Nordhaus–Gaddum type inequalities for Laplacian and signless Laplacian eigenvalues

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Abstract

Let G be a graph with n vertices. We denote the largest signless Laplacian eigenvalue of G by $q_1(G)$ and Laplacian eigenvalues of G by $\mu_1(G) \geqslant \cdots \geqslant \mu_{n-1}(G) \geqslant \mu_n(G) = 0$. It is a conjecture on Laplacian spread of graphs that $\mu_1(G) - \mu_{n-1}(G) \leqslant n-1$ or equivalently $\mu_1(G) + \mu_1(\overline{G}) \leqslant 2n-1$. We prove the conjecture for bipartite graphs. Also we show that for any bipartite graph G, $\mu_1(G)\mu_1(\overline{G}) \leqslant n(n-1)$. Aouchiche and Hansen [*Discrete Appl. Math.* 2013] conjectured that $q_1(G) + q_1(\overline{G}) \leqslant 3n-4$ and $q_1(G)q_1(\overline{G}) \leqslant 2n(n-2)$. We prove the former and disprove the latter by constructing a family of graphs H_n where $q_1(H_n)q_1(\overline{H_n})$ is about $2.15n^2 + O(n)$.

Keywords: Signless Laplacian eigenvalues of graphs; Laplacian eigenvalues of graphs; Nordhaus–Gaddum type inequalities; Laplacian spread

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. We denote the complement graph of G by $\overline{G}$, the adjacency matrix of G by $A(G)$, and the degree of a vertex $v \in V(G)$ by $d(v)$. The *Laplacian matrix* and the *signless Laplacian matrix* of G are the matrices $L(G) = A(G) - D(G)$ and $Q(G) = A(G) + D(G)$, respectively, where $D(G)$ is the diagonal matrix with $d(v_1), \ldots, d(v_n)$ on its main diagonal. It is well-known that

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$L(G)$ and $Q(G)$ are positive semidefinite and so their eigenvalues are nonnegative real numbers. The eigenvalues of $L(G)$ and $Q(G)$ are called the *Laplacian eigenvalues* and *signless Laplacian eigenvalues* of G, respectively, and are denoted by $\mu_1(G) \geqslant \cdots \geqslant \mu_n(G)$ and $q_1(G) \geqslant \cdots \geqslant q_n(G)$, respectively. We drop G from the notation when there is no danger of confusion. Note that each row sum of $L(G)$ is 0 and therefore $\mu_n(G) = 0$. In fact the multiplicity of 0 as an eigenvalue of $L(G)$ equals the number of conncted components of G.

Nordhaus and Gaddum [18] studied the chromatic number in a graph G and in its complement together. They proved lower and upper bounds on the sum and on the product of chromatic number of G and that of $\overline{G}$ in terms of the number of vertices G. Since then, any bound on the sum and/or the product of an invariant in a graph G and the same invariant in $\overline{G}$ is called a *Nordhaus–Gaddum type inequality.* In [2], Nordhaus–Gaddum type inequalities for graph parameters were surveyed. Many of those inequalities involve eigenvalues of adjacency, Laplacian and signless Laplacian matrices of graphs. The first known spectral Nordhaus–Gaddum results belong to Nosal [19], and to Amin and Hakimi [1], who showed that for every graph G of order n,

$$\lambda(G) + \lambda(\overline{G}) < \sqrt{2}(n-1),$$

where $\lambda(H)$ denotes the largest eigenvalues of $A(H)$. A minor improvement was obtained by Nikiforov [17]. In the same paper he conjectured that

$$\lambda(G) + \lambda(\overline{G}) < \frac{4}{3}n + O(1).$$

Csikvári [6] showed that $\lambda(G) + \lambda(\overline{G}) < \frac{1+\sqrt{3}}{2}n - 1$ and observed that any asymptomatic bound $\lambda(G) + \lambda(\overline{G}) < cn + o(n)$, for some constant c, implies $\lambda(G) + \lambda(\overline{G}) < cn - 1$. The conjecture was proved by Terpai [20] who showed that

$$\lambda(G) + \lambda(\overline{G}) < \frac{4}{3}n - 1.$$

In this paper we study Nordhaus–Gaddum type inequalities for Laplacian and signless Laplacian eigenvalues of graphs. For Laplacian eigenvalues, we have $\mu_1(G) + \mu_1(\overline{G}) = n + \mu_1(G) - \mu_{n-1}(G)$. The quantity $\mu_1(G) - \mu_{n-1}(G)$ is called *Laplacian spread* of G. It is a conjecture [22, 23] that $\mu_1(G) - \mu_{n-1}(G) \leqslant n-1$ or equivalently $\mu_1(G) + \mu_1(\overline{G}) \leqslant 2n-1$. We prove the conjecture for bipartite graphs. Partial results on this conjecture were obtained by several authors [3, 5, 9, 10, 13, 14, 15, 21]. Also we show that for any bipartite graph G, $\mu_1(G)\mu_1(\overline{G}) \leqslant n(n-1)$. Aouchiche and Hansen [2] conjectured that $q_1(G) + q_1(\overline{G}) \leqslant 3n-4$ and $q_1(G)q_1(\overline{G}) \leqslant 2n(n-2)$. We prove the former and disprove the latter by constructing a family of graphs H_n where $q_1(H_n)q_1(\overline{H_n})$ is about $2.15n^2 + O(n)$.

2 Preliminaries

We denote the number of edges of G by $e(G)$. We also denote the complete graph on n vertices by K_n and the complete bipartite graph with parts of sizes r and s by $K_{r,s}$.

The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a vertex $v \in V(G)$, $m(v)$ denotes the average degree of the neighbors of v, i.e. $m(v) = \frac{1}{d(v)} \sum_{u \sim v} d(u)$. For any graph parameter $p(G)$ we occasionally use $\overline{p}$ to denote $p(\overline{G})$.

Lemma 1. ([7, p. 222]) *Suppose that G is a graph and G' is obtained by removing one edge from G. Then the signless Laplacian eigenvalues of G and G' interlace:*

$$q_1(G) \geqslant q_1(G') \geqslant q_2(G) \geqslant q_2(G') \geqslant \cdots \geqslant q_n(G) \geqslant q_n(G').$$

Lemma 2. ([7, p. 217]) *If G is a bipartite graph, then the matrices $L(G)$ and $Q(G)$ are similar, i.e. the Laplacian and signless Laplacian eigenvalues of G are the same.*

The following lemma is easy to prove.

Lemma 3. (i) *The signless Laplacian eigenvalues of K_n are $2n-2$ with multiplicity 1 and $n-2$ with multiplicity $n-1$. The Laplacian eigenvalues of K_n are n with multiplicity $n-1$ and 0 with multiplicity 1.*

(ii) *The (signless) Laplacian eigenvalues of $K_{r,s}$ are $r+s$ with multiplicity 1, r with multiplicity $s-1$, s with multiplicity $r-1$, and 0 with multiplicity 1.*

Lemma 4. ([7, p. 185]) *For any graph G, $\mu_i(G) = n - \mu_{n-i}(\overline{G})$ for $i = 1, \ldots, n-1$.*

From Lemma 4, one concludes the following.

Lemma 5. *For any graph G with n vertices, $\mu_1(G) \leqslant n$ with equality if and only if $\overline{G}$ is disconnected.*

Lemma 6. (Das [8]) *Let G be a graph with n vertices, e edges, and Δ, δ be the largest degree and the smallest degree of G, respectively. Then for any vertex v,*

$$d(v) + m(v) \leqslant \frac{2e}{n-1} + \frac{n-2}{n-1}\Delta + (\Delta - \delta)\left(1 - \frac{\Delta}{n-1}\right). \tag{1}$$

If G is connected, then the equality holds if and only if either $d(v) = n-1$ or $d(v) = \Delta$ and all neighbors of v have degrees Δ and all non-neighbors of v have degrees δ.

From the Perron–Frobenius Theorem (see [4, p. 22]) it follows that the largest eigenvalue of any non-negative square matrix M does not exceed the maximum row-sum of M. Consider the matrix $M = D^{-1}QD = D^{-1}AD + D$. The row-sum of M corresponding with vertex v is $d(v) + m(v)$. Since $q_1(G)$ is the largest eigenvalue of M we have the following.

The following lemma is due to Merris [16]. The case of equality was obtained in [11].

Lemma 7. *For any graph G,*

$$q_1(G) \leqslant \max\{d(v) + m(v) \mid v \in V(G)\}. \tag{2}$$

If G is connected, then the equality holds if and only if G is regular or bipartite semiregular.

For a graph G, consider a partition $P = \{V_1, \dots, V_m\}$ of $V(G)$. The partition of P is *equitable* if each submatrix Q_{ij} of $Q(G)$ formed by the rows of V_i and the columns of V_j has constant row sums r_{ij}. The $m \times m$ matrix $R = (r_{ij})$ is called the *quotient matrix* of $Q(G)$ with respect to P. The proof of the following theorem is similar to the one given in [7, p. 187] where a similar result is presented for Laplacian matrix.

Lemma 8. *Any eigenvalue of the quotient matrix R is an eigenvalue of $Q(G)$. Moreover, the largest eigenvalue of R is the largest eigenvalue of $Q(G)$.*

We will use a variant of Lemma 8 as follows. We denote the largest eigenvalue of a matrix M by $\lambda_{\max}(M)$.

Lemma 9. *Let G be graph and $P = \{V_1, \dots, V_m\}$ be a partition of $V(G)$. Suppose that each submatrix Q_{ij} of $Q(G)$ formed by the rows of V_i and the columns of V_j has row sum at most r_{ij} and form the $m \times m$ matrix $R = (r_{ij})$. Then $q_1(G) \leqslant \lambda_{\max}(R)$.*

Proof. For all $1 \leqslant i, j \leqslant m$, we multiply each row of Q_{ij} by some number (larger than or equal to 1) so that all the row sums of the resulting matrix Q'_{ij} are equal to r_{ij}. Let Q' be the matrix obtained from $Q(G)$ by replacing the submatrices Q_{ij} by Q'_{ij} for all $1 \leqslant i, j \leqslant m$. Each entry of Q' is larger than or equal to the corresponding entry of Q, hence from Perron–Frobenius Theorem it follows that $\lambda_{\max}(Q) \leqslant \lambda_{\max}(Q')$. On the other hand, $\{V_1, \dots, V_m\}$ is an equitable partition for Q' with the qutiont matrix $R = (r_{ij})$. So, by Lemma 8, $\lambda_{\max}(Q') = \lambda_{\max}(R)$. This completes the proof. □

3 Nordhaus–Gaddum type inequalities for Laplacian eigenvalues

In this section we study Nordhaus–Gaddum type inequalities for both sum and product of Laplacian eigenvalues of a graph. In view of Lemma 4, $\mu_1(G) + \mu_1(\overline{G}) = n + \mu_1(G) - \mu_{n-1}(G)$. So to study $\mu_1 + \overline{\mu}_1$ it is enough to consider $\mu_1 - \mu_{n-1}$. Some authors have studied this quantity and it is called *Laplacian spread*. Most of the results on Laplacian spread of graphs are around the following conjecture which was posed in [23] and appeared as a question in [22].

Conjecture 10. *For any graph G with n vertices, $\mu_1(G) - \mu_{n-1}(G) \leqslant n-1$ (or equivalently $\mu_1(G) + \mu_1(\overline{G}) \leqslant 2n - 1$) and equality holds if and only if G or $\overline{G}$ is isomorphic to the join of K_1 and a disconnected graph of order $n - 1$.*

So far, the conjecture has been proved for trees [10], unicyclic graphs [3], bicyclic graphs [9, 13, 15], tricyclic graphs [5], cactus graphs [14], and quasi-tree graphs [21]. We remark that Conjecture 10 also holds if G or $\overline{G}$ is disconnected. To see this, assuming $\overline{G}$ is disconnected, by Lemma 5, $\mu_1(G) \leqslant n$ and $\mu_1(\overline{G}) \leqslant n - 1$ and the equality holds in the latter if and only if $\overline{G}$ has a connected component of order $n - 1$, say H, such that $\overline{H}$ is disconnected. In other words, $\mu_1(\overline{G}) = n - 1$ if and only if $G = K_1 \vee \overline{H}$ with $\overline{H}$ being disconnected where '$\vee$' shows the join of two graphs.

In this section we prove Conjecture 10 for bipartite graphs (and their complements). Therefore, our focus in this section will be on bipartite graphs. By Lemma 2, we may consider $Q(G)$ instead of $L(G)$ whenever is required, though we insist to state our results in terms of Laplacian eigenvalues.

We start with the following theorem which provides a lower bound on $\mu_{n-1}(G)$ (also called *algebraic connectivity* of G [12]). For a polynomial $f(x)$ with real zeros, we use the notation $z_{\min}(f)$ and $z_{\max}(f)$ to denote the smallest and largest real zeros of $f(x)$, respectively.

Theorem 11. *Let G be a bipartite graph with bipartition (X, Y) and $|Y| \geqslant |X|$. If X contains some vertices of degree $|Y|$ and Y contains ℓ vertices of degree $|X|$, then*

$$\mu_{n-1}(G) \geqslant \frac{\ell}{|Y|}.$$

Equality holds if and only if G is a star.

Proof. Let $k := |X| \leqslant n/2$ and so $|Y| = n - k$. Suppose that X_0 is the set of vertices in X of degree $n - k$ with $t := |X_0|$ and Y_0 is the set of vertices in Y of degree k, so $|Y_0| = \ell$. If $X = X_0$ or $Y = Y_0$, then G is the complete bipartite graph $K_{n-k,k}$ and by Lemma 3, $\mu_{n-1}(G) = k$. Thus assume that $1 \leqslant t \leqslant k - 1$ and $1 \leqslant \ell \leqslant n - k - 1$. Let H be the graph obtained from G by removing the edges between $X \setminus X_0$ and $Y \setminus Y_0$. By Lemma 1, $\mu_{n-1}(G) \geqslant \mu_{n-1}(H)$. So it suffices to prove the assertion for H. In H, any $v \in X \setminus X_0$ has degree ℓ and any $v \in Y \setminus Y_0$ has degree t. It turns out that the partition $(X_0, X \setminus X_0, Y_0, Y \setminus Y_0)$ is an equitable partition for H. The corresponding quotient matrix for $Q(H)$ is

$$R = \begin{pmatrix} n-k & 0 & \ell & n-k-\ell \\ 0 & \ell & \ell & 0 \\ t & k-t & k & 0 \\ t & 0 & 0 & t \end{pmatrix}.$$

The characteristic polynomial of R is $xf(x)$ where

$$f(x) := x^3 - (n + \ell + t)x^2 + (kt + nk + \ell n - \ell k + 2\ell t - k^2)x - \ell tn. \tag{3}$$

For simplicity, let $r := n-k$. Note that in the matrices $Q(H)-rI$, $Q(H)-\ell I$, $Q(H)-kI$ and $Q(H) - tI$, respectively, the rows corresponding to the vertices in X_0, $X \setminus X_0$, Y_0, and $Y \setminus Y_0$ are identical. So the four matrices above, respectively, have nullities at least $|X_0| - 1$, $|X \setminus X_0| - 1$, $|Y_0| - 1$, and $|Y \setminus Y_0| - 1$. It follows that $Q(H)$ has eigenvalues r, ℓ, k, and t with multiplicities at least $t - 1$, $k - t - 1$, $\ell - 1$, and $r - \ell - 1$, respectively. On the other hand, we have

$$\begin{aligned} f(r) &= -2t(r - n/2)(r - \ell), \\ f(k) &= -2\ell(k - n/2)(k - t), \\ f(\ell) &= t(r - \ell)(k - t), \\ f(t) &= \ell(r - \ell)(k - t). \end{aligned}$$

If $k < n/2$, then none of r, ℓ, k, t is a zero of f. It turns out that the polynomial

$$x(x-r)^{t-1}(x-k)^{\ell-1}(x-\ell)^{k-t-1}(x-t)^{r-\ell-1}f(x) \tag{4}$$

is the characteristic polynomial of $Q(H)$. If $k = n/2$, then $k = r$ is a zero of f. From the above argument, $x(x-n/2)^{t+\ell-3}(x-\ell)^{n/2-t-1}(x-t)^{n/2-\ell-1}f(x)$ is a factor of characteristic polynomial of $Q(H)$ and thus $n-1$ eigenvalues of $Q(H)$ are determined. To determine the remaining eigenvalue we use the fact that the sum of all eigenvalues of $Q(H)$ equals $2e(H)$; it turns out that the remaining eigenvalue is also $n/2$. So in this case also the characteristic polynomial of $Q(H)$ is given by (4). Note that as f is a cubic polynomial with a positive leading coefficient and as $f(t) > 0$, $f(\ell) > 0$, and $f(r) \leqslant 0$, it follows that two zeros of f are greater than ℓ and $z_{\min}(f) < \min\{\ell, t\}$. Therefore, $\mu_{n-1}(H) = z_{\min}(f)$ and further if we show that $f(\ell/r) < 0$, then we can conclude that $z_{\min}(f) \geqslant \ell/r$. We proceed to show that $f(\ell/r) < 0$.

We have

$$\begin{aligned}\frac{r^3}{\ell} f\left(\frac{\ell}{r}\right) &= -r^4 + (n - t + \ell - tn)r^3 + (2t\ell + tn)r^2 - (t\ell + \ell^2 + \ell n)r + \ell^2 \\ &= (-\ell r - r^3 + 2r^2\ell + r^2 n - r^3 n)t + \ell^2 - r^4 - \ell^2 r - \ell r n + r^3\ell + r^3 n.\end{aligned}$$

It is easy to see that the coefficient of t is always negative, so $f(\ell/r)$ is maximized only if $t = 1$. Therefore,

$$\frac{r^3}{\ell} f\left(\frac{\ell}{r}\right) \leqslant F(r, \ell) := -r^4 + (\ell - 1)r^3 + (n + 2\ell)r^2 - (\ell + \ell^2 + \ell n)r + \ell^2.$$

As $n \leqslant 2r$ and $\ell \leqslant r$, we have

$$\begin{aligned}\frac{\partial F}{\partial \ell} &= r^3 + 2r^2 - nr - 2\ell(r-1) - r \\ &\geqslant r^3 + 2r^2 - 2r^2 - 2r(r-1) - r = r(r-1)^2 > 0.\end{aligned}$$

So, F is increasing in ℓ and thus $F(r, \ell) < F(r, r) = 0$. Therefore, $f(\ell/r) < 0$ and so $\mu_{n-1}(G) \geqslant \mu_{n-1}(H) \geqslant \ell/r$.

Now we consider the case of equality. For a star obviously equality occurs. Conversely, suppose that $\mu_{n-1}(G) = \ell/|Y|$. Note that $\ell/|Y|$ is an algebraic integer only if $\ell = |Y|$. Hence $G = K_{k,n-k}$ and thus $k = \mu_{n-1}(G) = \ell/|Y| = 1$. This means that G is a star. □

Theorem 12. *Let G be a bipartite graph with n vertices, e edges, and bipartition (X, Y). If $|X| \leqslant |Y|$, then*

$$\mu_1(G) \leqslant |Y| + \frac{e}{|Y|}.$$

Moreover, if Y contains ℓ vertices of degree $|X|$, then

$$|Y| + \frac{e}{|Y|} \leqslant n - 1 + \frac{\ell}{|Y|}.$$

The equality $\mu_1(G) = n - 1 + \ell/|Y|$ holds if and only if G is a complete bipartite graph.

Proof. Let $k := |X| \leqslant n/2$ and so $|Y| = n - k$. Let $v \in V(G)$ and $d = d(v)$ and $m(v) = S/d$. Then

$$\begin{aligned} d + \frac{S}{d} \leqslant (n-k) + \frac{e}{n-k} \iff & (n-k)d + (n-k)\frac{S}{d} \leqslant (n-k)^2 + S + (e - S) \\ \iff & (n-k-d)\left(n - k - \frac{S}{d}\right) + (e - S) \geqslant 0, \end{aligned} \tag{5}$$

where the last inequality holds as d and S/d are at most $n-k$ and $S \leqslant e$. Hence, by Lemma 7, we have $\mu_1(G) \leqslant n - k + e/(n-k)$. Since Y contains ℓ vertices of degree k, we have $e \leqslant (n-k)(k-1) + \ell$ and so $n - k + e/(n-k) \leqslant n - 1 + \ell/(n-k)$.

Now we consider the equality case. If G is a complete bipartite graph, then $\mu_1 = n$ and the equality holds in both inequalities of the theorem. Conversely, if $\mu_1 = n - 1 + \ell/|Y|$, then we must have the equality in (5) which is possible only if either $d(v) = n - k$ or $m(v) = S/d = n - k$. This means that we must have some vertices in X (or in Y if $k = n/2$) of degree $n - k$. We must also have the equality in (2) which implies that all the vertices in X have degree $n - k$, and thus G is $K_{n-k,k}$. □

Combining Theorems 11 and 12, we conclude the following. We note that Theorem 11 may not hold in the case that $X_0 = \emptyset$ in which case we have $\mu_1(G) < n - 1$ by Part 4 of Theorem 14.

Theorem 13. *If G or $\overline{G}$ is a bipartite graph with n vertices, then the following equivalent inequalities hold:*

- $\mu_1(G) + \mu_1(\overline{G}) \leqslant 2n - 1;$
- $\mu_1(G) - \mu_{n-1}(G) \leqslant n - 1;$
- $\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) \geqslant 1.$

Equality holds if and only if G or $\overline{G}$ is a star.

We now proceed to establish a Nordhaus–Gaddum type inequality for the product of Laplacian eigenvalues. More precisely we will show that $\mu_1\overline{\mu}_1 \leqslant n(n-1)$ for bipartite graphs. However, this cannot be concluded from Theorems 11 and 12. We need to obtain sharper bounds on the eigenvalues. This will be done in the next theorem.

Theorem 14. *Let G be a bipartite graph with bipartition (X, Y) and $|Y| \geqslant |X| \geqslant 2$. Suppose that X contains t vertices of degree $|Y|$ and Y contains ℓ vertices of degree $|X|$.*

1. *If $t \geqslant 2$ and $\ell \geqslant 2$, then either $\mu_{n-1} > 1$ or $\mu_1 \leqslant n - 1$.*
2. *For $\ell = 1$: if $t \leqslant k - 2$, then $\mu_1 < n - 1$; if $t = k - 1 < n/2 - 1$, then $\mu_1 < n - 1 + 1/n$; if $t = k - 1 = n/2 - 1$, then $\mu_1 = \frac{1}{2}\left(n + \sqrt{n^2 - 4n + 8}\right)$ and $\mu_{n-1} = \frac{1}{2}\left(n - \sqrt{n^2 - 4n + 8}\right)$.*
3. *If $t = 1$ and $n \geqslant 7$, then $\mu_1 < n - 1 + \ell/n$.*

4. If either $t = 0$ or $\ell = 0$, then $\mu_1 < n-1$.

Proof. Let $k := |X| \leqslant n/2$ and $|Y| = n-k$. If $t = 0$, i.e. all the vertices in X have degree at most $n-k-1$, then by applying Lemma 9 for bipartition of G, we have

$$\mu_1 < \lambda_{\max}\begin{pmatrix} n-k-1 & n-k-1 \\ k & k \end{pmatrix} = n-1.$$

Similarly, if $\ell = 0$, then $\mu_1 < n-1$. This implies Case 4. In the rest of the proof we assume that $t \geqslant 1$ and $\ell \geqslant 1$.

Case 1. $t \geqslant 2$ and $\ell \geqslant 2$.

Suppose that X_0 is the set of vertices in X of degree $n-k$ and Y_0 is the set of vertices in Y of degree k. If we apply Lemma 9 on $Q(G)$ with the partition $(X_0, X\setminus X_0, Y_0, Y\setminus Y_0)$, it turns out that μ_1 does not exceed the largest eigenvalue of the matrix

$$R = \begin{pmatrix} n-k & 0 & \ell & n-k-\ell \\ 0 & n-k-1 & \ell & n-k-\ell-1 \\ t & k-t & k & 0 \\ t & k-t-1 & 0 & k-1 \end{pmatrix}.$$

The characteristic polynomial of R is $xg(x)$ where

$$g(x) = x^3 + (2-2n)x^2 + (n^2+nk-2n-k^2-\ell-t)x + \ell n + nk^2 + 2nk - \ell k + tk - n^2k - 2k^2.$$

We have

$$g(n-1) = (k-t-1)n - k^2 + \ell + t + tk - \ell k + 1.$$

On the other hand, from the proof of Theorem 11 we know that $\mu_{n-1} \geqslant z_{\min}(f)$ with $f(x)$ given in (3). Also, if we have $f(1) < 0$, then $z_{\min}(f) > 1$. Since $g(n-k) = t(2k-n) \leqslant 0$ and $g(k) = \ell(n-2k) \geqslant 0$, two zeros of g are at most $n-k$. It follows that if $g(n-1) > 0$, then $z_{\max}(g) < n-1$. Next, note that

$$f(1) = (\ell + k - t\ell - 1)n - t - \ell - \ell k + kt + 2t\ell - k^2 + 1,$$

and thus

$$g(n-1) - f(1) = (n-2)(t\ell - t - \ell).$$

By the assumptions $t \geqslant 2$ and $\ell \geqslant 2$, we have $t\ell - t - \ell \geqslant 0$. Hence $g(n-1) - f(1) \geqslant 0$. Now, if $\mu_{n-1} \leqslant 1$, then $z_{\min}(f) \leqslant 1$, hence $f(1) \geqslant 0$. From the above equation it follows that $g(n-1) \geqslant 0$, and so $\mu_1 \leqslant z_{\max}(g) \leqslant n-1$. Therefore, we have either $\mu_{n-1} > 1$ or $\mu_1 \leqslant n-1$.

Case 2. $\ell = 1$.

In this case we have $g(n-1) = (k-t)(n-1-k) + 2 - n$. It follows that $g(n-1) > 0$ provided that $k-t \geqslant 2$. Hence, if $t \leqslant k-2$, then $\mu_1 < n-1$, and we are done. So, assume that $t = k-1$. It turns out that in this subcase, there are no edges between $X \setminus X_0$ and

$Y \setminus Y_0$ and hence G is isomorphic to the graph H of Theorem 11. So $\mu_1 = z_{\max}(f)$. First assume that $k < n/2$. We have,

$$f\left(n-1+\frac{1}{n}\right) = \frac{1}{n^3}\Big(n^4 - 2(k+1)n^3 + (2k+3)n^2 - (k+3)n + 1\Big),$$

which is minimized when k is maximized, i.e. $k = (n-1)/2$. Thus $f(n-1+1/n) \geqslant (3n^2/2 - 5n/2 + 1)/n^3$ which is positive for $n \geqslant 2$. Therefore, $\mu_1 < n-1+1/n$. If $k = n/2$, then the three zeros of f are $\frac{n}{2}, \frac{1}{2}\left(n \pm \sqrt{n^2-4n+8}\right)$.

Case 3. $t = 1$.

In this case we show that $z_{\max}(g) < n-1+\ell/n$. It suffices to show that $g(n-1+\ell/n) > 0$. We have $g(n-1+\ell/n) = n^{-3}h(k,\ell)$ where

$$h(k,\ell) = (k-2)n^4 + (k-k^2+\ell+2)n^3 - \ell(k^2+2)n^2 - n\ell^2 + \ell^3.$$

By computation, for $n \geqslant 7$, we have

$$\begin{aligned} h(2,\ell) &= \ell(n^3 - 6n^2 - n\ell + \ell), \\ &\geqslant \ell(n^3 - 6n^2 - n^2 + \ell) > 0. \end{aligned}$$

So we may assume that $k \geqslant 3$. We may write $h(k,\ell) = h_1(\ell) + n^2 h_2(k,\ell)$ where

$$\begin{aligned} h_1(\ell) &= 2n^3 - 2\ell n^2 - n\ell^2 + \ell^3, \\ h_2(k,\ell) &= (k-2)n^2 + (k-k^2)n + \ell(n-k^2). \end{aligned}$$

The function h_1 is decreasing in ℓ, and so $h_1(\ell) > h_1(n) = 0$. Now for showing the positivity of h_2 we consider two subcases: $n \geqslant k^2$ and $n < k^2$. If $n \geqslant k^2$, then

$$h_2(k,\ell) > (k-2)n^2 - (k-1)nk$$

which is positive for $k \geqslant 3$. Next suppose that $n < k^2$. As $\ell \leqslant n-k-1$ (for if $\ell = n-k$, then $t = k \geqslant 2$), we have

$$\begin{aligned} h_2(k,\ell) &\geqslant (k-2)n^2 + (k-k^2)n + (n-k-1)(n-k^2) \\ &= (k-1)n^2 - (2k^2+1)n + k^2(k+1). \end{aligned} \tag{6}$$

If $k = 3$, then the right side of (6) becomes $2n^2 - 19n + 36$ which is positive for $n \geqslant 7$. The zeros of the quadratic form $(k-1)x^2 - (2k^2+1)x + k^2(k+1)$ are $(2k^2+1 \pm \sqrt{8k^2+1})/(2k-2)$, and for $k \geqslant 4$ we have

$$\frac{2k^2+1+\sqrt{8k^2+1}}{2k-2} < 2k.$$

Since $n \geqslant 2k$, it follows that $h_2(k,\ell) \geqslant 0$.

Therefore, $g(n-1+\ell/n) > 0$ and thus $\mu_1 \leqslant z_{\max}(g) < n-1+\ell/n$. □

Theorem 15. *Let G be a bipartite graph with n vertices. Then $\mu_1(G)\mu_1(\overline{G}) \leqslant n(n-1)$, equality holds if and only if either G or $\overline{G}$ is a star.*

Proof. As discussed in the paragraph following Conjecture 10, if either G or $\overline{G}$ is disconnected, then $\mu_1(G) \leqslant n$ and $\mu_1(\overline{G}) \leqslant n-1$ with equality if and only if G is a star. Hence, we may assume that both G and $\overline{G}$ are connected. We use the notation of Theorem 14. If $k=1$, then G is a star and we are done. So assume that $k \geqslant 2$. The connectedness of $\overline{G}$ also implies that $t \leqslant k-1$ and $\ell \leqslant n-k-1$. The theorem for bipartite graphs under the above conditions with $n \leqslant 6$ can be easily verified. So we may assume that $n \geqslant 7$.

By Lemma 4, we have $\overline{\mu}_1 = n - \mu_{n-1}$. Therefore, in view of Theorem 11,

$$\overline{\mu}_1 < n - \frac{\ell}{n-k}. \tag{7}$$

If one of the cases $t = \ell = 0$, or $t \geqslant 2, \ell \geqslant 2$, or $\ell = 1, t \leqslant k-2$ occurs, then by Theorem 14 we have either $\mu_1 \leqslant n-1$ or $\overline{\mu}_1 < n-1$, and we are done. If $\ell = 1$ and $t = k-1 < n/2 - 1$, then by Theorem 14 and (7),

$$\mu_1\overline{\mu}_1 < \left(n - 1 + \frac{1}{n}\right)\left(n - \frac{1}{n-k}\right) < n(n-1).$$

If $\ell = 1$ and $t = k-1 = n/2-1$, then by Theorem 14,

$$\mu_1\overline{\mu}_1 = \frac{1}{4}\left(n + \sqrt{n^2-4n+8}\right)^2$$

which is smaller than $n(n-1)$ for $n \geqslant 4$. If $t=1$, the result follows similarly from Theorem 14 and (7). □

Based on the above results, we pose the following conjecture.

Conjecture 16. *For any graph G with n vertices, $\mu_1(G)\mu_1(\overline{G}) \leqslant n(n-1)$ and equality holds if and only if G or $\overline{G}$ is isomorphic to the join of K_1 and a disconnected graph of order $n-1$.*

4 Nordhaus–Gaddum type inequalities for signless Laplacian eigenvalues

In this section we study Nordhaus–Gaddum type inequalities for signless Laplacian eigenvalues of a graph. Aouchiche and Hansen [2] surveyed Nordhaus–Gaddum type inequalities for graph parameters. Among other things, they gave the following two conjectures which is the subject of this section.

Conjecture 17. (Aouchiche and Hansen [2]) *Let G be a simple graph on $n \geqslant 2$ vertices. Then $q_1(G) + q_1(\overline{G}) \leqslant 3n-4$. Equality holds if and only if G is the star $K_{1,n-1}$.*

Conjecture 18. (Aouchiche and Hansen [2]) *Let G be a simple graph on $n \geqslant 2$ vertices. Then $q_1(G) \cdot q_1(\overline{G}) \leqslant 2n(n-2)$. Equality holds if and only if G is the star $K_{1,n-1}$.*

We remark that Conjecture 17 can be proved easily as follows. Das [8] showed that $d(v) + m(v) \leqslant 2e/(n-1) + n - 2$. Combining this with Lemma 7 imply the well-known bound $q_1(G) \leqslant 2e/(n-1) + n - 2$ with equality if G or $\overline{G}$ is a star. Applying this bound to $\overline{G}$ and adding the two inequalities implies Conjecture 17. In what follows we manage to prove a stronger inequality than Conjecture 17 and also disprove Conjecture 18.

Theorem 19. *Let G be a simple graph on $n \geqslant 2$ vertices. Then*

$$q_1(G) + q_1(\overline{G}) \leqslant 2n - 2 + (\Delta - \delta)\left(2 - \frac{\Delta - \delta + 1}{n-1}\right). \tag{8}$$

Equality holds if and only if G is either regular or the star $K_{1,n-1}$.

Proof. By Lemmas 6 and 7 we have

$$q_1 + \overline{q}_1 \leqslant \frac{2e + 2\overline{e}}{n-1} + \frac{n-2}{n-1}(\Delta + \overline{\Delta}) + (\Delta - \delta)\left(1 - \frac{\Delta}{n-1}\right) + (\overline{\Delta} - \overline{\delta})\left(1 - \frac{\overline{\Delta}}{n-1}\right).$$

By the fact that $e + \overline{e} = n(n-1)/2$, $\overline{\Delta} = n - 1 - \delta$ and $\overline{\delta} = n - 1 - \Delta$, we obtain

$$q_1 + \overline{q}_1 \leqslant n + \frac{n-2}{n-1}(\Delta - \delta + n - 1) + (\Delta - \delta)\left(1 - \frac{\Delta}{n-1} + \frac{\delta}{n-1}\right).$$

This implies (8).

Now we consider, the equality case in (8). If G is regular both side of (8) are equal $2n - 2$. If $G = K_{1,n-1}$, both sides of (8) are equal $3n - 4$. Now, assume that equality occurs in (8) for G. Then the equalities must hold in (1) and (2) for both G and $\overline{G}$. We may assume that G is not regular. By the case of equality in (2), G must be a bipartite semiregular for which the equality does not occur in (1) unless $\Delta = n - 1$. Hence $G = K_{1,n-1}$. □

Remark 20. The inequality of Theorem 19 is stronger than Conjecture 17 as shown below. By Theorem 19, it suffices to show that

$$(\Delta - \delta)\left(2 - \frac{\Delta - \delta + 1}{n-1}\right) \leqslant n - 2.$$

Note that for any graph G, $\Delta - \delta \leqslant n - 2$. Let

$$f(x) := x\left(2 - \frac{x+1}{n-1}\right).$$

Then $f'(x) = 2 - (2x+1)/(n-1)$ which is positive for $x < n - 3/2$. It follows that

$$f(\Delta - \delta) \leqslant f(n-2) = n - 2,$$

as desired. The equality holds if and only if $\Delta - \delta = n - 2$. From the equality case in Theorem 19, this is possible if and only if $G = K_{1,n-1}$.

Noting that $\frac{5}{18}\left(4+\sqrt{14}\right) \approx 2.15$, the following proposition disproves Conjecture 18.

Proposition 21. *For any positive integer n, there is a graph H_n on n vertices with*

$$q_1(H_n)q_1(\overline{H_n}) = \frac{5}{18}\left(4+\sqrt{14}\right)n^2 + O(n).$$

Proof. Let $n = 6k+s$ for some positive integers k and $-3 \leqslant s \leqslant 2$. Set $H_n = \overline{K_{n-k}} \vee K_k$, where '$\vee$' shows the join of two graphs. Then, $\overline{H_n} = K_{n-k} \cup \overline{K_k}$ with $q_1(\overline{H_n}) = 2(n-k-1)$. The partition of $V(H_n)$ into the k-clique and the $(n-k)$-independent set is an equitable partition with the quotient matrix

$$\begin{pmatrix} n+k-2 & n-k \\ k & k \end{pmatrix}.$$

It follows that

$$q_1(H_n) = \frac{n}{2} + k - 1 + \frac{1}{2}\sqrt{n^2+4nk-4n-4k^2+4}.$$

Substituting $k = (n-s)/6$ yields that

$$\begin{aligned} q_1(H_n)q_1(\overline{H_n}) =& \frac{5n+s-6}{18}\left(4n-s-6+\sqrt{14n^2-4ns-36n-s^2+36}\right) \\ =& \frac{5}{18}\left(4+\sqrt{14}\right)n^2 + O(n). \end{aligned}$$

□

Based on the above result, we pose the following.

Problem. Let $\rho(n) := \max\{q_1(G)q_1(\overline{G}) \mid G \text{ is a simple graph of order } n\}$. Is it true that $\lim_{n\to\infty} \frac{\rho(n)}{n^2}$ exists and equals $\frac{5}{18}\left(4+\sqrt{14}\right)$?

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References

[1] A.T. Amin and S.L. Hakimi, Upper bounds on the order of a clique of a graph, *SIAM J. Appl. Math.* **22** (1972), 569–573.

[2] M. Aouchiche and P. Hansen, A survey of Nordhaus–Gaddum type relations, *Discrete Appl. Math.* **161** (2013), 466–546.

[3] Y.-H. Bao, Y.-Y. Tan, and Y.-Z. Fan, The Laplacian spread of unicyclic graphs, *Appl. Math. Lett.* **22** (2009), 1011–1015.

[4] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.

[5] Y. Chen and L. Wang, The Laplacian spread of tricyclic graphs, *Electron. J. Combin.* **16** (2009), Research Paper 80, 18 pp.

[6] P. Csikvári, On a conjecture of V. Nikiforov, *Discrete Math.* **309** (2009), 4522–4526.

[7] D.M. Cvetković, P. Rowlinson, and S.K. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2010.

[8] K.Ch. Das, Maximizing the sum of squares of the degrees of a graph, *Discrete Math.* **285** (2004), 57–66.

[9] Y.Z. Fan, S.D. Li, and Y.Y. Tan, The Laplacian spread of bicyclic graphs, *J. Math. Res. Exposition* **30** (2010), 17–28.

[10] Y.-Z. Fan, J. Xu, Y. Wang, and D. Liang, The Laplacian spread of a tree, *Discrete Math. Theor. Comput. Sci.* **10** (2008), 79–86.

[11] L. Feng and G. Yu, On three conjectures involving the signless laplacian spectral radius of graphs, *Publ. Inst. Math. (Beograd) (N.S.)* **85** (2009), 35–38.

[12] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.* **23(98)** (1973), 298–305.

[13] P. Li, J.S. Shi, and R.L. Li, Laplacian spread of bicyclic graphs, (Chinese) *J. East China Norm. Univ. Natur. Sci. Ed.* **2010**, 6–9.

[14] Y. Liu, The Laplacian spread of cactuses, *Discrete Math. Theor. Comput. Sci.* **12** (2010), 35–40.

[15] Y. Liu and L. Wang, The Laplacian spread of bicyclic graphs, *Advances in Mathematics (China)* **40** (2011), 759–764.

[16] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra Appl.* **285** (1988), 33–35.

[17] V. Nikiforov, Eigenvalue problems of Nordhaus–Gaddum type, *Discrete Math.* **307** (2007), 774–780.

[18] E.A. Nordhaus and J. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956), 175–177.

[19] E. Nosal, *Eigenvalues of Graphs*, Master Thesis, University of Calgary, 1970.

[20] T. Terpai, Proof of a conjecture of V. Nikiforov, *Combinatorica* **31** (2011), 739–754.

[21] Y. Xu and J. Meng, The Laplacian spread of quasi-tree graphs, *Linear Algebra Appl.* **435** (2011), 60–66.

[22] Z. You and B. Liu, The Laplacian spread of graphs, *Czechoslovak Math. J.* **62(137)** (2012), 155–168.

[23] M. Zhai, J. Shu, and Y. Hong, On the Laplacian spread of graphs, *Appl. Math. Lett.* **24** (2011), 2097–2101.