Arc-transitive dihedral regular covers of cubic graphs

Jicheng Ma*

Department of Mathematics & KLDAIP
Chongqing University of Arts and Sciences
Chongqing 402160, China

ma_jicheng@hotmail.com

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Abstract

A regular covering projection is called *dihedral* or *abelian* if the covering transformation group is dihedral or abelian. A lot of work has been done with regard to the classification of arc-transitive abelian (or elementary abelian, or cyclic) covers of symmetric graphs. In this paper, we investigate arc-transitive dihedral regular covers of symmetric (arc-transitive) cubic graphs. In particular, we classify all arc-transitive dihedral regular covers of K_4, $K_{3,3}$, the 3-cube Q_3 and the Petersen graph.

Keywords: Arc-transitive graph; Regular cover; Dihedral cover; Cubic graph

1 Introduction

Covering techniques are known to be a useful tool in algebraic and topological graph theory. Application of these techniques has resulted in many important examples and classification of certain families of graphs with particular symmetry properties. For example, Djoković used graph covers to prove that there exist infinitely many 5-arc-transitive cubic graphs, as elementary abelian covers of Tutte's 8-cage.

Recently, quite a lot of attention has been paid to the classification of arc-transitive covers of symmetric graphs. Approaches have involved voltage graph techniques (see [9]) and universal group methods (see [3]). In most cases, the group of covering transformation is either cyclic or elementary abelian, or more generally abelian. These methods have been

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successfully applied in the classification of arc-transitive elementary abelian or abelian covers of symmetric cubic graphs, such as the complete graph K_4, the complete bipartite graph $K_{3,3}$, the 3-cube graph Q_3, the Petersen graph and the Heawood graph and so on.

In this paper, we are aiming to extend our research on arc-transitive abelian covers to non-abelian covers which is harder and has not been previously considered. We begin with some further background in Section 2, and determine the arc-transitive cyclic regular covers of the Möbius-Kantor graph and the Desargues graph in Sections 3 and 4, respectively. In Section 5, we deal with dihedral covers, and give a complete classification of arc-transitive dihedral covers of K_4, $K_{3,3}$, Q_3 and the Petersen graph.

2 Preliminaries

Throughout this paper, all the graphs are finite and simple. A *covering projection* is defined as a graph homomorphism $p : \tilde{X} \to X$ which is surjective and locally bijective, which means that the restriction $p : N(\tilde{v}) \to N(v)$ is a bijection whenever $\tilde{v}$ is a vertex of $\tilde{X}$ such that $p(\tilde{v}) = v \in V(X)$. We call X the base graph, $\tilde{X}$ a covering graph. A covering projection $p : \tilde{X} \to X$ is called *regular* if there exists a semi-regular subgroup N of the automorphism group $\mathrm{Aut}(\tilde{X})$ of $\tilde{X}$ such that the quotient graph $\tilde{X}/N$ (with vertices taken as the orbits of N, and two vertices adjacent whenever there exists an edge between these two N-orbits) is isomorphic to X. In that case we call $\tilde{X}$ a *regular cover* of X. The regular covering projection is called *dihedral* (or *cyclic*) if N is a dihedral (or cyclic) group. Similarly, we say a regular covering projection is *abelian* (or *elementary abelian*) when the group N is *abelian* (or *elementary abelian*).

Let $p : \tilde{X} \to X$ be a covering projection, and suppose α and β are automorphisms of X and $\tilde{X}$ such that $\alpha \circ p = p \circ \beta$, that is, such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \overset{\beta}{\longrightarrow} & \tilde{X} \\ p \downarrow & & \downarrow p \\ \tilde{X} & \underset{\alpha}{\longrightarrow} & X \end{array}$$

Then we say that α *lifts* along p to β, and β *projects* to α, and also we call β a *lift* of α, and α a *projection* of β. Note that α is uniquely determined by β, but β is not generally determined by α. The set of all lifts of a given $\alpha \in \mathrm{Aut}(X)$ is denoted by $\tilde{\alpha}$. If every automorphism of a subgroup G of $\mathrm{Aut}(X)$ lifts, then $\bigcup_{\alpha \in G} \tilde{\alpha}$ is a subgroup of $\mathrm{Aut}(\tilde{X})$, called the *lift* of G.

In particular, the lift of the identity subgroup of $\mathrm{Aut}(X)$ (or equivalently, the subgroup of all automorphisms of $\tilde{X}$ that project to the identity automorphism of X) is called the *group of covering transformations*, or *voltage group*, and is sometimes denoted by $\mathrm{CT}(p)$. The normalizer of $\mathrm{CT}(p)$ in $\mathrm{Aut}(\tilde{X})$ projects to the largest subgroup of $\mathrm{Aut}(X)$ that lifts. Hence in particular, if the latter subgroup is A, say, then the lift of A has a normal subgroup $\mathrm{CT}(p)$ with quotient isomorphic to A.

Two regular covering projections $p\colon Y \to X$ and $p'\colon Y' \to X$ are called *isomorphic* if there exist graph isomorphism $\tilde{\theta}\colon Y \to Y'$ and graph automorphism $\theta : X \to X$ such that $\theta p = p'\tilde{\theta}$. In particular, isomorphic covering projections p and p' are called *equivalent*, if θ is the trivial automorphism. Similarly, two regular covers Y and Y' are called *equivalent* if the two regular covering projections p and p' are equivalent. Usually, regular covers are studied up to equivalence.

For every symmetric cubic graph, we now know that the automorphism group is a quotient of one of seven finitely-presented groups, which can be listed as $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ and G_5, and presented as follows (see [5, 4])

$$G_1 = \langle\, h, a \mid h^3 = a^2 = 1 \,\rangle\ ;$$

$$G_2^1 = \langle\, h, p, a \mid h^3 = p^2 = a^2 = 1,\ php = h^{-1},\ a^{-1}pa = p \,\rangle;$$

$$G_2^2 = \langle\, h, p, a \mid h^3 = p^2 = 1,\ a^2 = p,\ php = h^{-1},\ a^{-1}pa = p \,\rangle;$$

$$G_3 = \langle\, h, p, q, a \mid h^3 = p^2 = q^2 = a^2 = 1,\ pq = qp,\ php = h,\ qhq = h^{-1},\ a^{-1}pa = q \,\rangle;$$

$$\begin{aligned} G_4^1 = \langle\, h, p, q, r, a \mid\ & h^3 = p^2 = q^2 = r^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ (qr)^2 = p,\\ & h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle; \end{aligned}$$

$$\begin{aligned} G_4^2 = \langle\, h, p, q, r, a \mid\ & h^3 = p^2 = q^2 = r^2 = 1,\ a^2 = p,\ pq = qp,\ pr = rp,\ (qr)^2 = p,\\ & h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle; \end{aligned}$$

$$\begin{aligned} G_5 = \langle\, h, p, q, r, s, a \mid\ & h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ ps = sp,\\ & qr = rq,\ qs = sq,\ (rs)^2 = pq,\ h^{-1}ph = p,\ h^{-1}qh = r,\\ & h^{-1}rh = pqr,\ shs = h^{-1},\ a^{-1}pa = q,\ a^{-1}ra = s \,\rangle. \end{aligned}$$

If a finite group G acts as an s-arc-regular group of automorphisms of a cubic graph X, then G is a smooth quotient of G_s or G_s^i, where $i = 1$ or 2 depending on whether or not the group contains an involution a that reverses an arc (in the cases where s is even). (Note, 'smooth' here means that the orders of the generators are preserved.) Let $\mathcal{U}$ be either G_s or G_s^i, then G is a smooth quotient $\mathcal{U}/K$ of $\mathcal{U}$ by some torsion-free normal subgroup K. If $\tilde{X}$ is a regular cover of X admitting a group action of the same type, then there exists a normal subgroup L of $\mathcal{U}$ contained in K, with $\mathcal{U}/L$ being the corresponding group automorphisms of $\tilde{X}$. Then the group $\mathcal{U}/L$ is an extension of the covering group K/L by the given group $G = \mathcal{U}/K$. In order to find all cyclic covers, we need to find all possibilities for L such that K/L is cyclic. The presentation of K can be found using Reidemeister-Schreier Theory, or by use of the `Rewrite` command in Magma [1]. In the cases we will consider, K is a free abelian group of finite rank d, namely the Betti number of the base graph X, with some basis $\{w_1, w_2, \cdots, w_d\}$. Algebraic or computational techniques can be applied to find the actions by conjugation of the generators of $\mathcal{U}$ on the generators of K. And these actions induce linear transformations of the free abelian group K. Equivalently, a d-dimensional matrix representation of the group $G = \mathcal{U}/K$ can be given. Therefore, in order to find all the cyclic covers, we need to find all the G-invariant subgroups L of rank $d-1$, equivalently we need to find all the $(d-1)$-dimensional representation of G.

More details of Conder and the author's universal group method can be seen in [3] and [8]. Here we introduce some computational techniques that involves using MAGMA. To find all finite cyclic regular covers with cyclic covering groups of exponent m, we may consider the action of G by conjugation on the generators of $K/K^{(m)}$, where $K^{(m)}$ is the characteristic subgroup of K generated by the mth powers of all elements of K. Since $K/K^{(m)}$ is G-invariant, we use MAGMA to construct a finite group $K/K^{(m)} \rtimes G$ which is the extension group of G by $K/K^{(m)}$. Note that this can be done, since both $K/K^{(m)}$ and G are finitely-presented subgroups of $\mathcal{U}$. With a finite group stored in MAGMA we can use the commands `NormalSubgroups` and `meet` to find all the subgroups L of $K/K^{(m)}$ which are normal in $K/K^{(m)} \rtimes G$. Note that the 'type' of group $K/K^{(m)} \rtimes G$ may not work for using the above commands in MAGMA, then what one needs to do is using the *double coset graph* construction method (more details can be seen in [3, Section 2]) to transform it into a permutation group (namely, an arc-transitive group of automorphisms). This method works successfully for 'small' integer m. Generally, if $m = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ is the prime-power factorisation of m with distinct primes p_i, then the factor K/L is a direct product of its Sylow subgroups. It follows that we need only consider the G-invariant subgroups of prime-power index in $K/K^{(m)}$.

Once all the possibilities for L have been found, we can determine additional information, such as uniqueness up to isomorphism and arc-transitivity of the covering graphs.

3 Arc-transitive cyclic regular covers of the Möbius-Kantor graph

In this section, we classify all the arc-transitive cyclic covering graphs of the Möbius-Kantor graph $GP(8,3)$. The automorphism group of $GP(8,3)$ is isomorphic to $\mathrm{GL}(2,3) \rtimes C_2$ and acts 2-arc-regularly on the arcs. There are two other 1-arc-regular subgroups $\mathrm{GL}(2,3)$ and $\mathrm{SL}(2,3) \rtimes C_2$.

Take the group G_2^1, with presentation $G = \langle h, a, p \mid h^3 = p^2 = a^2 = (ph)^2 = [a,p] = 1 \rangle$. The group G_2^1 has two normal subgroups of index 96, both with quotient $\mathrm{GL}(2,3) \rtimes C_2$, but these are interchanged by the outer automorphism that takes the three generators h, a and p to h, ap and p respectively, so without loss of generality we can take either one of them.

We will take the one that is contained in the subgroup $G_1 = \langle h, a \rangle$; this is a normal subgroup N of index 48 in G_1 with $G_1/N \cong \mathrm{GL}(2,3)$.

Using the `Rewrite` command in MAGMA, we find that the subgroup N is free of rank 9, on generators

$$\begin{aligned}
w_1 &= (h^{-1}ahaha)^2 & w_2 &= (h^{-1}ahah^{-1}a)^2 \\
w_3 &= (h^{-1}ah^{-1}aha)^2 & w_4 &= (hahah^{-1}a)^2 \\
w_5 &= (hah^{-1}aha)^2 & w_6 &= (hah^{-1}ah^{-1}a)^2 \\
w_7 &= h^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ah^{-1} & w_8 &= ahahahahah^{-1}ah^{-1}ah^{-1}ah^{-1} \\
w_9 &= ah^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ah^{-1}a
\end{aligned}$$

Easy calculations show that the generators h, a and p act by conjugation as below: (Note that the actions of generator ap is just the composition of a and p.)

$$\begin{array}{lllllllll}
{w_1}^a & = & {w_3}^{-1} & {w_1}^h & = & {w_7}^{-1} & {w_1}^p & = & w_6 \\
{w_2}^a & = & {w_5}^{-1} & {w_2}^h & = & {w_2}^{-1}w_4 & {w_2}^p & = & w_5 \\
{w_3}^a & = & {w_1}^{-1} & {w_3}^h & = & {w_3}^{-1}w_5 & {w_3}^p & = & w_4 \\
{w_4}^a & = & {w_6}^{-1} & {w_4}^h & = & {w_2}^{-1} & {w_4}^p & = & w_3 \\
{w_5}^a & = & {w_2}^{-1} & {w_5}^h & = & {w_3}^{-1} & {w_5}^p & = & w_2 \\
{w_6}^a & = & {w_4}^{-1} & {w_6}^h & = & {w_1}^{-1} & {w_6}^p & = & w_1 \\
{w_7}^a & = & w_9 & {w_7}^h & = & w_6 & {w_7}^p & = & {w_7}^{-1} \\
{w_8}^a & = & {w_8}^{-1} & {w_8}^h & = & w_1w_9 & {w_8}^p & = & {w_7}^{-1}w_8w_9 \\
{w_9}^a & = & w_7 & {w_9}^h & = & {w_1}^{-1}w_7{w_8}^{-1}{w_9}^{-1} & {w_9}^p & = & {w_9}^{-1}
\end{array}$$

Now take the quotient G_1/N' where N' is the derived subgroup of N, which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^9$ by the group $G_1/N \cong \mathrm{GL}(2,3)$, and replace the generators h, a and all w_i by their images in this group. Also let K denote the subgroup N/N', and let G be G_1/N'. Then, in particular, G is an extension of $\mathrm{GL}(2,3)$ by $\mathbb{Z}^9$.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^9$ as follows:

$$a \mapsto \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},$$

$$h \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1
\end{pmatrix},$$

and

$$p \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

These matrices generate a group isomorphic to $\mathrm{Aut}(GP(8,3))$, with the first two generating a subgroup isomorphic to $\mathrm{GL}(2,3)$; and the first and the product of the first and the third generating a subgroup isomorphic to $\mathrm{SL}(2,3) \rtimes C_2$. Note that the matrices of orders 3, 2 and 8 representing h, a and ha have traces $-3, -1$ and 1, respectively.

Next, the character table of the group $\mathrm{GL}(2,3)$ is given in Table 1, with γ being the zeroes of the polynomial $t^2 + 2t + 3$.

Table 1: The character table of the group $\mathrm{GL}(2,3)$

Element order	1	2	2	3	4	6	8	8
Class size	1	1	12	8	6	8	6	6
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	1	-1	-1
χ_3	2	2	0	-1	2	-1	0	0
χ_4	2	-2	0	-1	0	1	γ	$-\gamma$
χ_5	2	-2	0	-1	0	1	$-\gamma$	γ
χ_6	3	3	1	0	-1	0	-1	-1
χ_7	3	3	-1	0	-1	0	1	1
χ_6	4	-4	0	1	0	-1	0	0

By inspecting traces, we see that the character of the 9-dimensional representation of $\mathrm{GL}(2,3)$ over $\mathbb{Q}$ associated with the above action of $G = \langle h, a \rangle$ on K is the character $\chi_3 + \chi_4 + \chi_5 + \chi_7$, which is reducible to the sum of χ_3, $\chi_4 + \chi_5$ and χ_7, which are characters of three irreducible 2-, 4- and 3-dimensional representations over the rational field $\mathbb{Q}$. Especially, the 4-dimensional representation is reducible to two 2-dimensional irreducible representations over fields containing zeroes of the polynomial $t^2 + 2t + 3$. Therefore, for any prime k other than 2 and 3, there is no G-invariant subgroup of rank 8.

For prime integers 2 and 3, with the help of MAGMA by using the commands `GModule` and `Submodules` for matrix groups over prime fields, there is a unique G-invariant subgroup U of rank 8, which is generated by $w_1w_9, w_2w_9^{-1}, w_3w_9^{-1}, w_4w_9, w_5w_9, w_6w_9^{-1}, w_7w_9^{-1}$ and w_8 when $k = 3$. In particular, for prime integer 3 and exponent 3^2, using the `NormalSubgroups` command in MAGMA we can show that there is no normal subgroup of rank 8 and exponent 9.

Next, by calculation, we can see that the subgroup U is also p-invariant for the additional generator p. Hence the full automorphism group $\mathrm{GL}(2,3) \rtimes C_2$ can be lifted, and the covering graph is at least 2-arc-transitive.

Now we consider the lifting of $\mathrm{SL}(2,3) \rtimes C_2$, which is an 1-arc-regular subgroup generated by the cosets Nh and Nap of the quotient G_1/N.

With the help of MAGMA, a reduced character table of group $SL(2,3) \rtimes C_2$ is given in Table 2 where δ is a primitive 3rd root and ϕ is a primitive 4th root.

Note that the traces of matrices induced by h, ap, hap, $(haph)^2$ and $(haph)^3$ of orders 3, 2, 12, 6 and 4, respectively, are equal to -3, -3, 1, 1 and 1.

Table 2: The character table of the group $\mathrm{SL}(2,3) \rtimes C_2$

Element order	1	2	3	4	6	12
Class size	1	6	4	6	4	4
χ_1	1	1	1	1	1	1
χ_4	1	-1	δ	1	$-1-\delta$	$-\delta$
χ_5	1	-1	$-1-\delta$	1	δ	$1+\delta$
χ_7	2	0	-1	0	1	$-\phi$
χ_8	2	0	-1	0	1	ϕ
χ_{14}	3	-1	0	-1	0	0

Hence we can see that the character of the 9-dimensinal representation of $\mathrm{SL}(2,3) \rtimes C_2$ over $\mathbb{Q}$ associated with the above action of $\langle h, ap \rangle$ on K is $\chi_4+\chi_5+\chi_7+\chi_8+\chi_{14}$, which is reducible to the sum of $\chi_4+\chi_5$, $\chi_7+\chi_8$ and χ_{14}. In particular, if there exists a primitive 3rd root δ, then $\chi_4+\chi_5$ is reducible to χ_4 and χ_5; if there exists a primitive 4th root ϕ, then $\chi_7+\chi_8$ is reducible to χ_7 and χ_8.

Therefore, we can see that for prime $k \notin \{2,3\}$, if a primitive 3rd root δ exists, K is a direct sum of four G-invariant subgroups of ranks 1, 1, 3 and 4; and if a primitive 4th root ϕ exists, K is a direct sum of four G-invariant subgroups of ranks 2, 2, 2 and 3. (Note that, here we are only interested in the existence of G-invariant subgroups of rank 8.) In fact, with the help of MAGMA, if δ exists then these four G-invariant subgroups are generated by $\{w_1w_2^{\delta^2}w_3^{\delta}w_4w_5^{\delta^2}w_6^{\delta}w_7w_8^{\delta^2}w_9^{-\delta}\}$, $\{w_1w_2^{\delta}w_3^{\delta^2}w_4w_5^{\delta}w_6^{\delta^2}w_7w_8^{\delta}w_9^{-\delta^2}\}$, $\{w_1w_6^{-1}w_7^{-1},\ w_2w_4^{-1}w_5w_6^{-1}w_8^{-1},\ w_3w_4^{-1}w_9\}$, and $\{w_1w_6w_8, w_3w_4w_8,\ w_2w_4^{-1}w_5^{-1}w_6w_9^{-1},\ w_7w_9^{-1}\}$, respectively. Especially, by the conjugation action of generator a, we can see that a maps $w_1w_2^{\delta^2}w_3^{\delta}w_4w_5^{\delta^2}w_6^{\delta}w_7w_8^{\delta^2}w_9^{-\delta}$ to $(w_1w_2^{\delta}w_3^{\delta^2}w_4w_5^{\delta}w_6^{\delta^2}w_7w_8^{\delta}w_9^{-\delta^2})^{\delta}$. Hence these two cyclic covers are isomorphic, and we now take them as one cover. However, no cyclic covers exist for $k \notin \{2,3\}$ when lifting the subgroup $\mathrm{GL}(2,3)$. Therefore, the cyclic covering graph is 1-arc-transitive but not 2-arc-transitive.

By [4, Proposition 2.3], this covering graph can not be 3-arc-transitive. Suppose this graph is 4-arc-transitive, then it is a cover of the Heawood graph, by [4, Proposition 3.2]. Thus the cyclic covering group must be of order 7^e for some e. The full automorphism group of the cyclic cover is of order $16 \cdot 8 \cdot 7^e$. Since the 4-arc-transitive symmetric cubic graphs have vertex-stabilizer S_4, hence the order of cyclic covering graph is equal to $16 \cdot 8 \cdot 7^e/24$ which is not an integer, contradiction. Therefore the cyclic covering graph cannot be 4-arc-transitive. Finally, again by [4, Proposition 3.4], if the covering graph is 5-arc-transitive, then it is a cover of the Biggs-Conway graph which is of order 2352. Similar to the above argument, the covering graph cannot be 5-arc-transitive. Therefore, these cyclic covering graphs are 1-arc-transitive.

For either k equal to 2 or 3, similar to the lifting of $\mathrm{GL}(2,3)$ with the help of MAGMA, there is only one G-invariant subgroup of rank 8 of exponent 3. Hence not only the subgroup $\mathrm{SL}(2,3) \rtimes C_2$ can be lifted but also the full automorphism group $\mathrm{Aut}(GP(8,3))$ can be lifted. In particular, by Conder's list [2] we know that there is only one symmetric cubic graph of order 48; in which case the covering graph is 2-arc-regular.

Theorem 1. *Let $n = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive cyclic regular covers of the Möbius-Kantor graph with cyclic covering group of exponent n are as follows:*

(1) *For $k \equiv 1 \mod 3$, only the subgroup $SL(2,3) \rtimes C_2$ can be lifted, and there is one 1-arc-regular cover.*

(2) *For $k = 3$ and $e = 1$, there is a unique 2-arc-regular cover.*

4 Arc-transitive cyclic regular covers of the Desargues graph

In this section, we classify all the arc-transitive cyclic regular covering graphs of the Desargues graph $GP(10,3)$. The automorphism group of $GP(10,3)$ is isomorphic to $S_5 \times C_2$ and acts 3-arc-regularly on the arcs. There are two other 2-arc-regular subgroups S_5 and $A_5 \times C_2$.

Take the group G_3, with presentation $G = \langle h, a, p, q \mid h^3 = p^2 = q^2 = a^2 = (qh)^2 = [p,q] = [h,p] = [a,p] = apaq = 1 \rangle$. This group G_3 has a unique normal subgroup N of index 240, with quotient $S_5 \times C_2$.

Using the `Rewrite` command in MAGMA, we find that the subgroup N is free of rank 11, on generators

$$
\begin{array}{rclrcl}
w_1 &=& pqahah^{-1}ah^{-1}ahahah^{-1} & w_2 &=& pqah^{-1}ah^{-1}ahahah^{-1}ah^{-1} \\
w_3 &=& qhahah^{-1}ah^{-1}ahahap & w_4 &=& qhah^{-1}ah^{-1}ahahah^{-1}ap \\
w_5 &=& hpqahah^{-1}ah^{-1}ahahah & w_6 &=& hpqah^{-1}ah^{-1}ahahah^{-1}ah \\
w_7 &=& pahah^{-1}ahahah^{-1}ah^{-1}ah^{-1}aq & w_8 &=& qahah^{-1}ahahah^{-1}ah^{-1}ah^{-1}ap \\
w_9 &=& hqahah^{-1}ahahah^{-1}ah^{-1}ah^{-1}aph^{-1} & w_{10} &=& ahahahahahah^{-1}ah^{-1}ah^{-1}ah^{-1}ah^{-1} \\
w_{11} &=& (ahah^{-1})^5 & & &
\end{array}
$$

Easy calculations show that the generators h, a and p act by conjugation as below: (Note that the action of q can be given by the composition apa.)

$$
\begin{array}{rclrclrcl}
{w_1}^a &=& w_3 & {w_1}^h &=& w_4 & {w_1}^p &=& w_2 \\
{w_2}^a &=& {w_2}^{-1} & {w_2}^h &=& w_3 & {w_2}^p &=& w_1 \\
{w_3}^a &=& w_1 & {w_3}^h &=& w_6 & {w_3}^p &=& w_4 \\
{w_4}^a &=& {w_4}^{-1} & {w_4}^h &=& w_5 & {w_4}^p &=& w_3 \\
{w_5}^a &=& {w_7}^{-1} & {w_5}^h &=& w_1 & {w_5}^p &=& w_6 \\
{w_6}^a &=& w_8 & {w_6}^h &=& w_2 & {w_6}^p &=& w_5 \\
{w_7}^a &=& {w_5}^{-1} & {w_7}^h &=& {w_5}^{-1}w_6{w_9}^{-1} & {w_7}^p &=& {w_7}^{-1} \\
{w_8}^a &=& w_6 & {w_8}^h &=& w_3{w_4}^{-1}{w_7}^{-1} & {w_8}^p &=& {w_8}^{-1} \\
{w_9}^a &=& w_9w_{10}w_{11}^{-1} & {w_9}^h &=& w_8 & {w_9}^p &=& {w_9}^{-1} \\
{w_{10}}^a &=& w_{10}^{-1} & {w_{10}}^h &=& w_3{w_9}^{-1}{w_{10}}^{-1}w_{11} & {w_{10}}^p &=& {w_8}^{-1}w_9w_{10} \\
{w_{11}}^a &=& w_{11}^{-1} & {w_{11}}^h &=& {w_6}^{-1}w_8{w_{10}}^{-1} & {w_{11}}^p &=& w_7{w_9}^{-1}w_{11}
\end{array}
$$

Now take the quotient G_3/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^{11}$ by the group $G_3/N \cong S_5 \times C_2$, and replace the generators h, a, p and all w_i by their images in this group. Also let K denote the subgroup N/N', and let G be G_3/N'. Then, in particular, G is an extension of $S_5 \times C_2$ by $\mathbb{Z}^{11}$.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^{11}$ as follows:

$$
h \mapsto \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0
\end{pmatrix},
$$

$$a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and

$$p \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

These matrices generate a group isomorphic to $S_5 \times C_2$, with the first two generating a subgroup isomorphic to $A_5 \times C_2$; and the first and the product of the other two generating a subgroup isomorphic to S_5. Note that the matrices of orders 3, 2, 2, 6 and 6 representing h, a, ap, hap and $(ha)^2h^{-1}a$ have traces $-1, -3, -1, 1$ and 1, respectively.

By inspecting traces and the character tables (which can be easily given by the `CharacterTable` command in MAGMA) of groups $A_5 \times C_2$ and S_5, we see that the 11-dimensional representation of S_5 over $\mathbb{Q}$ associated with the above action of $\langle h, ap\rangle$ on K is a sum of U and V, which are two irreducible 6-dimensional and 5-dimensional representations over the rational field $\mathbb{Q}$. Also the 11-dimensional representation of $A_5 \times C_2$ over $\mathbb{Q}$ associated with the action of $\langle h, a\rangle$ on K is a sum of φ_1 and φ_2, which are characters of two irreducible 6-dimensional and 5-dimensional representations over the rational field $\mathbb{Q}$. However, in particular, if there exist zeros of the polynomial $t^2 - t - 1$, φ_1 is reducible to a sum of $\varphi_{1,1}$ and $\varphi_{1,2}$ each of which is a character of an irreducible 3-dimensional representation.

Therefore, for any prime k other than 2, 3 and 5, there is no $\langle h, ap\rangle$- and $\langle h, a\rangle$-invariant subgroup of rank 10; equivalently, no cyclic regular cover exists.

For prime $k = 3$ and 5, with the help of MAGMA, there is also no $\langle h, ap\rangle$- and $\langle h, a\rangle$-invariant subgroup of rank 10. Thus there are no cyclic regular covers.

For prime $k = 2$, with the help of MAGMA, there are only two $\langle h, a\rangle$-invariant subgroups of rank 10 of exponent 2 and 4, respectively. Thus, correspondingly, there are two cyclic covering graphs of order 40 and 80. Also there are only two $\langle h, ap\rangle$-invariant subgroups of rank 10 of exponent 2 and 4. By Conder's list [2], we know that there are unique symmetric cubic graphs of orders 40 and 80, respectively, each of which is 3-arc-regular. Hence the above two cyclic covering graphs are exactly these two graphs.

Theorem 2. *There are only two arc-transitive cyclic regular covers of the Desargues graph, both are 3-arc-transitive, with cyclic covering groups C_2 and C_4, respectively.*

5 Dihedral regular covers of cubic graphs

First of all, arc-transitive cubic graphs of small order like the complete graph K_4, the complete bipartite graph $K_{3,3}$, the 3-cube Q_3 and the Petersen graph are well known. The arc-transitive properties of each of the above graphs are as follows.

The complete graph K_4 is 2-arc-regular with automorphism group S_4, and the only arc-transitive subgroup of automorphisms of K_4 is the subgroup A_4, which acts regularly on the arcs. The complete bipartite graph $K_{3,3}$ is 3-arc-regular. Its automorphism group is the wreath product $S_3 \wr C_2$, and this contains three arc-transitive subgroups which act 1-, 2- and 2-arc-regularly on the arcs of $K_{3,3}$, respectively. In particular, two of these three subgroups are minimal, one is the group $A_3 \wr C_2$ which acts 1-arc-regularly, while the other is $(A_3 \times A_3) \rtimes C_4$ which acts 2-arc-regularly. The 3-cube Q_3 is 2-arc-regular, and its automorphism group is the direct product $S_4 \times C_2$. And the only arc-transitive proper subgroups of automorphisms are S_4 and $A_4 \times C_2$, each of which acts 1-arc-regularly on the arcs of Q_3. And finally, the Petersen graph is a 3-arc-regular graph. Its automorphism group is the symmetric group S_5, and the only other arc-transitive subgroup of automorphisms is the subgroup A_5, which acts 2-arc-regularly.

Before investigating the dihedral covers, we remind readers of the following useful result given by Gardiner and Praeger in [6].

Theorem 3. [6] *Let Γ be a connected G-symmetric graph of valency p a prime. For each normal subgroup N of G one of the following holds:*

(a) Γ is N-symmetric;

(b) N acts regularly on vertices, so Γ is a Cayley graph for N;

(c) N has just two orbits on vertices and Γ is bipartite; or

(d) N has $r \geqslant p+1$ orbits on vertices, the natural quotient graph Γ_N on N-orbits is G/N-symmetric of valency p, and Γ is a topological cover of Γ_N.

Now, suppose graph $\tilde{X}$ is an arc-transitive dihedral regular D_n-cover of cubic graph X where dihedral group D_n is of degree n (here, we always assume $n > 2$), then we have the following lemma:

Lemma 4. *$\tilde{X}$ is a cyclic regular cover of a 2-cover of X.*

Proof. Since $\tilde{X}$ is an arc-transitive dihedral cover of X, then there exists an arc-transitive subgroup $D_n \rtimes A$ of $\mathrm{Aut}(\tilde{X})$ which is the lifting subgroup of an arc-transitive subgroup A of $\mathrm{Aut}(X)$. Let C_n be the cyclic subgroup of D_n, then C_n is normal in $D_n \rtimes A$. Especially,

C_n is a semi-regular subgroup of $\mathrm{Aut}(\tilde{X})$. Thus by Theorem 3, the quotient graph $\tilde{X}/C_n$ is an arc-transitive 2-cover of X. □

For the complete graph K_4, we have the following classification of all arc-transitive dihedral regular D_n-covers.

Theorem 5. *For $n \neq 2$, graph X is an arc-transitive dihedral regular D_n-cover of K_4 if and only if it is an arc-transitive cyclic regular C_n-cover of the 3-cube Q_3.*

Before proving the above theorem, in [3], Conder and Ma gave the following results:

Theorem 6. [3] *Let $n = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive cyclic regular covers of the 3-cube Q_3 with cyclic covering groups of exponent n are as follows:*

(1) *if $k \equiv 1 \mod 3$, only the subgroup $A_4 \times C_2$ can be lifted and there is one 1-arc-regular cover.*

(2) *If $k = 3$ and $e = 1$, there is one 2-arc-regular cover.*

(3) *If $k = 2$ and $e = 1$, there is one 2-arc-regular cover.*

Proof of Theorem 5: By Lemma 4, we know that X is an arc-transitive cyclic cover of the Q_3 which is the only arc-transitive 2-cover of the K_4. From the above Theorem 6, we know that there are only three types of cyclic covers. The first type, namely when $k \equiv 1 \mod 3$, is of automorphism group $C_n \rtimes (A_4 \times C_2)$. Note that from [3, Section 6], we know that $C_n \rtimes (A_4 \times C_2)$ is generated by (the images of) elements v_t and h, ap where $\langle v_t \rangle \cong C_n$ and $\langle h, ap \rangle \cong A_4 \times C_2$. And also by [3, Page 235, Paragraph 8] we have ${v_t}^h = {v_t}^t$ and ${v_t}^{ap} = {v_t}^{-1}$, hence ${v_t}^{(hap)^3} = {v_t}^{-1}$ where $(hap)^3$ is of order 2 and $\langle (hap)^3 \rangle \cong C_2$ which is normal in $A_4 \times C_2$. Therefore we have $C_n \rtimes (A_4 \times C_2) \cong D_n \rtimes A_4$ which suggests that X is a dihedral regular cover of K_4.

For $k = 3$, similarly, by [3, Page 235, Paragraph -2] the order 3 covering group is generated by uv, and the conjugation action of h and ap are $(uv)^h = uv$ and $(uv)^{ap} = (uv)^{-1}$. Thus $(uv)^{(hap)^3} = (uv)^{-1}$. Hence $C_3 \rtimes (A_4 \times C_2) \cong D_3 \rtimes A_4$. (Note that the cyclic covering graph is of order 24, the structure of the automorphism group can also be easily checked by MAGMA.) □

Remark 7. We know that there is a unique symmetric cubic graph of order 16 which is the Möbius-Kantor graph. In particular, from [3], we know that it is a cyclic C_4-cover of the complete graph K_4 and also a 2-cover of the 3-cube Q_3.

Corollary 8. *Let X be an arc-transitive dihedral regular D_n-cover of K_4, then n is of the following possibilities:*

(1) *3 or 6; or*

(2) *$2^i 3^j k^e$ for $i, j \in \{0, 1\}$ and prime integer $k \equiv 1 \mod 3$ and $e > 0$.*

Note that the the product of integers $2, 3$ and k^e is just the order of each cyclic covering group which comes from the direct product of cyclic groups C_2, C_3 and C_{k^e}.

About the complete graph $K_{3,3}$, we have the following result.

Theorem 9. *The complete bipartite graph $K_{3,3}$ has no arc-transitive dihedral regular cover.*

Proof. Suppose $K_{3,3}$ has an arc-transitive dihedral regular cover $\mathcal{D}$, then by Lemma 4, $\mathcal{D}$ is a cyclic regular cover of a 2-cover of $K_{3,3}$. However, we know that $K_{3,3}$ has no arc-transitive 2-cover, and in fact, there is no arc-transitive cubic graph of order 12, contradiction. Hence $K_{3,3}$ has no arc-transitive dihedral covering graph. □

For the 3-cube graph Q_3, the classification of arc-transitive dihedral covers is as follows.

Theorem 10. *Let X be an arc-transitive dihedral regular D_n-cover of the Q_3, then n is equal to 3.*

Proof. We know that each dihedral regular cover of Q_3 is a cyclic regular cover of the Möbius-Kantor graph. In Theorem 1, there are two types of cyclic regular covers of the Möbius-Kantor graph. Firstly, if there exists a primitive 3rd root δ, then the cyclic covering groups of the cyclic covers are generated by $u = \{w_1 w_2^{\delta^2} w_3^{\delta} w_4 w_5^{\delta^2} w_6^{\delta} w_7 w_8^{\delta^2} w_9^{-\delta}\}$ and $v = \{w_1 w_2^{\delta} w_3^{\delta^2} w_4 w_5^{\delta} w_6^{\delta^2} w_7 w_8^{\delta} w_9^{-\delta^2}\}$, respectively. Since these two covering graphs are isomorphic, here we only consider the covering group generated by u. The images of u under the conjugation actions of generators h and ap are u^4 and u^{-1}. Since $SL(2,3) \rtimes C_2 = \langle h, ap\rangle$, there is a unique normal subgroup of order 2 which is generated by $(haph)^6$. And the image of u by the conjugation action of $(haph)^6$ is equal to u^{16^6}. Since $k \equiv 1 \mod 3$ but $16^6 + 1 \equiv 2 \mod 3$. Hence $u^{16^6} \neq u^{-1}$, which suggests there is no dihedral normal subgroup of $SL(2,3) \rtimes C_2$.

Secondly, for $k = 3$ and the cyclic covering group of order 3, the covering graph is 2-arc-regular and of order 48. With the help of MAGMA, we can easily verify that its a dihedral regular covering graph of the Q_3, with automorphism group isomorphic to $D_3 \rtimes (S_4 \times C_2)$. □

In [7], the author classified all the arc-transitive cyclic covers of the dodecahedron graph, and gave the following result.

Theorem 11. [7] *Let $n = k^\ell$ be any power of a prime k, with $\ell > 0$. Then the arc-transitive cyclic regular covers of the dodecahedron graph with covering group of exponent n are as follows:*

(a) *If $k = 2$, there are exactly two such covers, namely*
- *one 3-arc-transitive cover with covering group $\mathbb{Z}_2$ where $\ell = 1$,*
- *one 3-arc-transitive cover with covering group $\mathbb{Z}_4$ where $\ell = 2$.*

(b) *If $k = 3$, there is exactly one such cover, namely*
- *one 2-arc-transitive cover with covering group $\mathbb{Z}_3$ where $\ell = 1$.*

(c) *There is no arc-transitive cyclic cover for other prime integer $k \neq 2, 3$.*

Corollary 12. *All the arc-transitive cyclic regular covering graphs of the Desargues graph are also arc-transitive cyclic regular covers of the dodecahedron graph.*

Now, we can give the following results for arc-transitive dihedral regular D_n-covers of the Petersen graph.

Theorem 13. *Let X be an arc-transitive dihedral regular D_n-cover of the Petersen graph, then n is equal to either 3 or 6.*

Proof. First of all, every dihedral regular cover of the Petersen graph is a cyclic regular cover of a 2-cover of the Petersen graph. And we know that there are two 2-covers of the Petersen graph which are the dodecahedron graph and the Desargues graph. However by Corollary 12, we only need to consider the cyclic covers of the dodecahedron graph.

By Theorem 11, we know that there are only finitely many cyclic covers. For $n = 2$, by [3], we know that the Petersen graph has a $(C_2)^2$-cover. For $n = 4$, the covering graph is of order 80 with automorphism group isomorphic to $Q_8 \rtimes S_5$ where Q_8 is the quaternion group of order 8. Hence its a 'quaternion' Q_8-cover of the Petersen graph instead of a dihedral cover.

Similarly, for $n = 3$, we have a 2-arc-transitive C_3-covering graph of automorphism group $C_3 \rtimes (A_5 \times C_2)$. With the help of MAGMA, we have $C_3 \rtimes (A_5 \times C_2) \cong D_3 \rtimes A_5$ which suggests that its a dihedral regular D_3-cover of the Petersen graph.

Therefore, the Petersen graph only has two dihedral covers with covering groups D_3 and D_6. □

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