxes, *i.e.*, $w(T) = x_1^{t_1} x_2^{t_2} \cdots$. For example, the weight of the SSYT in Figure 4 is $x_1^2 x_2^4 x_3^2 x_4^4 x_6^3$. The skew Schur function $s_{\lambda/\mu}$ is defined combinatorially by the formal power series

$$s_{\lambda/\mu} = \sum_T w(T),$$

where the sum runs over all SSYTs of shape λ/μ. To obtain the usual Schur function one sets $\mu = \emptyset$. It follows directly from the combinatorial definition of Schur functions that $s_{\lambda \times \mu} = s_\lambda s_\mu$.

The space of homogeneous symmetric functions of degree n is denoted by Λ^n. A basis for this space is given by the Schur functions $\{s_\lambda \mid \lambda \vdash n\}$. The Hall inner product on Λ^n is denoted by $\langle \ , \ \rangle$ and it is defined by

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ denotes the Kronecker delta.

2.2 The Littlewood-Richardson rule

The *Littlewood-Richardson coefficients* are defined via the Hall inner product on symmetric functions as follows:

$$c^\lambda_{\mu,\nu} := \langle s_\lambda, s_\mu s_\nu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle.$$

That is, skewing is the adjoint operator of multiplication with respect to this inner product. The Littlewood-Richardson coefficients can be computed using the Littlewood-Richardson rule. Before presenting the rule we need to recall two additional definitions. A *lattice permutation* is a sequence $a_1 a_2 \cdots a_n$ such that in any initial factor $a_1 a_2 \cdots a_j$, for each $1 \leqslant i \leqslant n$, the number of labels i is at least the number of labels $(i+1)$. For example 11122321 is a lattice permutation. The *reverse reading word* of a tableau is the sequence of entries of T obtained by reading the entries from right to left and top to bottom, starting with the first row.

Example: The reverse reading word of the tableau 

is 218653974.

We denote by $LR(\lambda/\mu, \nu)$ the collection of SSYTs of shape λ/μ and type ν whose reverse reading word is a lattice permutation. We refer to a tableau in $LR(\lambda/\mu, \nu)$ as a *Littlewood-Richardson tableau* of shape λ/μ and type ν. The *Littlewood-Richardson rule* states that the Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is equal to the cardinality of $LR(\lambda/\mu,\nu)$.

We denote by $c^{\lambda}_{\mu,\nu,\eta}$ the multiplicity of s_λ in the product $s_\mu s_\nu s_\eta$, *i.e.*, $c^{\lambda}_{\mu,\nu,\eta} = \langle s_\lambda, s_\mu s_\nu s_\eta\rangle$.

In the next remark, we draw attention to a condition on tableaux in $LR(\lambda/\mu,\nu)$ which will be used later in our discussion. First we introduce a definition. Recall that given a partition ν, a *horizontal strip* in ν is a skew shape ν/ψ (for some partition $\psi \subseteq \nu$) such that no two boxes in ν/ψ are in the same column.

Remark 2.1. The lattice permutation condition on a tableau $T \in LR(\lambda/\mu,\nu)$ imposes the following condition. Let $\nu^{(1)} = \nu$. The labels in the last row of T must form a horizontal strip in the superstandard tableau of $\nu^{(1)}$. Denote this horizontal strip by h_1. Note that if we list the labels in the last row of T in order from left to right, they appear in h_1 in that order when read from right to left. Let $\nu^{(2)}$ be the partition obtained from $\nu^{(1)}$ by removing the boxes of h_1. Then, the labels in the second to last row of T form a horizontal strip in the superstandard tableau of $\nu^{(2)}$. This process continues recursively. We denote by h_j the horizontal strip in $\nu^{(j)}$ containing the labels of the j-th row from the bottom in T and let $\nu^{(j+1)}$ be the partition obtained from $\nu^{(j)}$ by removing the boxes of h_j. Then, the labels in the $(j+1)$-st row from the bottom in T must form a horizontal strip in the superstandard tableau of $\nu^{(j+1)}$.

We refer to the condition in Remark 2.1 as *the rows of T form horizontal strips in the superstandard tableau of ν*.

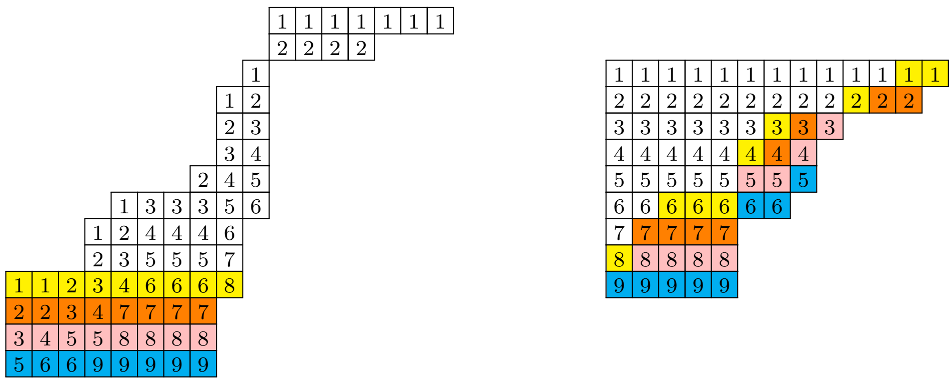


Figure 5: A Tableau T and the corresponding superstandard tableau

Figure 5 shows, on the left, a tableau $T \in LR(\lambda/\mu,\nu)$ with $\lambda = (17, 14, 10^6, 9^3, 8^3)$, $\mu = (10^2, 9, 8^3, 7, 4, 3^2)$, and $\nu = (13, 12, 9, 8^2, 7, 5^3)$. On the right, it shows the superstandard tableau of ν. The colors indicate the labels in some of the bottom rows of T and the corresponding horizontal strips in the superstandard tableau of ν.

2.3 Product of the conjugate and the complement of a partition

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ be a partition. We denote by ν' the *conjugate partition* of ν, *i.e.*, the partition whose rows are the columns of ν. If D is a skew-diagram, D^* denotes D

rotated by 180°.

If (m^m) is a square partition and $\mu \subseteq (m^m)$, the *complement partition* of μ in (m^m), denoted μ^c, is the partition $((m^m)/\mu)^*$. Whenever we need to emphasize m, we write $\mu^{c,m}$ for μ^c. See Figure 6 for the shape of the complement.

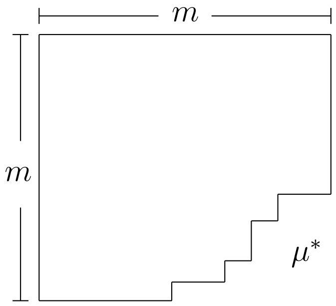


Figure 6: The complement of μ, *i.e.*, μ^c.

Given a partition μ and a square (m^m), we will be interested in the product $s_{\mu'}s_{\mu^c}$. Our interest in this particular product has its origin in the Kronecker product of a hook and a square shape, *i.e.*, $s_{(n-k,1^k)} * s_{(m^m)}$. We briefly explain this connection.

Using Pieri's rule and induction, one can easily see that

$$s_{(n-k,1^k)} = \sum_{i=0}^{k} (-1)^{k-i} s_{(n-i)} s_{(1^i)}.$$

Then, using Littlewood's formula [8], we obtain

$$\left(s_{(n-i)} s_{(1^i)}\right) * s_{(m^m)} = \sum_{\mu \vdash n-i, \nu \vdash i} c^{(m^m)}_{\mu,\nu} s_\mu s_{\nu'}.$$

Since $c^{(m^m)}_{\mu,\nu} = \begin{cases} 1 & \text{if } \mu,\nu \subset (m^m) \text{ and } \nu = \mu^c \\ 0 & \text{else.} \end{cases}$, we have

$$s_{(n-k,1^k)} * s_{(m^m)} = \sum_{\eta \vdash n} \left(\sum_{i=0}^{k} (-1)^{k-i} \sum_{\mu \vdash i} c^{\eta}_{\mu',\mu^c} \right) s_\eta.$$

The focus of this article article is the study of Schur-positivity of differences of products of the form $s_{\mu'}s_{\mu^c}$.

If we choose m large, then the partitions occurring in the expansion of $s_{\mu'}s_{\mu^c}$ have the shape shown in Figure 7. That is, if $c^{\lambda}_{\mu',\mu^c} \neq 0$, then $\lambda = (\eta^c + \gamma, \sigma)$, where $\eta \subseteq \mu$, $|\gamma| + |\sigma| = |\eta|$ and the complement of η is taken in (m^m).

Remark 2.2. In order for λ to occur with nonzero coefficient in the expansion of $s_{\mu'}s_{\mu^c}$, there must be LR tableaux of shape λ/μ^c and type μ'. With the notation of Figure 7, the conditions for a SSYT imply that $\ell(\gamma) \leqslant \ell(\mu') = \mu_1$ and $\sigma_1 \leqslant \mu'_1 = \ell(\mu)$. In

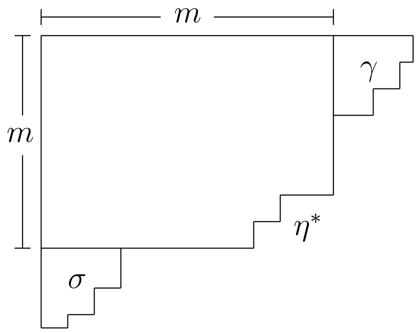


Figure 7: The shape of the partition $\lambda = (\eta^c + \gamma, \sigma)$ for large m.

addition, $\eta_1 \leqslant \mu_1$ and $\ell(\eta) \leqslant \ell(\mu)$. Therefore, if $m \geqslant \mu_1 + \ell(\mu)$, the skew diagram λ/μ^c is the disjoint union of three distinct diagrams, $\gamma, (\mu/\eta)^*$, and σ, and the label fillings in Littlewood-Richardson tableaux do not depend on m. Hence, the decomposition of the product $s_{\mu'}s_{\mu^c}$ becomes stable, *i.e.*, does not change for $m \geqslant \mu_1 + \ell(\mu)$.

For the rest of the article we assume that complements are taken in a square (m^m) with $m \geqslant \mu_1 + \ell(\mu)$. Thus, we assume that all partitions λ such that $c^{\lambda}_{\mu',\mu^c} \neq 0$, have the shape in Figure 7, *i.e.*, $\lambda = (\eta^c + \gamma, \sigma)$, where $\eta \subseteq \mu$ and $|\gamma| + |\sigma| = |\eta|$.

The following proposition describes a symmetry property for the expansion of the product $s_{\mu'}s_{\mu^c}$.

Proposition 2.3. *Let μ be a partition and $m \in \mathbb{Z}$ with $m \geqslant \mu_1 + \ell(\mu)$. Consider the partitions $\lambda = (\eta^c + \gamma, \sigma)$ and $\overline{\lambda} = (\eta^c + \sigma, \gamma)$ with η, γ, σ as above. The coefficients of s_λ and $s_{\overline{\lambda}}$ in $s_{\mu'}s_{\mu^c}$ are equal. Moreover, these coefficients are equal to $c^{\mu'}_{\gamma,\sigma,(\mu/\eta)^*}$.*

Proof. If λ, μ and m are as in the statement of the proposition, we have

$$\langle s_\lambda, s_{\mu'}s_{\mu^c}\rangle = \langle s_{\lambda/\mu^c}, s_{\mu'}\rangle = \langle s_\gamma s_{(\mu/\eta)^*} s_\sigma, s_{\mu'}\rangle = \langle s_\gamma s_\sigma s_{(\mu/\eta)^*}, s_{\mu'}\rangle.$$

Thus, $\langle s_\lambda, s_{\mu'}s_{\mu^c}\rangle = c^{\mu'}_{\gamma,\sigma,(\mu/\eta)^*}$. Moreover,

$$\langle s_\lambda, s_{\mu'}s_{\mu^c}\rangle = \langle s_\sigma s_{(\mu/\eta)^*} s_\gamma, s_{\mu'}\rangle = \langle s_{\overline{\lambda}/\mu^c}, s_{\mu'}\rangle = \langle s_{\overline{\lambda}}, s_{\mu'}s_{\mu^c}\rangle.$$

□

A consequence of Proposition 2.3 is the following combinatorial interpretation of the coefficient of s_λ in $s_{\mu'}s_{\mu^c}$.

Corollary 2.4. *If λ, μ, and m are as in Proposition 2.3, then*

$$\langle s_\lambda, s_{\mu'}s_{\mu^c}\rangle = |LR(\gamma \times \sigma \times (\mu/\eta)^*, \mu')|.$$

3 Schur-positivity for a difference of products

We say that a symmetric function f is *Schur-positive* if every coefficient in the expansion of f as a linear combination of Schur functions is a non-negative number. That is, if

$$f = \sum_{\lambda} c_\lambda s_\lambda,$$

then $c_\lambda \geqslant 0$ for all λ.

In this paper we are interested in the Schur-positivity of expressions of the form

$$s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c},$$

where ν^c and μ^c are complements in the same square, $\nu \subseteq \mu$, and $\mu \vdash |\nu|+1$. We introduce another definition to simplify the statements of the theorems.

Definition 3.1. We say that μ *covers* ν, if $\nu \subseteq \mu$, $|\mu| = |\nu|+1$, ν^c and μ^c are complements in the same square, and $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$ is Schur-positive.

For the rest of the article, whenever complements of different partitions appear in the same expression, it is understood that they are taken in the same square.

In the proof of the following proposition we use the *fundamental involution* ω on the ring of symmetric functions $\omega : \Lambda \to \Lambda$. This is defined on the Schur functions by $\omega(s_\lambda) = s_{\lambda'}$. It is a well-known fact that ω is an isometry with respect to the Hall inner product. For details see [15, Section 7.6]. This leads immediately to the following result.

Proposition 3.2. *If μ covers ν, then μ' covers ν'.*

3.1 Symmetry and stability

In this section we describe two properties of the coefficients occurring in the expansion of $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$.

Proposition 3.3 (Symmetry)**.** *Let μ be a partition and $m \in \mathbb{Z}$ with $m \geqslant \mu_1 + \ell(\mu)$. If $\lambda = (\eta^c + \gamma, \sigma)$ and $\overline{\lambda} = (\eta^c + \sigma, \gamma)$, then the coefficients of s_λ and $s_{\overline{\lambda}}$ in $s'_\mu s_{\mu^c} - s'_\nu s_{\nu^c}$ are equal.*

Proof. If $m \geqslant \mu_1 + \ell(\mu)$, by Proposition 2.3, both products $s_{\mu'} s_{\mu^c}$ and $s_{\nu'} s_{\nu^c}$ are symmetric. □

For fixed ν and μ, the differences $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$ satisfy the stability property given in the following proposition.

Proposition 3.4 (Stability)**.** *If ν and μ are partitions such that $\nu \subseteq \mu$ and $|\mu| = |\nu| + 1$, then $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$ is stable in the sense that if $m_1, m_2 \geqslant \mu_1 + \ell(\mu)$, then*

$$\langle s_{(\eta^{c,m_1} + \gamma, \sigma)}, s_{\mu'} s_{\mu^{c,m_1}} - s_{\nu'} s_{\nu^{c,m_1}} \rangle = \langle s_{(\eta^{c,m_2} + \gamma, \sigma)}, s_{\mu'} s_{\mu^{c,m_2}} - s_{\nu'} s_{\nu^{c,m_2}} \rangle,$$

for all η, γ, σ with $|\gamma| + |\sigma| = |\eta|$.

Proof. This follows directly from Remark 2.2 and Corollary 2.4 since $\langle s_\lambda, s_{\mu'} s_{\mu^c}\rangle = |LR(\gamma \times \sigma \times (\mu/\eta)^*, \mu')|$ does not depend on m if $m \geqslant \mu_1 + \ell(\mu)$. ☐

The following example shows that the bound $\mu_1 + \ell(\mu)$ on m in Proposition 3.4 is sharp.

Example 3.5. Suppose $\mu = (3, 2, 1)$ and $\nu = (3, 2)$. Thus, $\mu_1 = 3$ and $\ell(\mu) = 3$. Let $\eta = (2, 2, 1), \gamma = (1, 1, 1)$, and $\sigma = (1, 1)$. One can use Maple to check that, if $m_1 = 6$, then $\langle s_{(\eta^{c,m_1}+\gamma,\sigma)}, s_{\mu'} s_{\mu^{c,m_1}} - s_{\nu'} s_{\nu^{c,m_1}}\rangle = 1$. However, if $m_2 = 5$, then $(\eta^{c,m_2} + \gamma, \sigma) = (6, 6, 5, 3, 3, 1, 1)$ and, using Maple again, we have$\langle s_{(\eta^{c,m_2}+\gamma,\sigma)}, s_{\mu'} s_{\mu^{c,m_2}} - s_{\nu'} s_{\nu^{c,m_2}}\rangle = 2$.

3.2 Main conjecture

We now introduce our main conjecture. It gives a characterization of partitions ν and μ, where the diagram of μ differs from that of ν by a single box, for which $s_{\mu'} s_{\mu^c} - s_{\nu'} s_{\nu^c}$ is Schur-positive (assuming complements are taken in sufficiently large squares).

Conjecture 3.6. *Let $\nu \vdash n$ and $\mu \vdash n + 1$ be a partitions such that $\nu \subseteq \mu$. Suppose complements are taken in (m^m) with $m \geqslant \mu_1 + \ell(\mu)$. Then, μ covers ν if and only if the following conditions are satisfied.*

(C1) *ν or ν' is of the form $\beta + (s^s) + \alpha$, where β and α are partitions such that $\ell(\alpha) \leqslant s$, and*

- *(i) if $\beta_1 = i$, then β contains $(i^{s+2}, i - 1, i - 2, \ldots, 1)$,*
- *(ii) $s \neq 0$ if and only if $\alpha \neq \emptyset$.*

(C2) *If ν (respectively ν') is a partition $\beta + (s^s) + \alpha$ as in* (C1)*, then μ (respectively μ') is the partition $\beta + (s^s, 1) + \alpha$.*

Suppose ν satisfies (C1). We say that ν is of ***type* 1** if $s = 0$ and of ***type* 2** if $s \geqslant 1$. Note that in (C1) it is possible to have $\beta = \emptyset$.

In Figure 8, we show a general partition ν of type 1 together with the corresponding cover μ. Inside the diagram of ν we show (in blue) the smallest partition of type 1 of width equal to the width of ν. In Figure 9 we show a general partition of type 2 together with the corresponding cover μ. In both cases the red box represents the box added to ν to obtain μ.

Let δ_k be the kth *staircase partition* $(k, k - 1, k - 2, \ldots, 2, 1)$. Notice that in Figure 8 the red box $(1, i + 1)$ is the unique box in δ_{i+1} that is not in ν. In Figure 9 the red box $(s + 1, i + 1)$ is the unique box in δ_{s+i+1} that is not in ν.

Based on the observations we made on Figure 8 and Figure 9, we give the following alternate formulation of conditions (C1) and (C2) in Conjecture 3.6.

Theorem 3.7. *Condition (C1) is equivalent to*

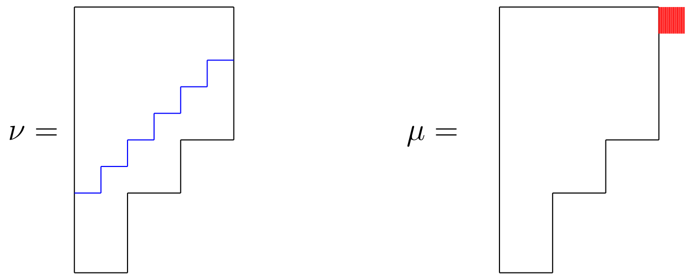


Figure 8: A partition ν of type 1 and the corresponding partition μ

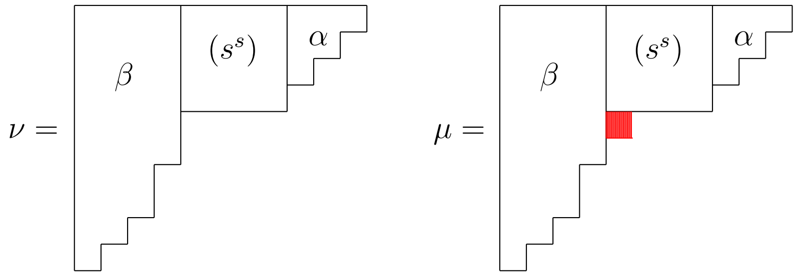


Figure 9: A partition ν of type 2 and the corresponding partition μ

(C1') The partition ν is such that, if δ_k is the smallest staircase partition which is not contained in ν, then there is a single box $(a+1, b+1)$ which is in δ_k but not in ν, and ν contains the box $(a, a+b)$ or the box $(a+b, b)$. (By convention, ν contains the boxes $(0, a+b)$ and $(a+b, 0)$.) Moreover, if $(a+1, b+1)$ is the box in δ_k not contained in ν, then $k = a+b+1$.

Moreover, condition (C2) is equivalent to

(C2') If ν is as in (C1'), then μ is the partition obtained from ν by adding the box $(a+1, b+1)$.

With the conditions of Theorem 3.7, type 1 corresponds to $a = 0$. Moreover, if $b = 0$ in (C1'), then ν' is of type 1.

Proof. We first show that (C1) is equivalent to (C1'). Suppose that ν is a partition satisfying (C1).

If ν is of type 1, then $\nu = \beta$ with $\beta_1 = i$ and β contains $(i^2, i-1, \ldots, 2, 1)$. Thus, ν contains δ_i but not δ_{i+1}. Moreover, the only box contained in δ_{i+1} but not in ν is $(1, i+1)$. Thus, ν satisfies (C1'). If ν is of type 2, then $\nu = \beta + (s^s) + \alpha$, $s \neq 0$, $\alpha \neq 0$ and, if $\beta_1 = i$, then β contains $(i^{s+2}, i-1, \ldots, 2, 1)$. Therefore, $\nu_1 \geqslant i+s+1$. Further, $\nu_{s+1} = i$ and $\nu_s \geqslant i+s$. Thus, δ_{i+s} is contained in ν since β contains $(i^{s+2}, i-1, \ldots, 1)$. However, δ_{i+s+1} is not contained in ν and the only box contained in δ_{i+s+1} but not in ν is $(s+1, i+1)$. Further, ν contains the box $(s, i+s)$. Therefore ν satisfies (C1'). Similarly, if ν' is of type 1 with $\nu' = \beta$ and $\beta_1 = i$, then $\delta_i \subseteq \nu$, $\delta_{i+1} \not\subseteq \nu$, and $(i+1, 1)$ is the only box in

δ_{i+1} not contained in ν. Thus, ν satisfies (C1'). If ν' is of type 2 with $\nu' = \beta + (s^s) + \alpha$ ($\beta_1 = i, s \geqslant 1, \alpha \neq \emptyset$), then $\delta_{i+s} \subseteq \nu$ but $\delta_{i+s+1} \not\subseteq \nu$, and $(i+1, s+1)$ is the only box in δ_{i+s+1} not contained in ν. Moreober, ν contains box $(i+s, s)$. Thus, ν satisfies (C1').

Now we assume that ν is a partition satisfying (C1'). Then for some k, the partition ν contains δ_{k-1} but it doesn't contain δ_k and there is only one box contained in δ_k but not in ν, $(a+1, b+1)$ for some positive integers a and b such that $a+b+1 = k$. Moreover, ν contains the box $(a, a+b)$ or the box $(a+b, b)$. If $a = 0$, then the box contained in δ_k but not in ν is $(1, b+1)$ and $k = b+1$. Thus, $\nu_1 = b$ and ν contains $(b, b, b-1, \ldots, 2, 1)$. Then, ν is a partition of type 1 and thus it satisfies (C1). If $b = 0$, then the box contained in δ_k but not in ν is $(a+1, 1)$ and $k = a+1$. In this case, ν' is of type 1. Now assume that $a, b \neq 0$. Suppose that ν contains box $(a, a+b)$. Then, ν contains $(b+a+1, b+a, \ldots, b+1, b, b, b-1, \ldots, 2, 1)$. Therefore, $\nu = \beta + (s^s) + \alpha$ with $i = b$, $s = a$, $\alpha \neq \emptyset$. Thus, ν is a partition of type 2 and therefore it satisfies (C1). A similar argument applies if $a, b \neq 0$ and ν contains the box $(a+b, b)$, in which case ν' is a partition of type 2.

It follows from the argument for the equivalence of (C1) with (C1') that (C2) is equivalent to (C2'). □

The next two lemmas list some properties of partitions satisfying (C1). They follow directly from condition (C1).

Lemma 3.8. *Suppose $\nu = \beta + (s^s) + \alpha$ satisfies the conditions in (C1). Then,*

(i) $\beta_1 = \beta_2 = \ldots = \beta_{s+1} = \beta_{s+2}$ *and* $\beta_j \geqslant s + \beta_1 - j + 2$ *for* $j \geqslant s+2$.

(ii) $\beta'_j \geqslant s + \beta_1 - j + 2$ *for* $1 \leqslant j \leqslant \beta_1$.

(iii) If $\beta \neq \emptyset$, *then* $\ell(\beta) \geqslant s + \beta_1 + 1$.

(iv) If $\lambda = (\eta^c + \gamma, \sigma)$ *is such that* $c^{\lambda}_{\nu'\nu^c} \neq 0$ *and* ν *is not self-conjugate, then* $\eta \neq \emptyset$.

(v) If $\nu \neq \emptyset$ *is of type 1 (i.e.,* $s = 0$*), and* $\lambda = (\eta^c + \gamma, \sigma)$ *is such that* $c^{\lambda}_{\nu'\nu^c} \neq 0$, *then* $\eta \neq \emptyset$.

In the next lemma we consider the effect of removing rows or columns from a diagram on the type of the partition.

Lemma 3.9.

(i) If ν *is a partition of type 1, then so is any partition* $\nu^{(k)}$ *obtained from* ν *by removing its first* k *columns,* $1 \leqslant k \leqslant \nu_1 - 1$.

(ii) If $\nu = \beta + (s^s) + \alpha$ *is a partition of type 2, then so is any partition* $\nu^{(k)}$ *obtained from* ν *by removing its first* k *columns,* $1 \leqslant k \leqslant \beta_1$.

(iii) If $\nu = \beta + (s^s) + \alpha$ *is a partition of type 2, then the partition* $\nu_{(s)}$ *obtained from* ν *by removing its first* s *rows is of type 1.*

Conjecture 3.6 gives a criterion for exactly when a partition covers another partition in the case that the partitions differ by one box. It might not be obvious that the conjecture implies that a partition ν satisfying condition (C1) is covered by a unique partition. We prove this in Theorem 3.13, but first we prove a few more properties of partitions satisfying (C1).

Proposition 3.10.

(a) If ν is a partition of type 2, then the expression given in (C1) *is unique.*

(b) If ν is a partition of type 1, then ν is not of type 2. I.e., ν cannot be of type 1 and type 2 simultaneously.

Proof. (a) Since ν satisfies (C1') and it is of type 2, then $a \neq 0$ and the unique box in δ_{a+b+1} that is not in ν is the box $(a+1, b+1)$. This box completely determines β, s and α. We have $\beta = (b^{a+2}, \nu_{a+3}, \ldots, \nu_{\ell(\nu)})$, $s = a$ and $\alpha = (\nu_1 - a - b, \nu_2 - a - b, \ldots, \nu_s - a - b)$.

(b) Suppose $\nu = \beta + (s^s) + \alpha$ is of type 2 (*i.e.*, $s \neq 0, \alpha \neq \emptyset$) and let δ_k be the smallest staircase not contained in ν. By the proof of Theorem 3.7, the unique box in δ_k that is not contained in ν is $(s+1, i+1)$. Since $s \neq 0$, this box is not in the first row. Thus, ν is not of type 1.

□

If ν is of type 2, since the decomposition $\nu = \beta + (s^s) + \alpha$ is unique, we refer to the Young diagram formed by the boxes in (s^s) as the *square in ν*.

Since the criterion in Conjecture 3.6 depends on the shape of ν or ν', we explore the relationship between partitions of type 1 and 2 and their conjugates.

Proposition 3.11.

(a) If ν is of type 1, then ν' is not of type 1.

(b) If $\nu = \beta + (s^s) + \alpha$ is of type 2 with $\beta_1 \geqslant 1$, then ν' is not of type 1.

(c) If $\nu = (s^s) + \alpha$ is of type 2 with $\beta_1 = 0$, then ν' is of type 1.

(d) If ν and ν' are both of type 2, then ν is of the form $\nu = \beta + (s^s) + \alpha$ with $s \geqslant 1$, $\alpha \neq \emptyset$ and β as in (C1) *satisfying $\beta_{s+\beta_1} = \beta_1$.*

Before we prove Proposition 3.11, we give an example illustrating part (d).

Example 3.12. Consider the partition $\nu = (6, 5, 3, 3, 3, 1)$. Then, $\nu' = (6, 5, 5, 2, 2, 1)$ and both both ν and ν' are of type 2. We have $\beta_1 = 3$, $s = 2$, and $\beta_{s+\beta_1} = \beta_5 = 3 = \beta_1$. The Young diagram of ν is given in Figure 10.

The condition $\beta_{s+\beta_1} = \beta_1$ ensures that the square in ν (marked with $\bullet$ above) and the square in ν' (marked with $\circ$ above) meet but do not have any common edge. Then, assuming Conjecture 3.6 is true, the only cover of ν is the partition $\mu = (6, 5, 4, 3, 3, 1)$. Moreover, μ' is the only cover of ν'.

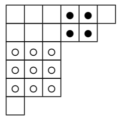


Figure 10: A partition of type 2 whose conjugate is of type 2

Proof. (a) If ν satisfies (C1'), the smallest staircase δ_k not contained in ν has only one box not in ν. If both ν and ν' were of type 1, δ_k would contain two boxes not in ν, the box in the first row and the box in the last row of δ_k.

(b) If $\nu = \beta + (s^s) + \alpha$ is of type 2 with $\beta_1 \geqslant 1$, the unique box $(s+1, \beta_1+1)$ in the smallest staircase δ_{β_1+s+1} that doesn't fit into ν is not in the first column. Hence in ν', the only box in δ_{β_1+s+1} not contained in ν' is not in the first row.

(c) If $\nu = (s^s) + \alpha$, $\alpha \neq \emptyset$, the smallest staircase not contained in ν is δ_{s+1} and the unique box in δ_{s+1} not contained in ν is $(s+1, 1)$. Hence $(1, s+1)$ is the only box contained in δ_{s+1} but not in ν'. Therefore, ν' is of type 1.

(d) If $\nu = \beta + (s^s) + \alpha$ is of type 2 and ν' is also of type 2, the smallest staircase that doesn't fit into ν and ν' is $\delta_{s+\beta_1+1}$ and the unique box in $\delta_{s+\beta_1+1}$ but not in ν or ν' is $(s+1, \beta_1+1)$. In addition ν must contain both (s, β_1+s) and (β_1+s, β_1). Therefore, $\beta_{\beta_1+s} = \beta_1$.

□

Note that parts (b) and (c) of Proposition 3.11 show that $\nu = \beta + (s^s) + \alpha$ is of type 2 with conjugate of type 1 if and only if $\beta = \emptyset$. Equivalently, a partition ν is of type 1 with conjugate of type 2 if and only if $(\nu_1^{\nu_1}) \subseteq \nu$.

Propositions 3.10 and 3.11 lead to the following result.

Corollary 3.13. *Assuming the validity of the main conjecture, a partition has either no cover or exactly one cover.*

We conclude this section by proving the interesting property that, if Conjecture 3.6 is true, then covering partitions μ are not covered by any partition.

Let $\mathcal{C}_1$ be the set of all partitions satisfying (C1), *i.e.*, $\mathcal{C}_1$ consists of partitions ν such that ν or ν' is equal to $\beta + (s^s) + \alpha$, with α, β satisfying the following conditions:

(i) If $\beta_1 = i \geqslant 0$, then β contains $(i^{s+2}, i-1, i-2, \ldots, 1)$.

(ii) $s \neq 0$ if and only if $\alpha \neq \emptyset$.

Let $\mathcal{C}_2$ be the set of all partitions satisfying (C2), *i.e.*, $\mathcal{C}_2$ consists of partitions μ such that μ or μ' is equal to $\beta + (s^s, 1) + \alpha$, where β and α are as above.

Theorem 3.14. *The sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint.*

Proof. Suppose $\lambda \in \mathcal{C}_1 \cap \mathcal{C}_2$. Since $\lambda \in \mathcal{C}_2$, λ satisfies (C2'). Thus, there exists a partition ν satisfying (C1') such that $\nu \subseteq \lambda$, $|\lambda| = |\nu| + 1$, and λ/ν consists of the unique box $(a+1, b+1)$ in δ_{k-1} that is not in ν, where δ_{k-1} is the smallest staircase partition not contained in ν. Then, $\delta_{k-1} \subseteq \lambda$. Since $(a+1, b+1)$ is not in ν, then both $(a+2, b+1)$ and $(a+1, b+2)$ are not in ν. Since $|\lambda| = |\nu| + 1$, the boxes $(a+2, b+1), (a+1, b+2)$

are not in λ. However, both $(a+2, b+1), (a+1, b+2)$ are in δ_k, the smallest staircase partition not contained in λ. This contradicts the fact that $\lambda \in \mathcal{C}_1$. □

We note that $\mathcal{C}_1 \cup \mathcal{C}_2$ is not equal to the set of all partitions. For example, partitions of the form (s^s) are neither in $\mathcal{C}_1$ nor in $\mathcal{C}_2$.

Corollary 3.15. *Assuming the validity of the Conjecture 3.6, if μ covers ν, then μ is not covered by any partition of size $|\mu| + 1$.*

4 Partitions of type 1

In this section we prove that Conjecture 3.6 holds for some special cases of partitions of type 1. In particular, we prove the conjecture for partitions ν of type 1 containing the rectangle $(\nu_1^{\nu_1 - 1})$ and those of width at most four.

4.1 Partitions of type 1 containing a large rectangle

Let $\nu = \beta$ be a partition of type 1 with $\beta_1 = i$ and suppose β contains the rectangle (i^{i-1}). Therefore, if $i \geqslant 2$, β contains $(i^{i-1}, 2, 1)$, *i.e.*, $\beta'_i \geqslant i - 1$. We will prove that ν is covered by $\mu = \nu + (1)$, the partition obtained from ν by adding a box at the end of its first row, by showing that $c^{\lambda}_{\nu'\nu^c} \leqslant c^{\lambda}_{\mu'\mu^c}$ for all partitions λ. We use Corollary 2.4 and consider Littlewood-Richardson tableau of shape $\gamma \times \sigma \times (\beta/\eta)^*$ (respectively $\gamma \times \sigma \times ((\beta + (1))/\eta)^*$) rather than of shape $\gamma \times (\beta/\eta)^* \times \sigma$ (respectively, $\gamma \times ((\beta + (1))/\eta)^* \times \sigma$). This simplifies considerably the description of tableaux and injections between sets of tableaux. For a partition $\lambda = (\eta^c + \gamma, \sigma)$ such that $c^{\lambda}_{\nu'\nu^c} \neq 0$, we give an algorithm that assigns to each $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ a distinct tableau T' in $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$. We denote by x the box in the diagram $\gamma \times \sigma \times ((\beta + (1))/\eta)^*$ that is not in $\gamma \times \sigma \times (\beta/\eta)^*$

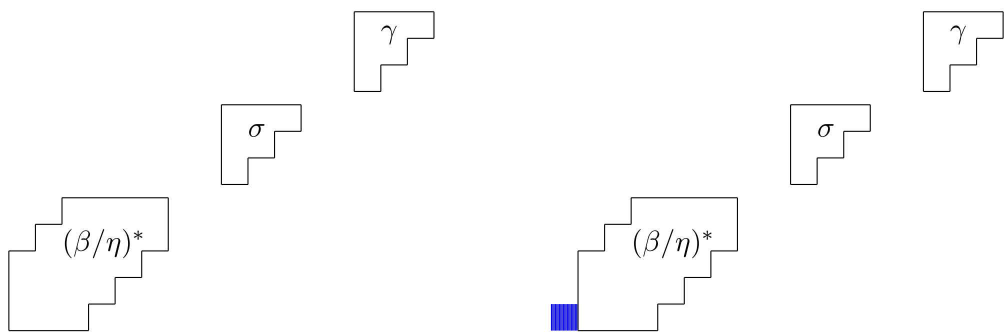


Figure 11: $\gamma \times \sigma \times (\beta/\eta)^*$ and $\gamma \times \sigma \times (\beta + (1)/\eta)^*$

Figure 11 illustrates the shape of the diagrams to be filled with the labels of the superstandard tableau of β' (for the diagram on the left) respectively of $(\beta', 1)$ (for the diagram on the right). The box x is shaded blue. The goal is to insert a label $i+1$ into a tableau of the shape shown on the left so the result is a Littlewood Richardson tableau of the shape on the right. When we count columns in a skew shape such as those in Figure 11 we do so starting with the leftmost non-empty column. This column becomes column 1. We only count non-empty columns.

Before describing the algorithm, we remark on several properties satisfied by some tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$. Remark 4.1 is true for all partitions of type 1.

Remark 4.1. Let β be a partition of type 1 with $\beta_1 = i$ and let $\lambda = (\eta^c + \gamma, \sigma)$ be such that $c^{\lambda}_{\beta'\beta^c} \neq 0$. Since β' has i rows and $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta') \neq \emptyset$, all columns in $\gamma \times \sigma \times (\beta/\eta)^*$ have length at most i. In addition, $\eta_1 \geqslant 1$. Moreover, if $\eta_1 = 1$, then $\beta_{i+1} = 1$ and every tableau in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ has the $(i-1)$st column filled, from the top to the bottom, with labels $1, 2, \ldots, i$.

Lemma 4.2. *Let β be a partition of type 1 with $\beta_1 = i$, $i \geqslant 4$, such that $(i^{i-1}) \subseteq \beta$ and $\beta_i \leqslant i-2$. Let $\lambda = (\eta^c + \gamma, \sigma)$ be such that $\eta_1 = 1$ and $c^{\lambda}_{\beta'\beta^c} \neq 0$, and let $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ be a tableau with all labels i in the last row. Let q be the label in the first column and in row $i - \beta_i$ from the bottom in T. Then, T satisfies the following properties.*

(a) *For each t such that $\beta_i + 2 \leqslant t \leqslant i$, row $i - t + 1$ from the bottom has each box filled with the label t (i.e., the last row is filled with i, and, if $\beta_i < i-2$, the second to last row is filled with $i-1$, and so on until we reach row $(i - \beta_i - 1)$ from the bottom which is filled with $\beta_i + 2$). Moreover, $\ell(\eta) \geqslant i - \beta_i$.*

(b) *For each label p in the $(i-1)$st column, the box directly to its right is either empty (i.e., not part of $(\beta/\eta)^*$) or it contains a label greater than p.*

(c) *The first column of T contains the labels $1, 2, \ldots, i$, with one label missing, in increasing order, from top to bottom. The missing label is $1, 2$, or $\beta_i + 1$.*

(d) *Row $i - \beta_i$ from the bottom in T is filled, from left to right, with the labels $\beta_i, \beta_i + 1, \beta_i+1, \ldots, \beta_i+1$ or with $\beta_i+1, \beta_i+1, \ldots, \beta_i+1$. Thus $q = \beta_i$ or β_i+1. If $\beta_i > 2$, the label directly above q is exactly $q-1$. Moreover, if $\beta_i > 2$ and $q = \beta_i + 1$, then each box in row $i - \beta_i + 1$ from the bottom in T is filled with β_i.*

Proof. (a) In the superstandard tableau of β', each label $i, i-1, \ldots, \beta_i + 1$ appears an equal number of times while the number of labels β_i is exactly one more. Since the reverse reading word of T is a lattice permutation, the last $i - \beta_i - 1$ labels in the first column of T, read from the bottom up must be $i, i-1, \ldots, \beta_i + 2$. The statement follows from the fact that the last row of T is filled with i.

(b) Since $\eta_1 = 1$, we have $\beta_{i+1} = 1$ by Remark 4.1. Therefore $\beta'_1 > \beta'_2$ and the statement follows.

(c) This follows from the fact that in the superstandard tableau of β', the number of labels j equals the number of labels $j-1$ for each $3 \leqslant j \leqslant i$ and $j \neq \beta_i + 1$. If label $j \neq 1, 2, \beta_i+1$ were missing from the first column of T, then in the reverse reading word of T there would be a label $j-1$ after the last label j. This violates the lattice permutation condition. For $j = 2, \beta_i + 1$, the number of labels j is strictly less than the number of labels $j - 1$.

(d) This follows directly from part (c). (Note that if $\beta_i = 2$ and $q = 3$ the label directly above it could be 1 or 2.) □

Note that if β, λ and T are as in Lemma 4.2, and $q = \beta_i + 1$, then $\ell(\eta) \geqslant i - \beta_i + 1$.

Remark 4.3. Let β be a partition of type 1 with $\beta_1 = i$, $i \geqslant 3$, such that $(i^{i-1}) \subseteq \beta$ and $\beta_i = i - 1$. If λ and T are as in Lemma 4.2, then part (b) of the lemma holds. In part (c), the missing label in the the first column of T can only be 1 or 2. In part (d), we have $q = \beta_i + 1$ and it is still true that, if $\beta_i > 2$, the label directly above q is $q - 1$.

In what follows we state the algorithm informally followed by an example and then proceed to give a formal statement of the algorithm.

Given $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$, the algoritm will assign to it a distinct tableau $T' \in LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$ as follows.

If $\eta_1 = i$, place label $i + 1$ in box x.

If $\eta_1 < i$ and at least one label i is in a row of T higher than the last row, insert label $i + 1$ in the rightmost box of the last row of T and bump all labels in the last row of T one position to the left. The leftmost label in the last row of T is placed in box x.

If all labels i are in the last row of T, let q be the label in the first column and in row $i - \beta_i$ from the bottom in T. We will insert label $i + 1$ in the rightmost box of the last row of T and bump labels along an up-then-left-hook path. If $q = \beta_i$, we bump up to the row of q and then left, placing label β_i in box x. If $q = \beta_i + 1$, we bump up to the first row directly above the row of q and then left, placing label β_i in box x.

The following example illustrates the last case.

Example 4.4. Consider the partition $\beta = (6, 6, 6, 6, 6, 3, 1, 1)$. Then $i = 6, \beta_i = 3$, and $\beta' = (8, 6, 6, 5, 5, 5)$. Let $\lambda = (\eta^c + \gamma, \sigma)$ with $\eta = (1, 1, 1, 1, 1), \gamma = (3, 1)$, and $\sigma = (1)$. We show the effect of the algorithm on two tableaux $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ in which all labels i are in the last row of T. In the first example, shown in Figure 12, the label q (marked in blue) in the first column and row $i - \beta_i = 3$ from the bottom in T is equal to $\beta_i = 3$. In the second example, shown in Figure 13, $q = \beta_i + 1 = 4$. In T, the path of the insertion is marked with a red line. In T' the affected labels are marked in red.

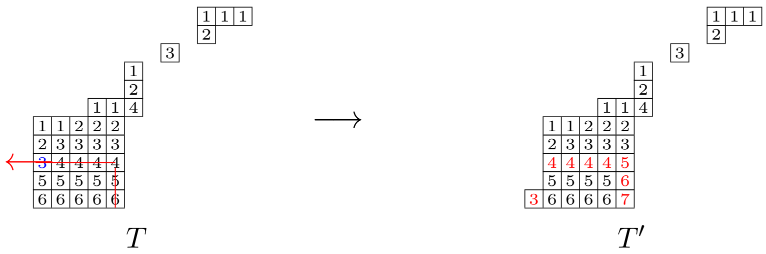


Figure 12: Example 4.4 with $q = \beta_i$

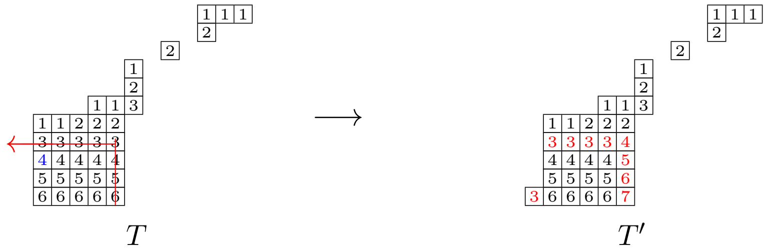


Figure 13: Example 4.4 with $q = \beta_i + 1$

Next we define two operations on tableaux. In this definition we use $-k$ to denote the kth row from the bottom of the tableau. These will help us formulate the algorithm. Let T be a Littlewood-Richardson tableau.

Definition 4.5. *Insert u and bump left in row $-k$*, denoted $\mathbf{R}_{-k} \leftarrow u$, is defined as follows. Label u replaces (bumps) the rightmost label in the kth row from the bottom in T. While there is a label to the left of it, the last bumped label replaces the label directly to its left. When the end of the row is reached, the last bumped label becomes the *evacuated* label.

Insert u and bump up-and-left in column k and row $-t$, denoted $\mathbf{H}_{k,-t} \leftarrow u$, is defined as follows. Label u replaces the last label in the kth column of T. While the last bumped label is in a row below the tth row from the bottom in T, the last bumped label replaces the label directly above it. If v is the label bumped from column k and row $t-1$ from the bottom in T, perform $\mathbf{R}_{-t} \leftarrow v$. When the end of the row is reached, the last bumped label becomes the *evacuated* label.

Note that, in the remainder of the article, we only use the operations of Definition 4.5 when the resulting tableau is a Littlewood-Richardson tableau.

Thus, in Figure 12, we performed $\mathbf{H}_{5,-3} \leftarrow 7$ and the evacuated label was placed in x. in Figure 13, we performed $\mathbf{H}_{5,-4} \leftarrow 7$ and the evacuated label was placed in x.

We are now ready to state the algorithm that will provide the desired injection of tableaux.

Algorithm 4.6. Input a tableau $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$.

(Initializing step) If $\eta_1 = 1$ and $\beta_i < i$, set q equal to the label in the first column and in row $i - \beta_i$ from the bottom in T.

(1) If $\eta_1 = i$, place label $i+1$ in box x.

(2) If $\eta_1 < i$,

- **(a)** If $\beta_i = i$ or $2 \leqslant \eta_1$ or the leftmost label in the last row of T is not i, perform $\mathbf{R}_{-1} \leftarrow (i+1)$. Place the evacuated label in box x.

(b) If $\beta_i < i$ and $\eta_1 = 1$ and the leftmost label in the last row of T equals i,

($\mathbf{b_1}$) if $q = \beta_i$, perform $\mathbf{H}_{i-1,-(i-\beta_i)}$.

($\mathbf{b_2}$) if $q = \beta_i + 1$, perform $\mathbf{H}_{i-1,-(i-\beta_i+1)}$.

Place the evacuated label in box x.

Output tableau T'.

Theorem 4.7. *Suppose $\nu = \beta$ is a partition of type 1 with $\beta_1 = i$ and such that $(i^{i-1}) \subseteq \beta$. Then, for each partition $\lambda = (\eta^c + \gamma, \sigma)$ the above algorithm provides an injection*

$$LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta') \hookrightarrow LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))').$$

Proof. Let $\lambda = (\eta^c + \gamma, \sigma)$ be a partition such that $c^{\lambda}_{\nu'\nu^c} \neq 0$. We input a tableau $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into the algorithm. The algorithm produces a tableau T'. By construction, the shape of T' is $\gamma \times \sigma \times ((\beta + (1))/\eta)^*$ and the type is $(\beta + (1))'$. We show that T' is a SSYT whose reverse reading word is a lattice permutation and the map $T \to T'$ is an injection from $LR(\gamma\times\sigma\times(\beta/\eta)^*, \beta')$ into $LR(\gamma\times\sigma\times((\beta+(1))/\eta)^*, (\beta+(1))')$.

If $\eta_1 = i$, then

$$\gamma \times \sigma \times ((\beta + (1))/\eta)^* = \gamma \times \sigma \times (\beta/\eta)^* \times (1).$$

The algorithm simply places the label $i + 1$ in x. Clearly $T' \in LR(\gamma \times \sigma \times (\beta/\eta)^* \times (1), (\beta + (1))')$ and $T \to T'$ is an injection from $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$. In particular, this settles the case $i = 1$.

Now suppose that $\eta_1 < i$. First consider the case in which $i = 2$. We have $\eta_1 = 1$ and $\beta_2 = 2$. Moreover, since the first column of T has length β'_2 and the only available labels are 1 and 2, the partition β must be of the form $(2^2, 1^k)$ and the only box in the last row of T has label 2. The algorithm replaces this label by 3 and places a label 2 in the box x. The resulting tableau is a SSYT. Since there are two labels 2 in the superstandard tableau of β', the reverse reading word of T' is a lattice permutation. If T_1, T_2 are two tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ such that $T_1 \neq T_2$, then they differ in a row higher than the last row. Since the algorithm only modifies the last row of a tableau in $LR(\gamma\times\sigma\times(\beta/\eta)^*, \beta')$, it maps T_1, T_2 to different tableaux. Therefore, $T \to T'$ is an injection. Together with the algorithm for the case $\eta_1 = 2$, this settles the case $i = 2$. For the remainder of the proof, we assume that $i \geqslant 3$.

If $\beta_i = i$ or $2 \leqslant \eta_1 < i$, the algorithm performs $\mathbf{R}_{-1} \leftarrow (i+1)$ and places the evacuated label in box x. This certainly creates a SSYT. In this case, it is impossible for T to contain all labels i in the last row. Therefore, in the reverse reading word of T', there is an i before $i + 1$ and thus the word is a lattice permutation. It is straightforward to see that, in this case, for two different tableaux $T_1, T_2 \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ the algorithm produces different tableaux in $LR(\gamma\times\sigma\times((\beta+(1))/\eta)^*, (\beta+(1))')$. Hence, we obtain an injection from $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$.

For the remainder of the proof assume that $\beta_i < i$ and $\eta_1 = 1$. Then, as noted in Remark 4.1, $\beta_{i+1} = 1$. Let T be a tableau in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with at least one label i in a row higher than the last. Then the algorithm performs $\mathbf{R}_{-1} \leftarrow (i+1)$ and places the evacuated label in box x. As in the preceding paragraph, the resulting tableau is a SSYT whose reverse reading word is a lattice permutation. Moreover, if the algorithm is applied to two different tableaux $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ such that each tableaux has at least one label i in a row higher than the last, the resulting tableaux are distinct. If T contains all labels i in the last row, the algorithm performs $\mathbf{H}_{i-1,-t}$, where

$$t = \begin{cases} i - \beta_i & \text{if } q = \beta_i \\ i - \beta_i + 1 & \text{if } q = \beta_i + 1, \end{cases}$$

and places the evacuated label in box x.

In the resulting tableau T', there is a label i directly above the label $i+1$. By the properties in Lemma 4.2 and Remark 4.3, T' is a SSTY. If $\beta_i \neq 2$, the label placed in x will always be β_i and, since we have one more label β_i than the number of labels $\beta_i + 1$, the lattice permutation condition is not violated. If $\beta_i = 2$, the label placed in x is either 2 or 1. In either case the lattice permutation condition is not violated. Thus $T' \in LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$.

Next, we will show that, if we apply the algorithm to two different tableaux in $T_1, T_2 \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with all labels i in the last row, we obtain two distinct tableaux T_1', T_2'. Consider first the case in which $\beta_i = i - 1$. If $i > 3$, the second to last row in both T_1 and T_2 must be filled with labels $i-1$. Therefore, in this case, T_1 and T_2 must differ in a row higher than the last two. Since the algorithm only affects labels in the last two rows, the resulting tableaux must be distinct. If $i = 3$, the second to last row in T_1, T_2 is $\boxed{1}\boxed{2}$ or $\boxed{2}\boxed{2}$. If T_1 and T_2 differ in the second row from the bottom, T_1' and T_2' differ in box x. If T_1 and T_2 differ in a row higher than the last two, T_1' and T_2' are distinct. Now assume that $\beta_i < i - 1$, $i \geqslant 4$. By property (a) in Lemma 4.2, T_1 and T_2 cannot differ in the last $i - \beta_i - 1$ rows. By property (d) in Lemma 4.2, if T_1 and T_2 differ in row $i - \beta_i$ from the bottom, then they can only differ in the label q (the leftmost label in this row). Suppose T_1 has $q = \beta_i$ while T_2 has $q = \beta_i + 1$. Then, the i-th column of T_1' is filled, from top to bottom, with the labels $1, 2, \ldots, \beta_i, \beta_i + 2, \beta_i + 3, \ldots i + 1$. The i-th column of T_2' is filled, from top to bottom, with the labels $1, 2, \ldots, \beta_i - 1, \beta_i + 1, \beta_i + 2, \ldots i + 1$. (Recall that in T_1' and T_2' the first column is the column of x.) Therefore, $T_1' \neq T_2'$. Now suppose T_1, T_2 differ only in rows higher than row $i - \beta_i$ from the bottom. If they both have $q = \beta_i$, the algorithm only affects the last $i - \beta_i$ rows of such a tableau and therefore $T_1' \neq T_2'$. If both T_1, T_2 have $q = \beta_i + 1$, the algorithm only affects the last $i - \beta_i + 1$ rows of such a tableau. Since tableau T_1 and T_2 do not differ in column $i - 1$, and the algorithm shifts up in column $i - 1$ and left in row $i - \beta_i + 1$ from the bottom, we must have $T_1' \neq T_2'$.

Finally suppose that T_1, T_2 are tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ such that T_1 has all labels i in the last row and T_2 does not. Then the rightmost label in the second to last row is equal to i in T_1' and is equal to $i - 1$ in T_2'.

Thus, if $\beta_i < i$ and $\eta_1 = 1$, the algorithm produces an injection from $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$. ☐

Corollary 4.8. *If ν is a partition of type 1 such that $(\nu_1^{\nu_1-1}) \subseteq \nu$, then ν is covered by $\nu+(1)$.*

In the remainder of the subsection, we show that if ν is of type 1 containing the rectangle $(\nu_1^{\nu_1-1})$ and μ does not satisfy (C2), then $s_{\mu'}s_{\mu^c} - s_{\nu'}s_{\nu^c}$ is not Schur-positive.

Theorem 4.9. *Suppose that $\nu = \beta$ is a partition of type 1 with $\beta_1 = i$ and such that $(i^{i-1}) \subseteq \beta$. Let η be the partition obtained by reordering the parts of $\beta/(\beta\cap\beta')$ to form a partition. Then, η is the smallest partition in lexicographic order such that $c_{\beta'\beta^c}^{(\eta^c+\gamma,\sigma)} \neq 0$ for some γ,σ.*

Proof. Let β be as in the statement of the theorem. If $i = 1, 2$, or 3, then $\beta\cap\beta'$ consists of the first i rows of β. If $i \geqslant 4$, then $(i^{i-1}, 2, 1) \subseteq \beta$, $(i+1, i, (i-1)^{i-2}) \subseteq \beta'$, and $(i^2, (i-1)^{i-3}, 2) \subseteq \beta\cap\beta'$. Since $\ell(\beta\cap\beta') = i$, the skew Young diagram $\beta/(\beta\cap\beta')$ contains rows $i+1, i+2, \ldots, \ell(\beta)$ of β. In addition, since $(i^{i-1}) \subseteq \beta$, if $i \geqslant 4$ and $\beta_i \leqslant i-2$, it will also contain the last box in rows $\beta_i+1, \beta_i+2, \ldots, i-1$ of β. Thus, the partition η obtained by reordering the the parts of $\beta/(\beta\cap\beta')$ to form a partition is

$$\eta = (\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)}, \xi),$$

where $\xi = \emptyset$ if $i = 1, 2, 3$ and, if $i \geqslant 4$, then

$$\xi = \begin{cases} \emptyset & \text{if } \ i-1 \leqslant \beta_i \leqslant i \\ & \\ (1^{i-1-\beta_i}) & \text{if } \ 2 \leqslant \beta_i < i-1. \end{cases}$$

We will prove that there exists γ and σ such that $c_{\beta'\beta^c}^{(\eta^c+\gamma,\sigma)} \neq 0$ and that if $\tilde{\eta}$ is smaller than η in lexicographic order, then $c_{\beta'\beta^c}^{(\tilde{\eta}^c+\gamma,\sigma)} = 0$ for all γ and σ. To prove that $c_{\beta'\beta^c}^{(\eta^c+\gamma,\sigma)} \neq 0$ we will produce a tableau in $LR(\gamma\times\sigma\times(\beta/\eta)^*, \beta')$.

Case 1: $\beta_i = i$ or $i-1$. Then $\xi = \emptyset$. Notice that for $i \leqslant 3$, $\beta_i = i$ or $i-1$. In this case we have $\eta = (\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)})$. Let $\gamma = \eta'$ and $\sigma = \emptyset$ and T be the tableau of shape $\eta'\times(\beta/\eta)^*$ and type β' filled so that the boxes of η' form a superstandard tableau and each column of $(\beta/\eta)^*$ is filled with consecutive integers from top to bottom starting with 1. Then, $T \in LR(\eta'\times(\beta/\eta)^*, \beta')$ and $c_{\beta'\beta^c}^{\eta^c+\eta'} \neq 0$.

Case 2: $2 \leqslant \beta_i < i-1$ for $i \geqslant 4$. Then $\xi = (1^{i-1-\beta_i}) \neq \emptyset$. In this case we have $\eta = (\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)}, 1^{i-1-\beta_i})$. Again let $\gamma = \eta'$ and $\sigma = \emptyset$ and T be the tableau of shape $\eta'\times(\beta/\eta)^*$ and type β' filled so that the boxes of η' form a superstandard tableau and the columns of $(\beta/\eta)^*$ are filled as follows from top to bottom: the first column is filled with $1, 2, \ldots, \beta_i, \beta_i+2, \ldots, i$, the next $i-\beta_i-1$ columns are filled with $2, 3, \ldots, i$, the next $\beta_i - 1$ columns are filled with $1, 2, \ldots, i$, and the right-most column is filled with $1, 2, \ldots, \beta_i+1$. Then, $T \in LR(\eta'\times(\beta/\eta)^*, \beta')$ and $c_{\beta'\beta^c}^{\eta^c+\eta'} \neq 0$.

If θ is any partition such that $c_{\beta'\beta^c}^{\theta^c+\gamma,\sigma} \neq 0$, *i.e.*, $LR(\gamma\times\sigma\times(\beta/\theta)^*, \beta') \neq \emptyset$, then there must be a tableau T of shape $\gamma\times\sigma\times(\beta/\theta)^*$ and type β'. Therefore, since we have exactly i distinct labels, the columns of $(\beta/\theta)^*$ must have length at most i. Then, $\theta_j \geqslant \beta_{i+j}$ and

1	1	1	1	1	1	1	1	1	1	**1**	**1**
2	2	2	2	2	2	2	2	2	**2**		
3	3	3	3	3	3	3	3	**3**			
4	4	4	4	4	4	4	4				
5	5	5	5	5	5	5	**5**				
6	6	6	6	6	6	6					
7	7	7	7	7	7	7					
8	**8**	**8**	**8**	**8**	**8**	**8**					

Figure 14: Example of the bottom longest possible horizontal strip (highlighted) of the superstandard tableau of $\beta' = (12, 10, 9, 8, 8, 7, 7, 7)$. These are the only possible values that can occur in the last row of $(\beta/\tilde{\eta})^*$.

therefore $(\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)}) \subseteq \theta$. Thus, if $\beta_i = i$ or $i-1$, then η is the smallest partition in lexicographic order such that $c^{(\eta^c+\gamma,\sigma)}_{\beta'\beta^c} \neq 0$ for some γ, σ.

Now suppose that $2 \leqslant \beta_i < i-1$, $\eta = (\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)}, 1^{i-1-\beta_i})$, and $\tilde{\eta}$ is strictly less than η in lexicographic order. Suppose that $c^{\tilde{\eta}^c+\tilde{\gamma},\tilde{\sigma}}_{\beta'\beta^c} \neq 0$ for some $\tilde{\gamma}, \tilde{\sigma}$. As previously noted, $(\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)}) \subseteq \tilde{\eta}$. Hence, $\tilde{\eta} = (\beta_{i+1}, \beta_{i+2}, \ldots, \beta_{\ell(\beta)}, 1^k)$ with $k < i-1-\beta_i$. Let $T \in LR(\tilde{\gamma} \times \tilde{\sigma} \times (\beta/\tilde{\eta})^*, \beta')$. Then, the ith column of T (*i.e.*, the last column of $(\beta/\tilde{\eta})^*$) has length $i-k > \beta_i + 1$. For easier reference, we divide the first i columns of tableau T into three parts: T_1 refers to the first $i-\beta_i$ columns of T, T_2 refers to the next $\beta_i - 1$ columns (*i.e.*, columns $i-\beta_i+1$ to $i-1$) of T, and T_3 refers to the ith column of T.

We will now show by contradiction that with the given labels we cannot fill $\tilde{\gamma} \times \tilde{\sigma} \times (\beta/\tilde{\eta})^*$ to form an LR tableau of type β'.

As explained in Remark 2.1, the rows of T form horizontal strips in the superstandard tableau of β'. The bottom longest possible horizontal strip of T in the superstandard tableau of β', read from right to left, ends in

$$\ldots, \beta_i, \underbrace{i, i, \ldots, i}_{i-1 \text{ times}}.$$

See Figure 14 for an example in which $\beta = (8^7, 5, 3, 2, 1, 1)$.

It follows that a label j such that $\beta_i + 1 \leqslant j \leqslant i-1$ cannot occur in the last row. In addition, since the first column in T has length $i-1$ and $2 \leqslant \beta_i < i-1$, then β_i cannot be in the first column of T in the bottom $i-\beta_i-1$ boxes. These boxes are filled, from the bottom up with $i, i-1, \ldots, \beta_i+2$ by the properties of LR tableaux. The conditions for a SSYT force the last row of T_1 to be filled with i, the second to last row of T_1 to be filled with $i-1$, and so on until row $i-\beta_i-1$ from the bottom in T_1 must be filled with β_i+2. Therefore, each label $i, i-1, \ldots, \beta_i+2$ has been used $i-\beta_i$ times in T_1. The columns of T_2 have length i and are thus filled from top to bottom with $1, 2, \ldots, i$. Thus, each label has been used $\beta_i - 1$ times in T_2. Therefore, labels $\beta_i+2, \beta_i+3, \ldots, i-1, i$ have each been used $i-1$ times in T_1 and T_2. Then, only labels $1, 2, \ldots, \beta_i+1$ can be used in T_3. This is impossible since the length of the ith column of T is at least β_i+2.

Therefore, $c^{\tilde{\eta}^c+\tilde{\gamma},\tilde{\sigma}}_{\beta'\beta^c} = 0$ for all $\tilde{\gamma}, \tilde{\sigma}$ and η is the smallest partition in lexicographic order such that $c^{(\eta^c+\gamma,\sigma)}_{\beta'\beta^c} \neq 0$ for some γ, σ.

□

Corollary 4.10. *If ν is a partition of type 1 such that $(\nu_1^{\nu_1-1}) \subseteq \nu$, then ν is not covered by $(\nu, 1)$, the partition obtained from ν by adding a box at the end of its first column.*

Proof. Let $\nu = \beta$ and η be as in the statement of the Theorem 4.9. Notice that $(\nu, 1)$ is also a partition satisfying the hypothesis of Theorem 4.9. Moreover, we have $\beta \cap \beta' = (\beta, 1) \cap (\beta, 1)'$. Then, by Theorem 4.9, the smallest partition η in lexicographic order so that $c^{\eta^c+\eta'}_{\nu'\nu^c} \neq 0$ is $\eta = \beta/\beta \cap \beta'$. However, if $\nu = (\beta, 1)$, the smallest partition $\hat{\eta}$ in lexicographic order so that $c^{\hat{\eta}^c+\hat{\eta}'}_{(\nu,1)'(\nu,1)^c} \neq 0$ is $\hat{\eta} = (\eta, 1)$. Therefore, by Theorem 4.9, $c^{\eta^c+\eta'}_{\nu'\nu^c} \neq 0$, but $c^{\eta^c+\eta'}_{(\nu,1)'(\nu,1)^c} = 0$. □

Theorem 4.11. *If ν is a partition of type 1 such that $(\nu_1^{\nu_1-1}) \subseteq \nu$, then ν is not covered by any partition μ with $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$, and $\mu \neq \nu + (1)$.*

Proof. Let $\nu = \beta$ be as in the statement of the theorem. If $\beta_1 = i$ and $i \geqslant 2$, we have $(i^{i-1}, 2, 1) \subseteq \beta$. Let μ be a partition such that $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$, and $\mu \neq \nu + (1)$. Then, μ is obtained from ν by adding a box at the end of its pth column, for some $1 \leqslant p \leqslant i$. If $p = 1$, we are in the case of Corollary 4.10 and the statement of the theorem is true. Suppose $p \geqslant 2$. Consider the partition $\tilde{\nu} = \nu^{(p-1)}$ obtained from ν by removing its first $p - 1$ columns. Then, by Corollary 4.10, $\tilde{\nu}$ is not covered by $(\tilde{\nu}, 1)$. Let $j = \tilde{\nu}_1$. Since $\tilde{\nu}_j = j$, it follows from the proof of Theorem 4.9 that $\tilde{\eta} = (\tilde{\nu}_{j+1}, \tilde{\nu}_{j+2} \dots, \tilde{\nu}_{\ell(\tilde{\nu})})$ is such that $c^{\tilde{\eta}^c+\tilde{\eta}'}_{\tilde{\nu}'\tilde{\nu}^c} = 1$ and $c^{\tilde{\eta}^c+\tilde{\eta}'}_{(\tilde{\nu},1)'(\tilde{\nu},1)^c} = 0$.

Let ε be the partition obtained by adjoining $\tilde{\eta}$ to the right of the first $p-1$ columns of ν, *i.e.*, $\varepsilon = (\nu'_1, \nu'_2, \dots, \nu'_{p-1}, (\tilde{\eta})')'$. Then, $\tilde{\nu}/\tilde{\eta} = \nu/\varepsilon$ and $|LR(\tilde{\eta}' \times (\tilde{\nu}/\tilde{\eta})^*, \tilde{\nu}')| = |LR(\varepsilon' \times (\nu/\varepsilon)^*, \nu')|$. Similarly, $|LR(\tilde{\eta}' \times ((\tilde{\nu}, 1)/\tilde{\eta})^*, (\tilde{\nu}, 1)')| = |LR(\varepsilon' \times (\mu/\varepsilon)^*, \mu')|$. Therefore, $c^{\varepsilon^c+\varepsilon'}_{\nu'\nu^c} = 1$ and $c^{\varepsilon^c+\varepsilon'}_{\mu'\mu^c} = 0$. Thus, μ does not cover ν. □

4.2 Partitions of type 1 with small width

Let $\nu = \beta$ be a partition of type 1. In this section we show that, if $\nu_1 \leqslant 4$, then Conjecture 3.6 holds.

Proposition 4.12. *If ν is a partition of type 1 with $\nu_1 = 1, 2,$ or 3, then Conjecture 3.6 is true. Moreover, if ν is a partition of type 1 with $\nu_1 = 4$ and $\nu'_4 \geqslant 3$, then Conjecture 3.6 is true.*

Proof. In each of these cases ν contains $(\nu_1^{\nu_1-1})$ and the conjecture follows from Theorems 4.8 and 4.11. □

We now consider partitions $\nu = \beta$ of type 1 with $\nu_1 = 4$ and $\nu'_4 = 2$. To show that ν is covered by $\nu + (1)$, for each partition $\lambda = (\eta^c + \gamma, \sigma)$ such that $c^{\lambda}_{\nu'\nu^c} \neq 0$, we give an algorithm that assigns to each $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ a distinct tableau in $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$. The labels in the superstandard tableau of ν' are $1, 2, 3, 4$ and there are exactly two labels 4.

In addition to the operations of Definition 4.5, we also use the operation *insert u and bump along the path* $(k_1, j_1) \to (k_2, j_2) \to (k_3, j_3) \to \cdots \to (k_t, j_t)$ defined as follows: u bumps the label in position (k_1, j_1) and for $1 \leqslant s \leqslant t - 1$, the label in position (k_s, j_s) bumps the label in position (k_{s+1}, j_{s+1}). The last bumped label becomes the *evacuated* label.

By box in position $(-k, j)$, we mean the box in the kth row from the bottom and the jth column.

Algorithm 4.13. Input a tableau $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$.

(Initializing step) If $1 \leqslant \eta_1 \leqslant 2$, set q equal to the label position $(-1, 1)$ in T, set b equal to the label in position $(-2, 2)$ in T, and set p equal to the label in position $(-2, 1)$ in T.

(1) If $\eta_1 = 4$, place label 5 in x.

(2) If $\eta_1 < 4$,

- **(a)** If there are less than two labels 4 in the last row of T, perform $\mathbf{R}_{-1} \leftarrow 5$ and place the evacuated label in x.
- **(b)** If there are two labels 4 in the last row of T,
  - **(b1)** If $q = 2$ or ($q = 3$ and $\beta_4 = 3$), insert 5 along the path $(-1, 3) \to (-2, 3) \to (-2, 2) \to (-1, 1)$.
  - **(b2)** If $q = 3$ and $\beta_4 = 2$, perform $\mathbf{H}_{3,-b} \leftarrow 5$.
  - **(b3)** If $q = 4$ and $\eta_2 = 2$, perform $\mathbf{H}_{2,-2} \leftarrow 5$.
  - **(b4)** If $q = 4$ and $\eta_2 = 1$ and ($\beta_4 = 3$ or $p < 3$), replace label in position $(-1, 2)$ by 5 and perform $\mathbf{R}_{-2} \leftarrow 4$.
  - **(b5)** If $q = 4$ and $\eta_2 = 1$ and $\beta_4 = 2$ and $p = 3$, replace label in position $(-1, 2)$ by 5 and perform $\mathbf{H}_{3,-3} \leftarrow 4$.

Place the evacuated label in x.

Output tableau T'.

Theorem 4.14. *Suppose $\nu = \beta$ is a partition of type 1 with $\beta_1 = 4$ and $\beta'_4 = 2$. Then, for each partition $\lambda = (\eta^c + \gamma, \sigma)$ such that $c^{\lambda}_{\nu'\nu^c} \neq 0$, the above algorithm provides an injection*

$$LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta') \hookrightarrow LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))').$$

Proof. Let $\lambda = (\eta^c+\gamma, \sigma)$, with $\eta \subseteq \beta$ and $|\gamma|+|\sigma| = |\eta|$, be a partition such that $c^{\lambda}_{\beta'\beta^c} \neq 0$. We input a tableau $T \in LR(\gamma\times\sigma\times(\beta/\eta)^*, \beta')$ into the algorithm. The algorithm produces a tableau T'. By construction, the shape of the tableau T' is $\gamma \times \sigma \times ((\beta + (1))/\eta)^*$ and the type is $(\beta + (1))'$. We show that T' is a SSYT whose reverse reading word is a lattice permutation and the map obtained is an injection from $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$.

If $\eta_1 = 4$, by the argument of Theorem 4.7 when $\eta_1 = i$, T' is a SSYT whose reverse reading word is a lattice permutation and $T \to T'$ is an injection from $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$.

If $\eta_1 < 4$ and T is a tableau with less than two labels 4 in the last row, the algorithm shifts all labels in the last row of T one position to the left. The leftmost label in this row is placed into the box x and label 5 is inserted into the rightmost box in the last row of T. As explained in the proof of Theorem 4.7, the resulting tableau is in $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$ and applying the algorithm to two different tableaux $T_1, T_2 \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with less than two labels 4 in the last row produces different tableaux in $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$.

We now consider the case in which $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ is a tableau with both labels 4 in the last row. We will explicitly perform the algorithm on all such tableaux to see that the resulting tableau T' is in $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$. Moreover, this will allow us to conclude that, if $\eta_1 < 4$, the algorithm produces an injection from $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ into $LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$.

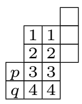
Note that in all diagrams below we only show the relevant labels. Since the algorithm does not affect γ and σ, we only show the $(\beta/\eta)^*$ part of T. It is possible for boxes without labels to not be part of the skew shape (the reader should imagine such boxes as not being part of the tableau) and columns of boxes without labels could be higher than shown, as long as the resulting diagram is a skew shape. Labels that are not marked do not change when the algorithm is performed.

If both labels 4 are in the last row of T, we have $\eta_1 = 1$ or 2. Since $\beta'_1 \geqslant 5$, we also have $\eta_2 \geqslant 1$. We consider several cases depending on the shape of η and β.

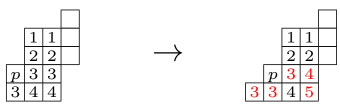
Case **(I) $\boldsymbol{\eta_1 = 1}$**. Then the last row of T is $q\,4\,4$. Since the third column of T ends in 4 and $\beta'_2 \geqslant 4$, we must have $\beta'_2 = 4$ (*i.e.*, $\beta_5 = 1$). Therefore, the number of labels 2 is equal to four and we have at least five labels 1. Moreover, $\beta_4 = 2$ or 3.

We consider these two cases separately.

(a) Case $\beta_4 = 3$. Then the first three columns of T are



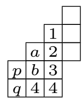
Since the number of labels 2 equals the number of labels 3, $q \neq 2$. Thus $q = 3$ and $p = 1$ or 2. The result of applying the algorithm is shown below. T' is clearly an LR tableau.



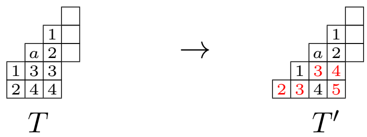
T $\qquad\qquad$ T'

If $\eta_1 = 1$, $\beta_4 = 3$, and T_1, T_2 are distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with both labels 4 in the last row, then T_1 and T_2 can only differ in the label p or in a row higher than the second row from the bottom. Since the algorithm only affects the last two rows but does not change the label p, the resulting tableaux T_1' and T_2' are different.

(b) Case $\beta_4 = 2$. Then the first three columns of T are

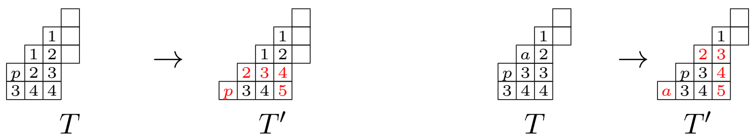


If $q = 2$, then $p = 1$. We must also have $b = 3$ because the number of labels 2 is one more than the number of labels 3. Then,



T' is clearly an LR tableau.

Suppose now that $q = 3$. Below we give the result of the algorithm depending on whether $b = 2$ or 3. Note that if $b = 3$, there are no labels 3 in a row higher than the second to last row in T.

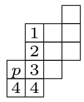


If T is an LR tableau, one can easily verify that in each case T' is also an LR tableau.

Suppose $\eta_1 = 1$, $\beta_4 = 2$, and T_1, T_2 are distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with both labels 4 in the last row. If T_1, T_2 differ in the label q, then T_1', T_2' either differ in the label in box $(-3, 4)$ or else in the label in box $(-2, 2)$ (recall that the first column of each T_1', T_2' consists of only the box x). If T_1 and T_2 have the same label q, and $q = 2$, then they differ only in rows higher than the second to last row. Since the algorithm in this case only affects the last two rows, the resulting tableaux are different. If T_1 and T_2 have the same label q, and $q = 3$, then T_1' and T_2' either differ in the label in box $(-3, 4)$ or else in at least one of the following labels: the label in box x or the label in box $(-2, 2)$ or a label in a row higher than the third to last row.

Case (II) $\eta_1 = 2$. Then, the last row of T consists entirely of 4. Because the type β' allows only for the labels $1, 2, 3$ and 4, we must have $\beta_4 = 2$ or 3. We distinguish between the two cases below.

(a) Case $\beta_4 = 3$. Then the first two columns of T are

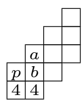


Below is the result of the algorithm depending on whether $\eta_2 = 1$ or 2.

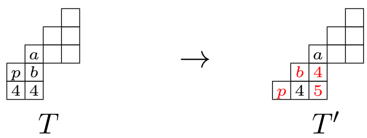
In each case T' is clearly an LR tableau.

Suppose $\eta_1 = 2$, $\eta_2 = 1$ or 2, $\beta_4 = 3$, and T_1, T_2 are distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with both labels 4 in the last row. Then T_1', T_2' differ in the label in box x or in a label in a row higher than the second to last row.

(b) Case $\beta_4 = 2$. Then the first two columns of T are



If $\eta_2 = 2$, we have



and T' is clearly an LR tableau.

Suppose $\eta_1 = \eta_2 = 2$, $\beta_4 = 2$, and T_1, T_2 are distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with both labels 4 in the last row. Then T_1', T_2' will differ in at least one of the following labels: the label in box x, or the label in box $(-2, 2)$, or a label in a row higher than the second to last row.

If $\eta_2 = 1$, then we must have $\beta_5 = 1$. Below we show the result of the algorithm depending on whether $p \neq 3$ or $p = 3$.

If T is an LR tableau, one can easily verify that in each case T' is also an LR tableau.

Suppose $\eta_1 = 2$, $\eta_2 = 1$, $\beta_4 = 2$, and T_1, T_2 are two distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with both labels 4 in the last row. Then T_1', T_2' will differ in the label in box $(-3, 4)$ or else in one of the following labels: the label in x, or the label in box $(-2, 2)$, or the label in box $(-3, 3)$, or a label in a row higher than the third to last row.

The discussion above shows that if $T \in LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$, the algorithm produces a tableau $T' \in LR(\gamma \times \sigma \times ((\beta + (1))/\eta)^*, (\beta + (1))')$. Moreover, we showed that if T_1, T_2 are two distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with both labels 4 in the last row, then the resulting tableaux T_1', T_2' are distinct. We also showed that if T_1, T_2 are two distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ with less than two labels 4 in the last row, then the resulting tableaux T_1', T_2' are distinct.

Now suppose T_1, T_2 are two distinct tableaux in $LR(\gamma \times \sigma \times (\beta/\eta)^*, \beta')$ such that T_1 has both labels 4 in the last row and T_2 has at most one label 4 in the last row. Then, if $\eta_1 = 1$ or $\eta_1 = \eta_2 = 2$, T_1' has a label 4 directly above the label 5 while T_2' has a label 2 or 3 directly above the label 5. If $\eta_1 = 2$ and $\eta_2 = 1$, then in T_1' the label in x is less than or equal to the label in box $(-2, 2)$ while in T_2' the label in x is strictly greater than the label in box $(-2, 2)$. Therefore, in either case $T_1' \neq T_2'$. $\square$

Corollary 4.15. *If ν is a partition of type 1 with $\nu_1 = 4$ and $\nu_4' = 2$, then $\nu + (1)$ covers ν.*

Proposition 4.16. *If ν is a partition of type 1 with $\nu_1 = 4$ and $\nu_4' = 2$, then ν is not covered by any partition μ such that $\nu \subset \mu$, $|\mu| = |\nu| + 1$, and $\mu \neq \nu + (1)$.*

Proof. To prove the proposition, it is enough to show that $(\nu, 1)$ does not cover ν by finding a partition η such that $c^{\eta^c+\eta'}_{\nu'\nu^c} \neq 0$ and $c^{\eta^c+\eta'}_{(\nu,1)'(\nu,1)^c} = 0$. If we can show this, the argument in the proof of Theorem 4.11 shows that ν is not covered by any partition other than $\nu + (1)$.

Let $\eta = (\beta_5, \beta_6, \ldots, \beta_{\ell(\beta)}, \xi)$, where $\xi = \emptyset$ if $\beta_3' = 3$, and $\xi = (1)$ if $\beta_3' \geqslant 4$. (Note that η is obtained by reordering the parts of $\beta/(\beta \cap \beta')$ to form a partition.) One can easily check that $c^{\eta^c+\eta'}_{\beta'\beta^c} = 1$ and $c^{\eta^c+\eta'}_{(\beta,1)'(\beta,1)^c} = 0$. $\square$

The results above lead to the following theorem.

Theorem 4.17. *If ν is a partition of type 1 with $\nu_1 = 4$, then Conjecture 3.6 is true.*

The proof of Theorem 4.14 gives a glimpse into the difficulty of proving the conjecture in general by matching Littlewood-Richardson tableaux.

There are certain similarities between the algorithm above (when $\nu_1 = 4$ and $\nu'_4 = 2$) and the algorithm given at the beginning of this section for the case when ν is a partition of type 1 such that $(\nu_1^{\nu_1 - 1}) \subseteq \nu$. In both cases, if all labels i (*i.e.*, highest label) are in the last row of a tableau T, one creates T' by bumping labels up and then to the left (and possibly southwest) according to certain rules. Otherwise, one just bumps labels to the left in the last row. One might ask why such an algorithm would not work for partitions of higher width. Below we give an example where the natural generalization of the algorithm above fails.

Example 4.18. Consider the partition $\nu = (5, 5, 5, 3, 3, 1)$ of type 1 with $\nu_1 = 5$. Let $\eta = (1^3)$, $\gamma = (2, 1)$, and $\sigma = \emptyset$. In Figure 15 we show a particular tableau $T \in LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu')$.

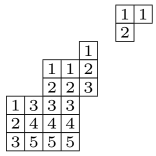


Figure 15: A tableau T for which the corresponding T' is not LR

Notice that all labels 5 are in the last row. We could try to perform insertions as in the algorithm for $\nu_1 = 4$. Neither $\mathbf{H}_{4,-2} \leftarrow 6$ nor $\mathbf{H}_{4,-3} \leftarrow 6$ produce SSYT. We could also try up-left-SW bumping paths as in **(b1)** of the algorithm. If we insert 6 along the path $(-1, 4) \to (-2, 4) \to (-2, 3) \to (-2, 2) \to (-1, 1)$ and place the evacuated box in x, the reverse reading word of the resulting tableau is not a lattice permutation. If we insert 6 along the path $(-1, 4) \to (-2, 4) \to (-3, 4) \to (-3, 3) \to (-3, 2) \to (-2, 1)$ and place the evacuated box in x, the resulting tableau is not a SSYT.

While one can figure out a rule for assigning a tableau T' to the specific tableau T in Example 4.18, ($\mathbf{H}_{4,-5} \leftarrow 6$ works), this just confirms the need for a myriad of rules for specific cases as the size of ν grows.

4.3 Failure of Schur-positivity

The *only if* part of Conjecture 3.6 for partitions of type 1 states that, if ν is of type 1 and μ is a partition such that $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$, $\mu \neq \nu + (1)$, then $s_{\nu'\nu^c} - s_{\mu'\mu^c}$ is not Schur-positive.

Using Maple and Sage, we checked the only if part of the conjecture for many partitions ν of size up to 25. As the size of ν grows, so does the size of the square in which one takes the complement. The computations become expensive very quickly. The largest example we could examine, thanks to a computation by Nicolas Thiéry, was for a partition $\nu \vdash 35$ with the complement taken in the square (14^{14}).

Based on the proofs of Theorem 4.9 and Proposition 4.16, and many explicit examples, we make the following conjecture.

Lex-minimality Conjecture: *If $\nu = \beta$ is of type 1, the smallest partition η in lexicographic order such that $c^{\eta^c+\gamma,\sigma}_{\beta'\beta^c} \neq 0$, for some γ, σ is given by reordering the rows of $\beta/(\beta \cap \beta')$ to form a partition. Moreover, for this η, we have $c^{\eta^c+\eta'}_{\beta'\beta^c} = 1$.*

The Lex-minimality conjecture implies that a partition ν of type 1 is not covered by $(\nu, 1)$. Using the same argument as in Theorem 4.11, this would be enough to show that a partition ν of type 1 could only be covered by a partition μ satisfying (C2).

In the remainder of this section we prove the failure of Schur-positivity of the expression $s_{\nu'\nu^c} - s_{\mu'\mu^c}$ for partitions ν of type 1 satisfying a symmetry condition and partitions μ such that $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$, $\mu \neq \nu + (1)$.

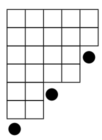
Recall that an outer corner of a Young diagram, and thus a partition, is a position (outside the diagram) such that, if we add a box in that position we still obtain a Young diagram.

Following the notation of Lemma 3.9, we denote by $\nu^{(k)}_{(s)}$ the diagram obtained from ν by removing its first k columns and its first s rows. If $k = 0$ or $s = 0$, the notation means that no columns, respectively rows, were removed from ν.

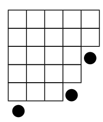
Definition 4.19. A partition ν is called *corner-symmetric* if for every outer corner (k, j) of ν with $k > 1$, there exists $0 \leqslant t < k - 1$ such that $\nu^{(j-1)}_{(t)}$ is self-conjugate.

Definition 4.20. Equivalently, ν is *corner-symmetric* if for every outer corner (k, j) of ν with $k > 1$, there exists a partition $\eta_{k,j}$ such that $\nu/\eta_{k,j}$ is a self-conjugate (non-skew) partition and $\nu/\eta_{k,j}$ contains box $(k-1, j)$ of ν but does not contain any box from the first $j - 1$ columns of ν.

The partition in Figure 16 (a) is corner-symmetric. It has three outer corners (marked with •) in positions $(7, 1)$, $(5, 3)$, and $(3, 5)$. We can take $\eta_{7,1} = (5, 5)$, $\eta_{5,3} = (5, 5, 2, 2, 2, 2, 2)$, and $\eta_{3,5} = (5, 4, 4, 4, 2, 2)$. The partition in Figure 16 (b) is not corner-symmetric. For the outer corner $(6, 1)$ there is no partition $\eta_{6,1}$ satisfying Definition 4.20.



(a)



(b)

Figure 16: (a) Corner-symmetric and (b) non-corner-symmetric partitions

Theorem 4.21. *If ν is corner-symmetric and μ is such that $|\mu| = |\nu| + 1$, $\nu \subseteq \mu$, $\mu \neq \nu + (1)$, then μ does not cover ν.*

Proof. Let $\mu \neq \nu + (1)$ be a partition such that $|\mu| = |\nu| + 1$ and $\nu \subseteq \mu$. Then μ is obtained from ν by adding a box at an outer corner (k, j) with $k > 1$. Let $\eta_{k,j}$ be the partition from Definition 4.20 (which is not necessarily unique) and let $\lambda = \eta^c_{k,j} + \eta'_{k,j}$. Then, $\lambda/\nu^c = \eta'_{k,j} \times (\nu/\eta_{k,j})^*$ and the partition λ appears in $s_{\nu'}s_{\nu^c}$ with multiplicity one but does not appear in $s_{\mu'}s_{\mu^c}$. Therefore μ does not cover ν. □

Corollary 4.22. *A corner-symmetric partition of type* 1 *is covered by at most one partition.*

5 Partitions of type 2

Recall that ν is a partition of type 2 if $\nu = \beta + (s^s) + \alpha$, where $s \geqslant 1$, $\alpha \neq \emptyset$. Moreover, by Proposition 3.10 (a), this decomposition is unique. For the rest of the section we set $i = \beta_1$. Then, $(i^{s+2}, i-1, i-2, \ldots, 1) \subseteq \beta$. Consider the partition $\mu = \beta + (s^s, 1) + \alpha$. In this section, we prove that, for some particular partitions ν of type 2, μ is the only partition covering ν.

Theorem 5.1. *Suppose ν is a partition of type* 2 *with $\beta = \emptyset$, i.e., $\nu = (s^s) + \alpha$, with $s \geqslant 1$, $\alpha \neq \emptyset$. Then Conjecture 3.6 is true.*

Proof. By Proposition 3.11 (c), if ν is as in the statement of the theorem, then ν' is of type 1. Moreover, $\nu'_1 = s$ and $(s^{s-1}) \subseteq \nu'$. The result now follows from Corollary 4.8, Theorem 4.11, and Proposition 3.2. □

Before considering additional cases of partitions of type 2, we introduce more notation and prove two helpful lemmas.

Let $\lambda = (\eta^c + \gamma, \sigma)$ be a partition such that $c^{\lambda}_{\nu'\nu^c} \neq 0$. As we did with partitions of type 1 in the previous section, we attempt to show that $c^{\lambda}_{\mu'\mu^c} \geqslant c^{\lambda}_{\nu'\nu^c}$ by matching each tableau $T \in LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu')$ with a distinct tableau $T' \in LR(\gamma \times \sigma \times (\mu/\eta)^*, \mu')$.

The shape μ' is obtained from ν' by adding a box in position $(i + 1, s + 1)$. Thus, the type μ' provides us with the same labels as the type ν' plus an additional label $i + 1$. In $\gamma \times \sigma \times (\mu/\eta)^*$, we denote the box $(\gamma \times \sigma \times (\mu/\eta)^*)/(\gamma \times \sigma \times (\nu/\eta)^*)$ by x. As before, we refer to x as the added box. We refer to the row of x in T or T' as r_x. Thus, r_x is the $(s + 1)$-st row from the bottom. In T, we denote by a the label directly to the right of x (if it exists) and by b the label directly below x (if it exists). Depending on the shape η, there might not be a box directly below or directly to the right of x.

Example 5.2. In Figure 17, we consider the diagram in which $\nu = \beta + (s^s) + \alpha$ with $\beta = (2^5, 1)$, $s = 3$ and $\alpha = (2, 1, 1)$. Moreover, $\eta = (3, 1)$, $\gamma = \emptyset$ and $\sigma = (2, 2)$. The figure shows $\gamma \times \sigma \times (\nu/\eta)^*$ with the place where the added box, x, would be placed. The label in the box to the right of x is a and the label in the box below x is b.

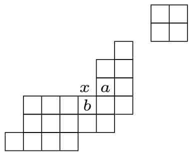


Figure 17: $\gamma \times \sigma \times (\beta/\eta)^*$ with the relevant boxes marked

If $\eta = (2^4)$, $\sigma = (4,4)$, and $\gamma = \emptyset$, there is no box to the right of x in $\gamma \times \sigma \times (\nu/\eta)^*$. If $\eta = (3,3,3,2)$, $\sigma = (4,4,3)$, and $\gamma = \emptyset$, there is no box directly below or directly to the right of x in $\gamma \times \sigma \times (\nu/\eta)^*$.

Lemma 5.3. *Suppose $\nu = \beta + (s^s) + \alpha$ is of type 2 with $\beta_1 = i$ and $\lambda = (\eta^c + \gamma, \sigma)$ is such that $c^\lambda_{\nu'\nu^c} \neq 0$. Let $T \in LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu')$. If $1 \leqslant j \leqslant s$ and label $i + j$ appears in row r of T, then r is at most j rows under r_x. In particular, the lowest row in which label $i + 1$ can appear is the s-th row from the bottom.*

Proof. Since T is of type ν', each label $i + j$, $1 \leqslant j \leqslant s$, appears exactly s times in T. Therefore, the lattice permutation condition forces T to contain label $i + j + 1$, $1 \leqslant j \leqslant s - 1$, in a row below the row of the last label $i + j$. The statement of the lemma follows from the fact that there are at most s rows under r_x. □

Note that it is possible for all labels $i + j$ (for some $1 \leqslant j \leqslant s$) to be above r_x.

Corollary 5.4. *Suppose $\nu = \beta + (s^s) + \alpha$ is of type 2 with $\beta_1 = i$. For $\lambda = (\eta^c + \gamma, \sigma)$ such that $c^\lambda_{\nu'\nu^c} \neq 0$, let $T \in LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu')$. In T there are at least s labels i occurring in row r_x or a row above it.*

Lemma 5.5. *Suppose $\nu = \beta + (s^s) + \alpha$ is of type 2 with $\beta_1 = i$. For $\lambda = (\eta^c + \gamma, \sigma)$ such that $c^\lambda_{\nu'\nu^c} \neq 0$, let $T \in LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu')$. In T there is at least one label $i + 1$ in row r_x or a row above it.*

Proof. If all $i + 1$ labels appear in the first row after r_x, then all labels $i + s$ appear in the last row of T. Since $\alpha \neq \emptyset$, there is a label $i + s + 1$ and it would have to be placed below the last row. □

Theorem 5.6. *Suppose $\nu = \beta + (s^s) + \alpha$ is a partition of type 2 and $\beta_1 = i = 1$. Then, $\mu = \beta + (s^s, 1) + \alpha$ covers ν.*

Proof. Suppose $\lambda = (\eta^c + \gamma, \sigma)$ is such that $\eta \subseteq \nu$ and $|\gamma| + |\sigma| = |\eta|$ and $c^\lambda_{\nu'\nu^c} \neq 0$. Let $T \in LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu')$. We will assign to T a distinct tableau $T' \in LR(\gamma \times \sigma \times (\mu/\eta)^*, \mu')$. The tableau T contains s labels 2 whereas the tableau T' contains $s + 1$ labels 2.

Since the partition (1^{s+2}) is contained in β, if there is a box in T directly to the right of x, then $a \geqslant 2$. If there is no label 1 in a row below r_x, then either there is no box directly below x or $b > 2$. (If $b = 2$ and all labels 1 appear in row r_x or a row above it, then all s boxes in the row of b and to the left of b must be filled with label 2. This contradicts Lemma 5.5.)

The tableau T' is constructed from T in the following way. If there is no label 1 in a row below r_x, place a label 2 into the box x. Otherwise, find the highest, rightmost label

1 below r_x and replace it by 2. Then, place a label 1 into the box x. By Lemma 5.3, there is no label 2 in a row lower than the first row below r_x. Thus, the tableau T' obtained above is a SSYT. By construction, its reverse reading word is a lattice permutation.

It is straightforward to see that this construction gives an injection

$$LR(\gamma \times \sigma \times (\nu/\eta)^*, \nu') \hookrightarrow LR(\gamma \times \sigma \times (\mu/\eta)^*, \mu').$$

□

Recall that $\nu_{(s)}$ denotes the partition obtained from ν by removing its first s rows.

Proposition 5.7. *Let $\nu = \beta + (s^s) + \alpha$ be a partition of type 2 with $\beta_1 = i$ and such that $\nu_{(s)}$ is corner-symmetric. Then ν is not covered by any partition μ with $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$, and $\mu \neq \beta + (s^s, 1) + \alpha$.*

Proof. If μ is obtained from ν by adding a box at an outer corner (k, j) at the end of the jth column with $1 \leqslant j \leqslant i$, let $\eta_{k,j}$ be the partition obtained by applying Definition 4.20 to $\nu_{(s)}$, and let $\eta = (\nu_1, \nu_2, \ldots, \nu_s, \eta_{k,j})$. Then, $c^{\eta^c+\eta'}_{\nu'\nu^c} = 1$ and $c^{\eta^c+\eta'}_{\mu'\mu^c} = 0$. Therefore, μ does not cover ν.

Suppose now that μ is obtained from ν by adding a box at an outer corner at the end of the jth column with $i + s + 1 \leqslant j \leqslant \nu_1$, *i.e.*, the added box is at the end of one of the columns of α. We denote by $\tilde{\alpha}$ the partition obtained from α by adding this box. We will show that ν' is not covered by μ'. Then, Proposition 3.2 will imply that ν is not covered by μ. Consider $\nu' = (\beta', s^s, \alpha')$ and let $\chi = (\nu')_{(i)}$. Then, $\chi = (s^s, \alpha')$ is a partition of type 1 with $\chi'_{\chi_1} \geqslant \chi_1$. By Proposition 4.11, χ is not covered by $\tilde{\chi} = (s^s, \tilde{\alpha}')$. Moreover, the proof of Proposition 4.11 gives a partition η for which $c^{\eta^c+\eta'}_{\chi'\chi^c} = 1$ and $c^{\eta^c+\eta'}_{\tilde{\chi}'\tilde{\chi}^c} = 0$. Let $\tilde{\eta} = (\beta', \eta)$. It is easy to see that $c^{\tilde{\eta}^c+\tilde{\eta}'}_{\nu(\nu')^c} = 1$ and $c^{\tilde{\eta}^c+\tilde{\eta}'}_{\mu(\mu')^c} = 0$. Therefore μ' does not cover ν'. □

Corollary 5.8. *Suppose $\nu = \beta + (s^s) + \alpha$ is a partition of type 2 and $\beta_1 = i = 1$. Then ν is not covered by any partition μ with $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$, and $\mu \neq \beta + (s^s, 1) + \alpha$.*

We remark that the proof of Proposition 5.7 shows that if we can prove that a partition of type 1 is not covered by any partition not satisfying (C2), then it follows that a partition of type 2 is not covered by any partition not satisfying (C2). Therefore, establishing the Lex-minimality conjecture would prove the *only if* part of Conjecture 3.6 for partitions of type 1 and 2.

6 Non-type 1 or 2 partitions

In this section we consider some partitions ν such that neither ν nor ν' is of type 1 or 2, *i.e.*, partitions not satisfying (C1). The next proposition sheds light on the necessity of $\alpha \neq \emptyset$ in the definition of type 2 partitions.

Proposition 6.1. *Let $\nu = \beta + (s^s)$ with $s \geqslant 1$, $\beta_1 = i \geqslant 0$ and such that β contains $(i^{s+2}, i-1, i-2, \ldots, 1)$. Then, ν is not covered by $\mu = \beta + (s^s, 1)$. Moreover, ν is not covered by $\mu = \beta + (s^s) + (1)$ either.*

Proof. If $\mu = \beta + (s^s, 1)$ or $\mu = \beta + (s^s) + (1)$, then $\lambda = \beta^c + \beta'$ appears with multiplicity 1 in $s_{\nu'} s_{\nu^c}$ but does not appear in $s_{\mu'} s_{\mu^c}$. (Note that if $\beta = \emptyset$, then $\lambda = (m^m)$ is the square in which we take the complement.) □

Corollary 6.2. *Let $\nu = (s^s)$, $s \geqslant 1$. Then ν is not covered by any partition μ.*

Proof. This follows from Proposition 6.1 with $\beta = \emptyset$ and the fact that the only partitions μ such that $\nu \subseteq \mu$, $|\mu| = |\nu| + 1$ are $\mu = (s^s, 1)$ and $\mu = (s^s) + (1)$. □

We now consider partitions ν such that both ν and ν' are corner symmetric. We show that they are neither of type 1 nor of type 2 and are not covered by any partition μ with $\nu \subseteq \mu$ and $|\mu| = |\nu| + 1$.

Proposition 6.3. *If ν is a partition such that both ν and ν' are corner-symmetric, then ν is neither of type 1 nor of type 2.*

Proof. First we show that, if ν of type 2, then ν is not corner-symmetric. Let ν be a partition of type 2, $\nu = \beta + (s^s) + \alpha$, with $\beta_1 = i$, and consider the outer corner $(s+1, i+1)$. Let $0 \leqslant t \leqslant s-1$. Because of the shape of ν, the partition $\nu_{(t)}^{(i)}$ has last part at least s and length at most s. Since $\alpha \neq \emptyset$, $\nu_{(t)}^{(i)}$ is not self-conjugate. Then, by Definition 4.19, ν is not corner-symmetric.

Next, we show that, if ν is of type 1, then ν' is not corner-symmetric. Suppose that $\nu = \beta$ is of type 1with $\beta_1 = i$, and let $j = \beta'_i$. Then ν' has an outer corner at $(i+1, 1)$. For each $0 \leqslant t < i$, the partition $\nu'_{(t)}$ has last part equal to j. To be self-conjugate, $\nu'_{(t)}$, and therefore ν', must end in j rows of length j, *i.e.*, $\nu'_{i-j+1} = j$. However, since ν is of type 1, by Lemma 3.8 (ii), we have $\nu'_{i-j+1} \geqslant i - (i-j+1) + 2 = j+1$. Then, by Definition 4.19, ν' is not corner-symmetric. □

The next proposition follows directly from Theorem 4.21 and Proposition 3.2.

Proposition 6.4. *If ν is a partition such that both ν and ν' are corner-symmetric, then ν is not covered by any partition μ such that $\nu \subseteq \mu$ and $|\mu| = |\nu| + 1$.*

Corollary 6.5. *If ν is self-conjugate, then ν is not covered by any partition μ such that $\mu \subseteq \nu$ and $|\mu| = |\nu| + 1$.*

Notice that a partition ν such that both ν and ν' are corner-symmetric is not necessarily self-conjugate. For example $\nu = (6, 5, 5, 5, 4, 4, 3, 3, 3)$ is corner-symmetric and so is ν'. However, ν is not self-conjugate.

Corollary 6.6. *The staircase partition $\delta_i = (i, i-1, \ldots, 2, 1)$, with $i \geqslant 1$, is not covered by any partition.*

7 Final remarks

The proof of the main conjecture in complete generality will likely involve some new combinatorial ideas. As the proofs of Theorem 4.7 and Theorem 4.14 suggest, trying to match tableaux to show Schur-positivity seems to involve complicated insertion algorithms. The larger the width of the shape β, the more exception rules need to be introduced. In an attempt to prove the main conjecture we have also tried (unsuccessfully) using the Jacobi-Trudi identity (see [15]) and, separately, the Plücker relations (see [6]).

We note that if ν and μ are such that $\nu \subseteq \mu$ and $|\mu| = |\nu| + 1$, then McNamara's necessary conditions for Schur-positivity [10] for $s_{\mu'}s_{\mu^c} - s_{\nu'}s_{\nu^c}$ are satisfied. The particular cases of Conjecture 3.6 proved in this article provide another set of examples showing that the conditions are not sufficient.

References

[1] F. Bergeron, R. Biagioli and M. Rosas, *Inequalities between Littlewood-Richardson coefficients*, J. Comb. Theory, Series A. Vol. 113, issue 4, (2006), 567- 590.

[2] F. Bergeron and P. McNamara, *Some positive differences of products of Schur functions*, arXiv:math/0412289 [math.CO], Dec 2004.

[3] S. Fomin, and W. Fulton, C.-K Li, and Y.-T. Poon, *Eigenvalues, singular values, and Littlewood-Richardson coefficients*, Amer. J. Math. 127 (2005), 101-127.

[4] R. C. King, T. A. Welsh, S. J. van Willigenburg, *Schur positivity of skew Schur function differences and applications to ribbons and Schubert classes*, J. Algebraic Combin. 28 (2008), no. 1, 139 - 167.

[5] A. N. Kirillov. *An invitation to the generalized saturation conjecture*. Publ. Res. Inst. Math. Sci., 40(4):1147–1239, 2004.

[6] M. Kleber, *Plücker Relations on Schur Functions*, J. Algebraic Combin. 13 (2001), no. 2, 199–211.

[7] T. Lam and A. Postnikov and P. Pylyavskyy, *Schur positivity and Schur log-concavity*, Amer. J. Math., 129 (2007), 1611-1622.

[8] D. E. Littlewood, *The Kronecker product of symmetric group representations*, J. London Math. Soc. 31 (1956), 89-93.

[9] I. G. Macdonald, "Symmetric Functions and Hall polynomials", sec. ed. *Oxford University Press*, 1995.

[10] P. W. McNamara, *Necessary conditions for Schur-positivity*. J. Algebraic Combin. 28 (2008), no. 4, 495-507.

[11] P. W. McNamara and S. van Willigenburg, *Positivity results on ribbon Schur funtions diffrences*, European J. Combin. 30, (2009), no. 5, 1352-2369.

[12] P. McNamara and S. van Willigenburg, *Maximal supports and Schur-positivity among connected skew shapes*. European J. Combin. 33 (2012), no. 6, 1190–1206.

[13] A. Okounkov, *Log-concavity of multiplicities with applications to characters of* $U(\infty)$, Adv. Math., 127 no. 2 (1997), 258-282.

[14] T. Scharf, J-Y Thibon, and B.G. Wybourne. *Powers of the Vandermonde determinant and the quantum Hall effect*, J. Phys. A: Math. Gen. 27 (1994) 4211-4219.

[15] R. Stanley. *Enumerative Combinatorics* Vol 2, *Cambridge Univ. Press*, 1999.