Ascent-Descent Young Diagrams and Pattern Avoidance in Alternating Permutations

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Abstract

We investigate pattern avoidance in alternating permutations and an alternating analogue of Young diagrams. In particular, using an extension of Babson and West's notion of shape-Wilf equivalence described in our recent paper (with N. Gowravaram), we generalize results of Backelin, West, and Xin and Ouchterlony to alternating permutations. Unlike Ouchterlony and Bóna's bijections, our bijections are not the restrictions of Backelin, West, and Xin's bijections to alternating permutations. This paper is the second of a two-paper series presenting the work of *Beyond alternating permutations: Pattern avoidance in Young diagrams and tableaux* (with N. Gowravaram, arXiv:1301.6796v1). The first paper in the series is *Beyond alternating permutations: Pattern avoidance in Young diagrams and tableaux* (with N. Gowravaram, *Electronic Journal of Combinatorics* 20(4):#P17, 2013).

Keywords: pattern avoidance; alternating permutation; shape-Wilf equivalence

1 Introduction

This paper proves a special case of a conjecture of Gowravaram and the author [5, 6]. We now review the definitions of AD-Young diagrams and the alternating and semi-alternating conditions from the recent paper [6] in order to state our main result.

1.1 AD-Young diagrams and the shape-equivalence of permutation matrices

For a nonnegative integer n, let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$. Given a permutation p, let $M(p)$ denote its permutation matrix, and given matrices A and B, let $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. We assume that the reader is familiar with the basic terminology of Young diagrams and tableaux; see, for example, [3, Chapters 2 and 6]. We draw Young diagrams in English notation and use matrix coordinates, and for example $(1, 2)$ is the second square in the first row of a Young diagram. Furthermore, we require all Young diagrams to have the same number of rows and columns.

Definition 1.1 ([6], Definition 2.1). Let Y be a Young diagram with k rows. If A and D are disjoint subsets of $[k-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length, then we call the triple $\mathcal{Y} = (Y, A, D)$ an *AD-Young diagram*. We call Y the Young diagram of $\mathcal{Y}$, A the *required ascent set* of $\mathcal{Y}$, and D the *required descent set* of $\mathcal{Y}$. Figure 1 gives an example of an AD-Young diagram.

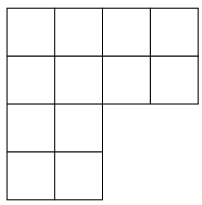

Figure 1: If $Y = (4^2, 2^2)$, $A = \emptyset$, and $D = \{3\}$, then (Y, A, D) is an AD-Young diagram.

As in [1, 2], a *transversal* of Young diagram Y is a set of squares $T = \{(i, t_i)\}$ such that every row and every column of Y contains exactly one member of T.

Definition 1.2 ([6], Definition 2.2). Given a transversal $T = \{(i, t_i)\}$, let $\mathrm{Asc}(T) = \{i \in [k-1] \mid t_i < t_{i+1}\}$ and $\mathrm{Des}(T) = \{i \in [k-1] \mid t_i > t_{i+1}\}$. We call $\mathrm{Asc}(T)$ the *ascent set* of T and $\mathrm{Des}(T)$ the *descent set* of T. If $A \subseteq \mathrm{Asc}(T)$ and $D \subseteq \mathrm{Des}(T)$, then we say that T a *valid transversal* of $\mathcal{Y}$.

Example 1.3 ([6], Example 2.3). If T is a transversal of a Young diagram Y, then T is a valid transversal of the AD-Young diagram $(Y, \emptyset, \emptyset)$.

In this paper, we restrict ourselves to the AD-Young analogues of alternating and reverse alternating permutations, as defined below.

Definition 1.4 ([6], Definition 2.4). Given positive integers x, y and an AD-Young diagram (Y, A, D) such that Y has k rows, we say that (Y, A, D) is *x, y-alternating* if A, D satisfy the property that if $x - 1 \leqslant i \leqslant k - y$, then $i \in A$ if and only if $i + 1 \in D$.

If $\mathcal{Y}$ is x, y-alternating, then $\mathcal{Y}$ is a, b-alternating for all a, b with $a \geqslant x$ and $b \geqslant y$.

Definition 1.5 ([6], Definition 2.5)**.** If $\mathcal{Y}$ is $1, y$-alternating, then we say that $\mathcal{Y}$ is y-alternating, while if $\mathcal{Y}$ is $2, y$-alternating, then we say that $\mathcal{Y}$ is y-semialternating.

In particular, if $\mathcal{Y} = (Y, A, D)$ is an AD-Young diagram with $k \geqslant y$ columns and $1 \in D$, then $\mathcal{Y}$ cannot be y-alternating, but $\mathcal{Y}$ can be y-semialternating. Alternating AD-Young diagrams are the counterpart of alternating permutations, while semialternating AD-Young diagrams allow reverse alternating permutations.

Example 1.6 ([6], Example 2.6). Let $Y = (4^4)$. Then, $(Y, \{1\}, \{2\})$ is 1-alternating, while $(Y, \{1, 3\}, \{2\})$ is 2-alternating but not 1-alternating. Furthermore, $(Y, \{2, 4\}, \{1, 3\})$ is 1-semialternating but not y-alternating for $y \leqslant 4$.

The notion of pattern avoidance is exactly as in [1, 2]. A transversal $T = \{(i, t_i)\}$ of a Young diagram Y *contains* a $r \times r$ permutation matrix M if there are rows $a_1 < a_2 < \cdots < a_r$ and columns $b_1 < b_2 < \cdots < b_r$ of Y such that $(a_r, b_r) \in Y$ and the restriction of T to the rows a_i and the columns b_i has contains exactly the squares where M has ones. If T does not contain M, then T *avoids* M (see Figure 2).

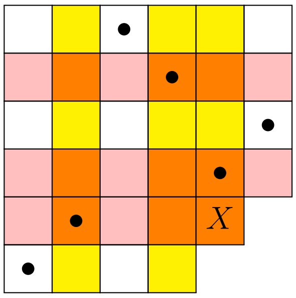

Figure 2: The transversal $T = \{(1, 3), (2, 4), (3, 6), (4, 5), (5, 2), (6, 1)\}$ of $Y = (6^4, 5, 4)$ contains $M(231)$ because the restriction of T to the yellow columns and the pink rows rows is a copy of $M(231)$ in T. We require that $X \in Y$. However, T does not contain $M(4321)$: for example, the restriction of T to rows $3, 4, 5, 6$ and columns $1, 2, 5, 6$ is not a copy of $M(4321)$ in T because $(6, 6) \notin Y$.

Given an AD-Young diagram $\mathcal{Y}$ and a permutation matrix M, let $S_{\mathcal{Y}}(M)$ denote the set of valid transversals of $\mathcal{Y}$ that avoid M.

Definition 1.7 ([6], Definition 2.7)**.** If M and N are permutation matrices such that $|S_{\mathcal{Y}}(M)| = |S_{\mathcal{Y}}(N)|$ for all x-alternating AD-Young diagrams $\mathcal{Y}$, we say that M and N are *shape-equivalent for x-alternating AD-Young diagrams* and write $M \underset{x-\mathrm{ASE}}{\sim} N$. If we have $|S_{\mathcal{Y}}(M)| = |S_{\mathcal{Y}}(N)|$ for all x-semialternating AD-Young diagrams $\mathcal{Y}$, then we say that M and N are *shape-equivalent for x-semialternating AD-Young diagrams* and write $M \underset{x-\mathrm{SASE}}{\sim} N$.

1.2 The main result of this paper

For a positive integer r, let F_r denote the permutation matrix $M((r-1)(r-2)\cdots 1r)$. In this paper, we prove the following result.

Theorem 1.8. *We have* $F_3 \underset{1-SASE}{\sim} J_3$.

Due to the following result of Gowravaram and the author, Theorem 1.8 in fact yields an infinite family of shape-equivalences.

Theorem 1.9 ([6], Extension Theorem 2.9)**.** *If permutation matrices* M *and* M' *are shape-equivalent for* x*-alternating (resp.* x*-semialternating) AD-Young diagrams and* C *is an* $r \times r$ *permutation matrix, then we have* $M \oplus C \underset{(x+r)-ASE}{\sim} M' \oplus C$ *(resp.* $\underset{(x+r)-SASE}{\sim}$ *).*

Theorem 1.8 has consequences in the theory of pattern-avoiding alternating permutations. We now recall the basic definitions. Let S_n denote the set of permutations of $[n]$. We treat a permutation $w \in S_n$ as a sequence $w_1 w_2 w_3 \cdots w_n$ that contains every element of $[n]$ exactly once. A permutation w is said to *contain* a permutation q if there is a subsequence of w that is order-isomorphic to q. For example, the subsequence 246 of 214536 shows that 214536 contains 123. If w does not contain q, we say that w *avoids* q. A permutation $w \in S_n$ is called *alternating* if $w_1 < w_2 > w_3 < \cdots$ and *reverse alternating* if $w_1 > w_2 < w_3 > \cdots$. Reverse alternating permutations can be transformed into alternating permutations (and vice versa) by the *complementation* map that sends a permutation $w = w_1 w_2 \cdots w_n$ to $w^c = (n+1-w_1)(n+1-w_2)\cdots(n+1-w_n)$. Given a pattern q, let $A_n(q)$ (resp. $A'_n(q)$) denote the set of alternating (resp. reverse alternating) permutations of length n that avoid q. If p and q are such that $|A_n(p)| = |A_n(q)|$ (resp. $|A'_n(p)| = |A'_n(q)|$) for all even n, we say that p and q are *equivalent for even-length alternating* (resp. *reverse alternating*) permutations and we write $p \underset{\text{even}}{\sim} q$ (resp. $p \underset{\text{even}}{\sim^r} q$). We make similar definitions for odd-length permutations.

Gowravaram and the author [6] proved the following two results that link shape-equivalence for AD-Young diagrams to equivalence for alternating permutations.

Proposition 1.10 ([6], Proposition 2.8)**.** *Let* p *and* q *be permutations.*

(a) If $M(p) \underset{1-ASE}{\sim} M(q)$*, then* $p \underset{odd}{\sim} q$.

(b) If $M(p) \underset{1-SASE}{\sim} M(q)$*, then* $p \underset{even}{\sim^r} q$.

(c) If $M(p) \underset{2-ASE}{\sim} M(q)$*, then* $p \underset{even}{\sim} q$.

(d) If $M(p) \underset{2-SASE}{\sim} M(q)$*, then* $p \underset{odd}{\sim^r} q$.

The following corollaries of Theorem 1.8 follow immediately from Propositions 1.10 and 1.8, and the Extension Theorem 1.9.

Corollary 1.11. *For all* $t > 3$ *and all permutations* q *of* $[t] \setminus [3]$*, the patterns* $213q$*, and* $321q$ *are equivalent for even- and odd-length alternating and reverse-alternating permutations.*

Corollary 1.12. *For all* $t \geqslant 3$ *and all permutations* q *of* $[t]$*, the patterns* $(t-1)t(t-2)q$ *and* $(t-2)(t-1)tq$ *are equivalent for even- and odd-length alternating permutations.*

1.3 Relationship to other work

Babson and West [1] proved that F_3 and J_3 are shape-Wilf equivalent, which is the analogue of Theorem 1.8 for ordinary Young diagrams. Backelin, West, and Xin [2] proved that F_r and J_r are shape-Wilf equivalent. We apply the ideas behind Backelin, West, and Xin's work in defining bijections to prove Theorem 1.8.

Gowravaram and the author [5, 6] proved the equivalence $F_2 \underset{1-\mathrm{ASE}}{\sim} J_2$ and conjectured that $F_r \underset{1-\mathrm{SASE}}{\sim} J_r$ holds for $r \geqslant 3$. This conjecture has recently been fully proven by Yan [9]. Bóna [4] proved special cases of the consequences of Gowravaram and the author's result that $F_2 \underset{1-\mathrm{ASE}}{\sim} J_2$, while Ouchterlony [8] proved a variant of the shape-equivalence $F_2 \underset{1-\mathrm{ASE}}{\sim} J_2$ for doubly alternating permutations, which are alternating permutations whose inverses are alternating. Several related conjectures were posed by Lewis [7].

Our bijection between the sets $S_{\mathcal{Y}}(F_3)$ and $S_{\mathcal{Y}}(J_3)$ differs from Backelin, West, and Xin's and Yan's bijections in many cases when $\mathcal{Y}$ has non-empty sets of required ascents or descents. On the other hand, the bijections of Bóna [4], Ouchterlony [8], and Gowravaram and the author [6] are restrictions of Backelin, West, and Xin's bijections to alternating permutations. The essential difficulty is that the bijection that Backelin, West, and Xin use to prove that F_3 and J_3 are shape-Wilf equivalent does not preserve ascents and descents, and therefore the induced bijection between $S_n(321q)$ and $S_n(213q)$ does not preserve the alternating property.

This paper is the second of a two-paper series presenting the work of [5] (joint work with N. Gowravaram); the first paper is [6].

1.4 Outline of the paper

The remainder of this paper is devoted to the proof of Theorem 1.8. The idea of the proof is to establish a bijection between $S_{\mathcal{Y}}(F_3)$ and $S_{\mathcal{Y}}(J_3)$ for $\mathcal{Y}$ a 1-alternating AD-Young diagram. Similar to the first proof of [2, Proposition 3.1], our bijection selects a copy of J_3 (resp. F_3) in a transversal and removes it, but significant complications arise due to the required ascent and descent sets. We divide into cases based on the locations of required ascents and descents near the rightmost entry of the copy of J_3 (resp. F_3) and convert the copy to an instance of F_3 (resp. J_3) in a manner that maintains required ascents and descents. The fact that rows of Y have equal size at required ascents and descents of $\mathcal{Y}$ plays a critical role in ensuring that the replacement algorithm returns a valid transversal of $\mathcal{Y}$. Similar to Backelin, West, and Xin [2]'s method, we restrict ourselves to so-called *separable* transversals (a class of transversals that contains any transversal that avoids F_3 or J_3) because the two replacement procedures are not inverse in general for non-separable transverals. Due to the more elaborate process of removing copies of J_3 and F_3, our notion of separability becomes slightly more technical than the notion implicitly used by Backelin, West, and Xin [2].

The organization of the paper is as follows. In Section 2, we fix some useful notation that we use to describe our bijections. In Section 3, we state our bijection, and in Section 4, we prove Theorem 1.8. In Section 5, we explain a difference between our bijection and that of Backelin, West, and Xin. The proofs of two technical results used in the proof of Theorem 1.8 are deferred to Sections 6 and 7.

2 Cyclic Shifts

Fix a Young diagram Y with n columns for the entirety of this section and let $T = \{(i, b_i)\}$ be a transversal of Y. Let $\mathcal{T}$ denote the set of transversals of Y. We define a function $\omega_M^P : \mathcal{T} \to \mathcal{T}$ for sets $M, P \subseteq [1, n]$ with $m = \max M$ and $p = \max P$, such that the mth row of Y has at least p squares. Let $i_1 < i_2 < \cdots < i_k$ denote the indices $i_j \in M$ with $b_{i_j} \in P$. Take the index j of i_j modulo k, and let $\Gamma_M^P(T) = \{i_j \mid j \in [k]\}$. Then, we define

$$\omega_M^P(T) = T \setminus \{(i_j, b_{i_j}) \mid i \in [k+1]\} \cup \{(i_j, b_{i_{j-1}}) \mid j \in [k]\}.$$

We now define the function $\theta_M^P : \mathcal{T} \to \mathcal{T}$, which takes the same arguments as ω_M^P and will be proven to be the inverse of ω_M^P. Let $\{i_1 < i_2 < \cdots < i_k\} = \Gamma_M^P(T)$, and we define

$$\theta_M^P(T) = T \setminus \{(i_j, b_{i_j}) \mid i \in [k+1]\} \cup \{(i_j, b_{i_{j+1}}) \mid j \in [k]\}.$$

Because the mth row of Y has at least p boxes, ω_M^P and θ_M^P return transversals of Y. Notice that

$$\Gamma_M^P(T) = \Gamma_M^P\left(\omega_M^P(T)\right),$$

and hence $\omega_M^P(\cdot)$ and $\theta_M^P(\cdot)$ are inverses of each other. Furthermore, if $i \notin M$, the position of the element of T in the ith column of Y is the same as that of $\omega_M^P(T)$ and in $\theta_M^P(T)$. If $M \times P$ and $M' \times P'$ are disjoint, then it is clear that $\omega_M^P(\cdot)$ and $\theta_M^P(\cdot)$ each commute with $\omega_{M'}^{P'}(\cdot)$ and $\theta_{M'}^{P'}(\cdot)$. The functions ω_M^P and θ_M^P cyclically alter certain entries b_i of a transversal $T = \{(i, b_i)\}$. See Figure 3 for an example of cyclic shifts.

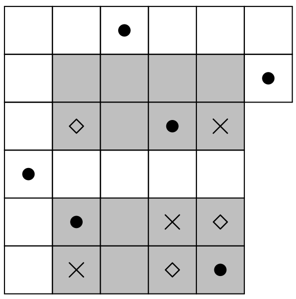

Figure 3: Let $Y = (6^2, 5^4)$ and let $T = \{(1,3), (2,6), (3,4), (4,1), (5,2), (6,5)\}$. Then, $\Gamma_{[2,3]\cup[5,6]}^{[2,5]}(T) = \{3, 5, 6\}$, and thus $\omega_{[2,3]\cup[5,6]}^{[2,5]}(T) = \{(1,3), (2,6), (3,5), (4,1), (5,4), (6,2)\}$ and $\theta_{[2,3]\cup[5,6]}^{[2,5]}(T) = \{(1,3), (2,6), (3,2), (4,1), (5,5), (6,4)\}$. Bullets mark elements of T and crosses mark elements of $\omega_{[2,3]\cup[5,6]}^{[2,5]}(T) \setminus T$, while diamonds mark elements of $\theta_{[2,3]\cup[5,6]}^{[2,5]}(T) \setminus T$.

3 Statement of the Bijection

We first prove that $F_3 \underset{1-\text{ASE}}{\sim} J_3$. To this end, suppose that $\mathcal{Y} = (Y, A, D)$ is a 1-alternating AD-Young diagram with n rows. We shall define a bijection $\Phi : S_{\mathcal{Y}}(F_3) \to S_{\mathcal{Y}}(J_3)$ and its inverse $\Psi : S_{\mathcal{Y}}(J_3) \to S_{\mathcal{Y}}(F_3)$. To define Φ and Ψ, we first define functions ϕ and ψ, and then obtain Φ and Ψ by iterating ϕ and ψ, respectively. Let $T = \{(i, b_i)\}$ be a transversal of $\mathcal{Y}$. If $a_1 < a_2 < a_3 \in [n]$, then we say that (a_1, a_2, a_3) is a copy of J_3 (resp. F_3) in T if $\{(a_i, b_{a_i}) \mid i \in [3]\}$ is a copy of J_3 (resp. F_3) in T.

Let $T = \{(i, b_i)\}$ be a transversal of $\mathcal{Y}$ that contains J_3. Suppose that (a_1, a_2, a_3) is a copy of J_3 in T. We define auxiliary functions $\phi_\ell^{(a_1,a_2,a_3)}(T)$ for $\ell \in [3]$ (the functions take arguments (a_1, a_2, a_3) and T, and return only transversals of the Young diagram (n^n) *à priori*). We define

$$\begin{aligned}
\phi_1^{(a_1,a_2,a_3)}(T) &= \theta_{\{a_1,a_2,a_3\}}^{[1,b_{a_1}]}(T) \\
\phi_2^{(a_1,a_2,a_3)}(T) &= \omega_{\{a_1,a_3-1\}}^{[1,b_{a_1}]}(T) \\
\phi_3^{(a_1,a_2,a_3)}(T) &= \omega_{[1,a_1]\cup\{a_3+1\}}^{[b_{a_3},b_{a_1}]}\left(\omega_{[a_2,a_3]}^{[b_{a_3},b_{a_1}]}(T)\right).
\end{aligned}$$

The operation ϕ_1 is the one used by Backelin, West, and Xin in their proof of [2, Proposition 3.1].

Let $U(T)$ denote the set of triples $a \in [n]^3$ that are copies of J_3 in T. If $a = (a_1, a_2, a_3) \in U(T)$, then define the J-type of a in the following cases.

Case 1: If ($a_3 - 1 \notin D$ or $b_{a_1} < b_{a_3-1}$) and $a_3 \notin A$, we say that (a_1, a_2, a_3) is of J-type 1.

Case 2: If $a_3 - 1 \in D$ and $b_{a_3-1} < b_{a_1}$, we say that (a_1, a_2, a_3) is of J-type 2.

Case 3: If ($a_3 - 1 \notin D$ or $b_{a_3-1} > b_{a_1}$) and $a_3 \in A$, we say that (a_1, a_2, a_3) is of J-type 3.

See Figures 4, 5, and 6 for geometric descriptions of the functions ϕ_ℓ.

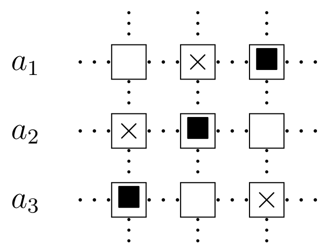

Figure 4: We show the effect of $\phi_1^{(a_1,a_2,a_3)}$ on a transversal T. Black boxes mark the selected elements of T while crosses mark elements of $\phi_1^{(a_1,a_2,a_3)}(T) \setminus T$.

For $u = (u_1, u_2, u_3) \in \mathbb{N}^3$, let $\#(u) = (u_3, u_1, u_2)$. Let $h_J(T)$ be the triple $a \in U(T)$ that minimizes $\#(a)$ in the lexicographic order. This is exactly the way in which a copy

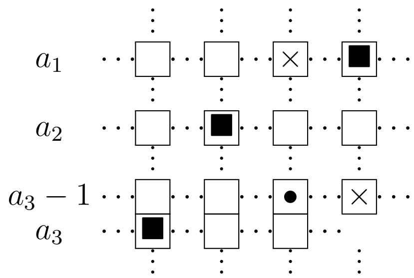

Figure 5: We show the effect of $\phi_2^{(a_1,a_2,a_3)}$ on a transversal T. Black boxes mark the selected elements of T and bullets mark other elements of T, while crosses mark elements of $\phi_1^{(a_1,a_2,a_3)}(T) \setminus T$.

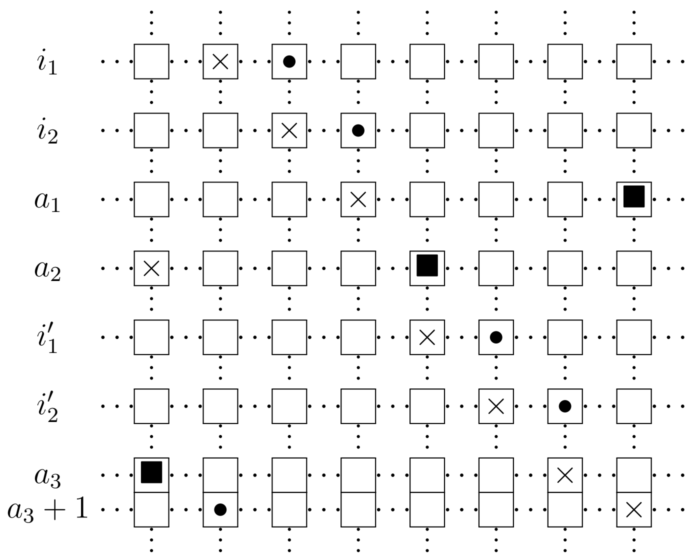

Figure 6: We show the effect of $\phi_3^{(a_1,a_2,a_3)}(T)$ on a transversal T, supposing that $\{i_1, i_2\} = \Gamma_{[1,a_1)}^{[b_{a_3},b_{a_1}]}(T)$ and $\{i'_1, i'_2\} = \Gamma_{(a_2,a_3)}^{[b_{a_3},b_{a_1}]}(T)$. Black boxes mark the selected elements of T and bullets mark other elements of T, while crosses mark elements of $\phi_1^{(a_1,a_2,a_3)} \setminus T$.

of J_3 is chosen to be removed in the proof of [2, Proposition 3.1]. If $h_J(T)$ is of J-type t, let $\phi(T) = \phi_t^{h_J(T)}(T)$, and we say that T is of J-type t.

We define the functions ψ_ℓ, which take the same arguments as the ϕ_ℓ and return only transversals of (n^n) *á priori*. For $\ell \in [3]$, let

$$\begin{aligned}
\psi_1^{(a_1,a_2,a_3)} &= \omega_{\{a_1,a_2,a_3\}}^{[1,b_{a_3}]}(T) \\
\psi_2^{(a_1,a_2,a_3)} &= \theta_{\{a_1,a_3\}}^{[1,b_{a_3}]}(T) \\
\psi_3^{(a_1,a_2,a_3)} &= \theta_{[a_2,a_3-1]}^{[b_{a_2},b_{a_3}]}\left(\theta_{[1,a_1]\cup\{a_3\}}^{[b_{a_2},b_{a_3}]}(T)\right).
\end{aligned}$$

The operation ψ_1 is the one used by Backelin, West, and Xin in their proof of [2, Proposition 3.1].

Let $V(T)$ denote the set of triples $a = (a_1, a_2, a_3) \in [n]^3$ that are copies of F_3 in T such that $a_3 \notin A$. For each $a \in V(T)$, we define a triple $S(a)$ that will depend on the F-type of a. The triple $S(a)$ will be independent of T, and ψ will convert a copy of F_3 at a to a copy of J_3 at $S(a)$. For $a = (a_1, a_2, a_3) \in V(T)$, we define the F-type of a in the in the following cases.

Case 1: If $a_3 - 1 \notin A$, let $S(a) = (a_3, a_1, a_2)$. We say that a is of F-type 1.

Case 2: If $a_3 - 1 \in A$ and $a_2 = a_3 - 1$, let $S(a) = (a_3 + 1, a_1, 0)$. We say that a is of F-type 2.

Case 3: If $a_3 - 1 \in A$ and $a_2 \neq a_3 - 1$, let $S(a) = (a_3 - 1, a_1, a_2)$. We say that a is of F-type 3.

See Figures 7, 8, and 9 for geometric descriptions of the functions ψ_ℓ.

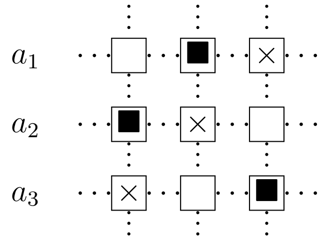

Figure 7: We show the effect of $\psi_1^{(a_1,a_2,a_3)}$ on a transversal T. Black boxes mark the selected elements of T, while crosses mark elements of $\psi_1^{(a_1,a_2,a_3)}(T) \setminus T$.

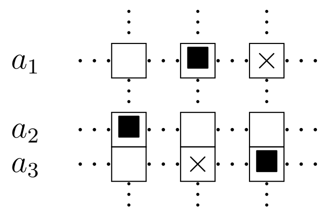

Figure 8: We show the effect of $\psi_2^{(a_1,a_2,a_3)}$ on a transversal T. Black boxes mark the selected elements of T, while crosses mark elements of $\psi_2^{(a_1,a_2,a_3)}(T) \setminus T$.

For $u, u' \in V(T)$, we write $u \triangleright u'$ if $S(u) \geqslant S(u')$ in the lexicographic order. We will select a copy of F_3 to eliminate by treating $\triangleright$ as a total order on $V(T)$. To do so, we require the following lemma.

Lemma 3.1. *S is injective, and thus $\triangleright$ is a total order on $V(T)$.*

Proof. Suppose that $S(u) = (d_3, d_1, d_2)$. If $d_2 = 0$, then u and u' are of F-type 2, and thus $u = (d_1, d_3 - 2, d_3 - 1)$. If $d_2 \neq 0$ and $d_3 \in A$, then u and u' are of F-type 3, and hence $u = (d_1, d_2, d_3 + 1)$. Otherwise, we have $d_2 \neq 0$ and $d_3 \notin A$, which implies that u is of F-type 1 and $u = (d_1, d_2, d_3)$. It follows that S has a left inverse and is therefore injective. The fact that $\triangleright$ is a total order follows. $\square$

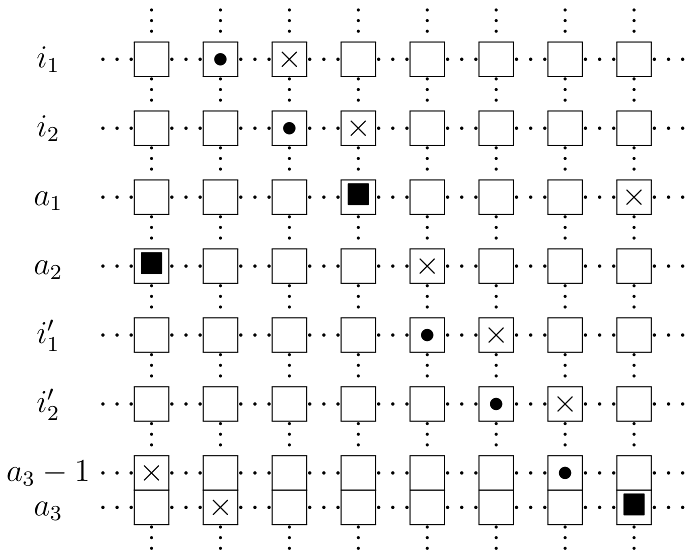

Figure 9: We show the effect of $\psi_3^{(a_1,a_2,a_3)}$ on a transversal T, supposing that $\{i_1, i_2\} = \Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1)}(T)$ and $\{i'_1, i'_2\} = \Gamma^{[b_{a_2}, b_{a_3}]}_{(a_2,a_3)}(T)$. Black boxes mark the selected elements of T and bullets mark other elements of T, while crosses mark elements of $\psi_3^{(a_1,a_2,a_3)}(T) \setminus T$.

Let $h_F(T)$ denote the maximum of $V(T)$ with respect to the restriction of $\triangleright$. If $A = D = \emptyset$, then $h_F(T)$ agrees with Backelin, West, and Xin's selection of a copy of F_3 to remove in their proof of [2, Proposition 3.1], but in general, $h_F(T)$ differs from Backelin, West, and Xin's selection. If $h_F(T)$ is of F-type t, let $\psi(T) = \psi_t^{h_F(T)}(T)$, and we say that T is of F-type t.

4 Proof of Theorem 1.8

The functions ϕ and ψ are not inverses on general transversals of $\mathcal{Y}$, but we are only interested in the transversals $\phi^m(T)$ and $\psi^m(T)$ for $T \in S_{\mathcal{Y}}(F_3) \cup S_{\mathcal{Y}}(J_3)$. We now define the class of transversals that we will consider. A transversal T is said to be *separable* if it satisfies the property that if $u \in U(T)$ and $u' = S\,(V(T))$, then $\#(u) \geqslant u'$ in the lexicographic order. Any element of $S_{\mathcal{Y}}(J_3)$ (resp. $S_{\mathcal{Y}}(F_3)$) is separable, as $U(T)$ (resp. $V(T)$) is empty. We restrict our attention to separable transversals.

The critical properties of ϕ and ψ are the following two propositions.

Proposition 4.1. *If T is a separable valid transversal of $\mathcal{Y}$ that contains J_3, then $\phi(T)$ is a separable valid transversal of $\mathcal{Y}$ and we have $\psi(\phi(T)) = T$. Furthermore, if $T = \{(i, b_i)\}$ and $\phi(T) = \{(i, c_i)\}$, then we have $(b_1, b_2, \ldots, b_n) > (c_1, c_2, \ldots, c_n)$ in the lexicographic order.*

Proposition 4.2. *If T is a separable valid transversal of $\mathcal{Y}$ that contains F_3, then $\psi(T)$ is a separable valid transversal of $\mathcal{Y}$ and we have $\phi(\psi(T)) = T$. Furthermore, if $T = \{(i, b_i)\}$*

and $\psi(T) = \{(i, c_i)\}$, then we have $(b_1, b_2, \ldots, b_n) < (c_1, c_2, \ldots, c_n)$ in the lexicographic order.

We defer the proofs of Propositions 4.1 and 4.2 to Sections 6 and 7, respectively. To complete the proof of Theorem 1.8, we require a simple technical lemma to extend from alternating to semialternating AD-Young diagrams. The lemma follows immediately from the definitions of ϕ and ψ, and so we omit its proof.

Lemma 4.3. *Let $T = \{(i, b_i)\}$ be a valid transversal of $\mathcal{Y}$ with $b_1 = 1$. If T contains J_3 and $\phi(T) = \{(i, c_i)\}$, then we have $c_1 = 1$. If T contains F_3 and $\psi(T) = \{(i, c_i)\}$, then we have $c_1 = 1$.*

Proof of Theorem 1.8 assuming Propositions 4.1 and 4.2. Let $\mathcal{Y}$ be a 1-alternating AD-Young diagram with n rows and let $T \in S_{\mathcal{Y}}(F_3)$. Because $V(T) = \emptyset$, the transversal T is separable. We claim that there exists an integer m such that $\phi^m(T) \in S_{\mathcal{Y}}(J_3)$. It follows from by Proposition 4.1 that for all $m \geqslant 1$, the iterate $\phi^m(T)$ exists and is a separable, valid transversal of $\mathcal{Y}$ provided that the iterate $\phi^{m-1}(T)$ exists, is a separable valid transversal of $\mathcal{Y}$, and contains J_3. Assume for sake of deriving a contradiction that $\phi^m(T) \notin S_{\mathcal{Y}}(T)$ for all m. In this case, $\phi^m(T)$ exists and is a separable, valid transversal of $\mathcal{Y}$ that contains J_3, for all $m \geqslant 0$. The finiteness of the set of valid transversals of $\mathcal{Y}$ implies that there exist integers $p < q$ such that $\phi^p(T) = \phi^q(T)$. Let $\phi^m(T) = \{(i, b_i^{(m)}) \mid i \in [n]\}$. Proposition 4.1 implies that $\left(b_1^{(m)}, b_2^{(m)}, \ldots, b_n^{(m)}\right) > \left(b_1^{(m+1)}, b_2^{(m+2)}, \ldots, b_n^{(m+1)}\right)$ in the lexicographic order for all $m \geqslant 0$. Therefore, we have $\left(b_1^{(p)}, b_2^{(p)}, \ldots, b_n^{(p)}\right) > \left(b_1^{(q)}, b_2^{(q)}, \ldots, b_n^{(q)}\right)$ in the lexicographic order, which contradicts the assumption that $\phi^p(T) = \phi^q(T)$. Hence, we can conclude that there exists an integer m such that $\phi^m(T) \in S_{\mathcal{Y}}(J_3)$.

Let M be the smallest integer m such that $\phi^m(T) \in S_{\mathcal{Y}}(J_3)$. Then, let $\Phi(T) = \phi^m(T)$, and Φ defines a function from $S_{\mathcal{Y}}(F_3)$ to $S_{\mathcal{Y}}(J_3)$. Define $\Psi : S_{\mathcal{Y}}(J_3) \to S_{\mathcal{Y}}(F_3)$ analogously, using Proposition 4.2. We claim that Φ and Ψ are inverses of each other. Let $T \in S_{\mathcal{Y}}(F_3)$, and suppose that $\Phi(T) = \phi^m(T)$. By m applications of Proposition 4.1, we have $\psi^m(\Phi(T)) = T$, and because $\psi(W)$ is defined only for $W \notin S_{\mathcal{Y}}(F_3)$, we have $\Psi(\Phi(T)) = \psi^m(\Phi(T)) = T$. A similar argument using Proposition 4.2 demonstrates that $\Phi(\Psi(T)) = T$ for all $T \in S_{\mathcal{Y}}(J_3)$, and therefore Φ and Ψ define inverse bijections.

Suppose that $\mathcal{Y}' = (Y', A', D')$ is a 1-semialternating AD-Young diagram, and let $Y' = (Y_1', Y_2', \ldots, Y_n')$. If $1 \notin D'$, then it is clear that $\mathcal{Y}'$ is 1-alternating and $|S_{\mathcal{Y}}(F_3)| = |S_{\mathcal{Y}}(J_3)|$. If $1 \in D$, then let $Y = (Y_1'+1, Y_1'+1, Y_2'+1, Y_3'+1, \ldots, Y_n'+1)$, let $A = \{1\} \cup (A'+1)$, and let $D' = D+1$. The AD-Young diagram $\mathcal{Y} = (Y, A, D)$ is 1-alternating. For $T' = \{(i, b_i')\}$ a valid transversal of $\mathcal{Y}'$, let $\alpha(T) = \{(1,1)\} \cup \{(i+1, b_i'+1) \mid i \in [n]\}$. It is clear that $\alpha(T)$ is a valid transversal of $\mathcal{Y}$, and that α is injective. Furthermore, if $T' \in S_{\mathcal{Y}'}(F_3)$ (resp. $S_{\mathcal{Y}'}(J_3)$), then $\alpha(T') \in S_{\mathcal{Y}}(F_3)$ (resp. $S_{\mathcal{Y}}(F_3)$) because $(1,1)$ cannot be an element of any copy of F_3 (resp. J_3) in $\alpha(T')$. Define $\Phi' = \alpha^{-1} \circ \Phi \circ \alpha$. Lemma 4.3 implies that Φ sends the range of α to the range of α. Together with the fact that α is injective, it follows that $\Phi'(T)$ is defined (and well-defined) for all $T \in S_{\mathcal{Y}}(F_3)$. It is clear that Φ' sends $S_{\mathcal{Y}'}(F_3)$ to $S_{\mathcal{Y}'}(J_3)$. We define Ψ' analogously. Because Φ and Ψ are inverses of each other, so are Φ' and Ψ'. Hence, we have that $|S_{\mathcal{Y}'}(F_3)| = |S_{\mathcal{Y}'}(J_3)|$, and the fact that $F_3 \underset{1-\text{SASE}}{\sim} J_3$ follows. $\square$

5 Comparison with Backelin *et al.*'s Bijection

Let $Y = (5^5)$, $A = \{3\}$, and $D = \{4\}$, and let $T = \{(1,5),(2,4),(3,1),(4,3),(5,2)\}$ be a transversal of $\mathcal{Y} = (Y, A, D)$. Then, T avoids F_3 but contains J_3. Indeed, we have $h_J(T) = (1,2,3)$, the transversal is of J-type 3, and $\phi(T) = \{(1,3),(2,1),(3,4),(4,5),(5,2)\}$ avoids J_3. Thus, we have $\Phi(T) = \phi(T)$. However, Backelin, West, and Xin [2]'s version of Φ would send T to $T' = \{(1,3),(2,1),(3,5),(4,2),(5,4)\}$ after two iterations of the corresponding version of ϕ. Because $\Phi(T) \neq T'$, our bijections are not the restrictions of those of Backelin, West, and Xin [2]. Indeed, T' is not a valid transversal of $\mathcal{Y}$. In the case when the ascent and descent sets of an AD-Young diagram are empty, then all transversals are of J-type and F-type 1, and our bijections agree with those of Backelin, West, and Xin [2].

6 Proof of Proposition 4.1

Backelin, West, and Xin's proof of [2, Proposition 3.1] involves a subboard E. We consider a similar board, and it plays a substantial role in the following proofs. Let T be a separable valid transversal of $\mathcal{Y}$ that contains J_3, and let $h_J(T) = (a_1, a_2, a_3)$. We define a subset of Y called $E_\phi(T)$ that will be free of elements of T by the definition of h_J. Let

$$E_\phi(T) = \left(\begin{array}{l} ([1,a_1) \times [b_{a_2}, Y_{a_3}]) \cup ((a_1,a_2) \times [b_{a_3}, b_{a_1}]) \\ \cup\, ((a_2,a_3) \times [1, b_{a_2}]) \cup ((a_3,\infty) \times (b_{a_2},\infty)) \end{array} \right) \cap Y.$$

The critical property of $E_\phi(T)$ is the following lemma, which plays a critical role in the proof of Proposition 4.1.

Lemma 6.1. *If T is a separable valid transversal of $\mathcal{Y}$ that contains J_3, then $E_\phi(T)$ does not contain any element of T.*

Proof. If $(i,b_i) \in [1,a_1) \times [b_{a_2}, Y_{a_3}]$, then $(i,a_2,a_3) \in U(T)$. If $(i,b_i) \in (a_1,a_2) \times [b_{a_3}, b_{a_1}]$, then $(a_1,i,a_3) \in U(T)$. If $(i,b_i) \in (a_2,a_3) \times [1,b_{a_2}]$, then $(a_1,a_2,i) \in U(T)$. All three contradict the definition of h_J.

If $(i,b_i) \in (a_3,\infty) \times (b_{a_2},\infty)$, then $v = (a_2,a_3,i)$ is a copy of F_3 in T. If $i \in A$, replace v by $(a_2,a_3,i{+}1)$. Then, we have $v \in V(T)$, and $S(v) \geqslant (a_3,a_2,0) > (a_3,a_1,a_2) = \#(h_J(T))$ in the lexicographic order, which contradicts the separability of T. $\square$

The following lemma will be used in the proof of Proposition 4.1 for the case in which T is of J-type 2.

Lemma 6.2. *Let T be a separable, valid transversal of $\mathcal{Y}$ of J-type 2, and let $h_J(T) = (a_1,a_2,a_3)$. Then, $b_{a_2} \leqslant b_{a_3-1}$ and $a_3 - a_1 \geqslant 3$.*

Proof. If $b_{a_2} > b_{a_3-1}$, then $(a_1, a_2, a_3 - 1) \in U(T)$, which contradicts the definition of h_J.

If $a_3 - a_1 \leqslant 2$, then we have $a_3 = a_1 + 1$ and $a_2 = a_1 + 1$. Because $\mathcal{Y}$ is 1-alternating and $a_3 - 1 \in D$, we have $a_1 = a_3 - 2 \in A$. Therefore, we have $b_{a_1} < b_{a_2}$, which contradicts the validity of T. $\square$

The following lemma will be used repeatedly in the proof of Proposition 4.1 for the case in which T is of J-type 3.

Lemma 6.3. *Let $T = \{(i, b_i)\}$ be a separable, valid transversal of $\mathcal{Y}$ of J-type 3, and let $\phi(T) = \{(i, c_i)\}$.*

(a) If $i \in \Gamma^{[b_{a_3}, b_{a_1}]}_{[1,a_1)}(T)$, then $b_{a_3+1} < b_i < b_{a_2}$. Let $\Gamma^{[b_{a_3}, b_{a_1}]}_{[1,a_1]}(T) = \{i_1 < i_2 < \cdots < i_k\}$. Then $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ and $c_{i_1} < c_{i_2} < \cdots < c_{i_k}$.

(b) Let $\Gamma^{[b_{a_3}, b_{a_1}]}_{[a_2,a_3)}(T) = \{i_1 < i_2 < \cdots < i_k\}$, then $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ and $c_{i_1} < c_{i_2} < \cdots < c_{i_k} < c_{a_3}$. In particular, if $i \in \Gamma^{[b_{a_3}, b_{a_1}]}_{[a_2,a_3)}(T)$, then $b_{a_2} \leqslant b_i$.

Proof. First, we prove part (a). Let $i \in \Gamma^{[b_{a_3}, b_{a_1}]}_{[1,a_1)}(T)$. If $b_i \leqslant b_{a_3+1}$, then $(i, a_3, a_3 + 1) \in V(T)$ and $S(i, a_3, a_3 + 1) = (a_3 + 2, i, 0) > (a_3, a_1, a_2)$ in the lexicographic order, which contradicts the separability of T. The fact that $b_i \geqslant b_{a_2}$ follows from Lemma 6.1. If $j < j'$ with $b_{i_j} > b_{i_{j'}}$, then $(i_j, i_{j'}, a_3) \in U(T)$, which contradicts the definition of h_J. Because $c_{i_j} = b_{i_{j+1}}$ for $j \in [k - 1]$, it suffices to prove that $c_{i_1} < c_{i_2}$. This follows from $c_{i_1} = b_{a_3+1} < b_{i_1} = c_{i_2}$.

The proof of part (b) is similar. If $j < j'$ with $b_{i_j} < b_{i_{j'}}$ then $(a_1, i_j, i_{j'}) \in U(T)$, which contradicts the definition of h_J. Once again, to finish it suffices to prove that $c_{i_1} < c_{i_2}$, but this follows from $c_{i_1} = b_{a_3} < b_{i_1} = c_{i_2}$. $\square$

Proof of Proposition 4.1. We divide into cases based on the J-type of T. Let $h_J(T) = a = (a_1, a_2, a_3)$.

6.1 T is of J-type 1

See Figure 10. First, we prove that $\phi(T)$ is a valid transversal of $\mathcal{Y}$. To verify that if $i \in A$ (resp. $i \in D$) then $i \in \mathrm{Asc}(\phi(T))$ (resp. $i \in \mathrm{Des}(\phi(T))$), we divide into cases based on the value of i.

Case 1: $\{i, i + 1\} \cap \{a_1, a_2, a_3\} = \emptyset$. Because (a_1, a_2, a_3) is a copy of J_3 in T, we have $(a_3, b_{a_1}) \in Y$, which implies that $\phi(T)$ is a transversal of Y. If $\{i, i + 1\} \cap \{a_1, a_2, a_3\} = \emptyset$, then we have $b_i = c_i$ and $b_{i+1} = c_{i+1}$, so i is an ascent (resp. descent) of T if and only if i is an ascent (resp. descent) of $\phi(T)$.

Case 2: $i = a_1 - 1$. By Lemma 6.1, we have that $b_{a_1-1} \notin [b_{a_2}, b_{a_1}]$, which implies that $a_1 - 1$ is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T.

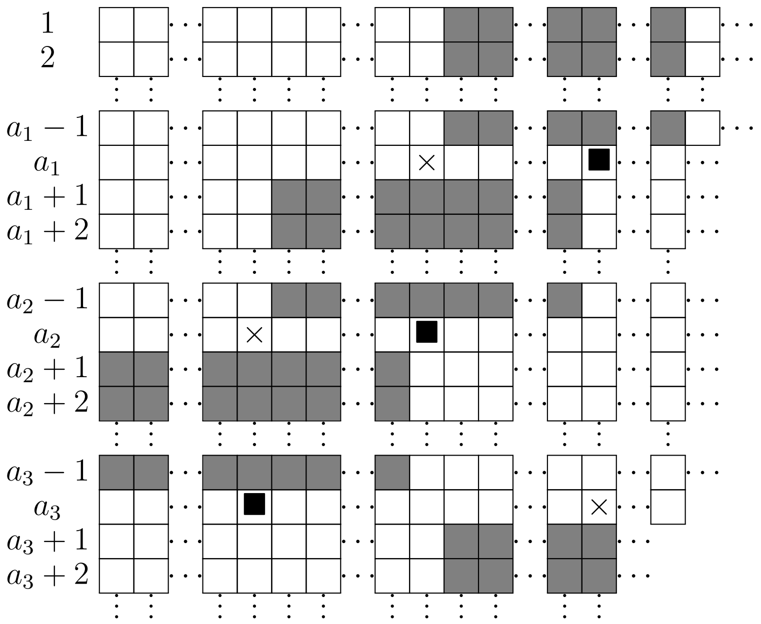

Figure 10: The squares marked with a solid black box are the elements of the chosen copy of J_3 for a separable, valid transversal T of J-type 1. The crosses mark new elements of $\phi(T)$, i.e. elements of $\phi(T) \setminus T$, while the gray squares are free of elements of T (and $\phi(T)$).

Case 3: $i = a_1$. By Lemma 6.1, we have $b_{a_1+1} \notin [b_{a_3}, b_{a_1}]$. Provided that $a_2 \neq a_1 + 1$, this implies that a_1 is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T. However, if $a_2 = a_1 + 1$, then it is clear a_1 is a descent of both T and $\phi(T)$.

Case 4: $i = a_2 - 1 \neq a_1$. Because $a_2 \neq a_1 + 1$, Lemma 6.1 implies that $b_{a_2-1} < b_{a_3}(= c_{a_2}) < b_{a_2}$. Therefore, $a_2 - 1$ is an ascent of both T and $\phi(T)$.

Case 5: $i = a_2$. If $a_3 \neq a_2 + 1$, then by Lemma 6.1, we have $b_{a_2+1} > b_{a_2} > b_{a_3} = c_{a_2}$, which implies that $a_2 + 1$ is an ascent of both T and $\phi(T)$. If $a_3 = a_2 + 1$, then we have $a_2 \notin D$ by definition, and because $b_{a_2} > b_{a_3}$, we have $a_2 \notin A$.

Case 6: $i = a_3 - 1$. By Lemma 6.1, we have $b_{a_3-1} \geqslant b_{a_2} > b_{a_3}$, which implies that $a_3 - 1 \notin A$.

Case 7: $i = a_3$. Because $\mathcal{Y}$ is 1-alternating, we have $a_3 \notin D$, and we also have $a_3 \notin A$ by the definition of J-type.

It follows that $\phi(T)$ is a valid transversal of $\mathcal{Y}$.

Next, we prove that $h_F(\phi(T)) = a$. It is clear that $a \in V(\phi(T))$, and because $a_3 - 1 \notin A$, we have $S(a) = (a_3, a_1, a_2)$. Suppose that $a' = (a'_1, a'_2, a'_3) \in U(T)$ with $S(a') > S(a)$ in the lexicographic order. Because $a_3 - 1 \notin A$, we must have $a'_3 \geqslant a_3$. If $a'_3 > a_3$, then we have $c_{a'_3} = b_{a'_3} < b_{a_2}$ by Lemma 6.1. For $i \in [2]$, let

$$d_i = \begin{cases} a_3 & \text{if } a'_i = a_2 \\ a'_i & \text{otherwise.} \end{cases}$$

Because $b_{a'_2} < b_{a_2}$, we have $a'_2 \notin (a_2, a_3)$ by Lemma 6.1, from which it follows that $d_1 < d_2$. We have $b_{d_i} = b_{a'_i}$ for $i \in [2]$, which implies that $v = (d_1, d_2, a'_3) \in V(T)$. Because $a_3 \notin A$, the first component of $S(v)$ is greater than a_3, and this contradicts the definition of h_F. Hence, we may assume that $a'_3 = a_3$, and because $a_3 - 1 \notin A$, the first component of $S(a')$ is a_3. If $a'_1 > a_1$, then $(a_1, a'_1, a'_2) \in U(T)$, which contradicts the definition of h_J. If $a'_1 = a_1$, then Lemma 6.1 implies that $a'_2 \leqslant a_2$. The fact that $h_F(T) = a$ follows by the definition of h_F. It is clear that $\phi(T)$ is of F-type 1 and that $\psi(\phi(T)) = T$.

We prove that if $e = (e_1, e_2, e_3) \in U(\phi(T))$, then we have $\#(e) > \#(a)$ in the lexicographic order. Assume for sake of contradiction that $\#(e) \leqslant \#(a)$ in the lexicographic order. If $e_3 < a_3$, then we derive contradictions by dividing into cases based on the relative values of c_{e_3} and c_{a_1}.

Case 1: $c_{e_3} > c_{a_1}$. We have $c_{e_i} = b_{e_i}$ for all i. Therefore, $(e_1, e_2, e_3) \in U(T)$, which contradicts the definition of h_J.

Case 2: $c_{e_3} = c_{a_1}$. We have $e_3 = a_1$ and $b_{e_2} = c_{e_2} > Y_{a_3} \geqslant b_{a_1}$ by Lemma 6.1. Hence, $e \in U(T)$, which contradicts the choice of a.

Case 3: $c_{e_3} < c_{a_1}$. We have $e_3 \leqslant a_2$ by Lemma 6.1. Therefore, we can include this case in the following cases.

Case 4: $e_3 < a_2$. Assume for sake of deriving a contradiction that $b_{e_3} \neq c_{e_3}$. Then, we have $e_3 = a_1$ and $c_{e_3} = c_{a_1}$, which we have shown to be impossible in a prior case. It follows that $b_{e_3} = c_{e_3}$.

Assume for sake of deriving a contradiction that $e_2 = a_1$. Then, we have

$$b_{e_1} = c_{e_1} > Y_{a_3} \geqslant b_{a_1} > b_{e_3} = c_{e_3}$$

by Lemma 6.1. Because $e_2 = a_1$, we have $e \in U(T)$, which contradicts the choice of a. Hence, we may assume that $e_2 \neq a_1$.

Because $Y_{e_3} \geqslant Y_{a_3} \geqslant b_{a_1} > c_{a_1}$ holds, we have $(e_1, e_2, e_3) \in U(T)$ regardless of whether a_1 equals e_1. This contradicts the choice of a.

Case 5: $e_3 = a_2$ and $c_{e_1} < c_{a_1}$. We have $(e_1, e_2, a_3) \in U(T)$, which contradicts the choice of a.

Case 6: $e_3 = a_2$ and $c_{e_1} = c_{a_1}$. By Lemma 6.1 and because $e_1 < e_2 < e_3$, we have $c_{e_2} \leqslant c_{a_2} = c_{a_3}$, which contradicts the assumption that $e \in U(\phi(T))$.

Case 7: $e_3 = a_2$ and $c_{e_1} > c_{a_1}$. then we have $c_{e_1} = b_{e_1} > Y_{a_3} \geqslant b_{a_1}$ by Lemma 6.1, which implies that $(e_1, a_1, a_2) \in U(T)$, contradiction.

Hence, we may assume that $e_3 = a_3$. If $e_1 \leqslant a_1$, we have $c_{e_1} \leqslant c_{a_1} < c_{a_3}$ by Lemma 6.1, which contradicts the assumption that $e \in U(\phi(T))$. Because $c_{a_1} < c_{a_3}$, it is impossible for e_1 to equal a_1. Therefore, we have $e_1 > a_1$, and the separability of $\phi(T)$ follows.

We have $b_i = c_i$ for all $i < a_1$, and $b_{a_1} > b_{a_2} = c_{a_1}$. Therefore, $(b_1, b_2, \ldots, b_n) > (c_1, c_2, \ldots, c_n)$ in the lexicographic order.

6.2 T is of J-type 2

See Figure 11. First, we prove that $\phi(T)$ is a valid transversal of $\mathcal{Y}$. Because $Y_{a_3-1} \geqslant Y_{a_3} \geqslant b_{a_1}$, the set $\phi(T)$ is a transversal of Y. To verify that if $i \in A$ (resp. $i \in D$) then $i \in \mathrm{Asc}(\phi(T))$ (resp. $i \in \mathrm{Des}(\phi(T))$), we divide into cases based on the value of i.

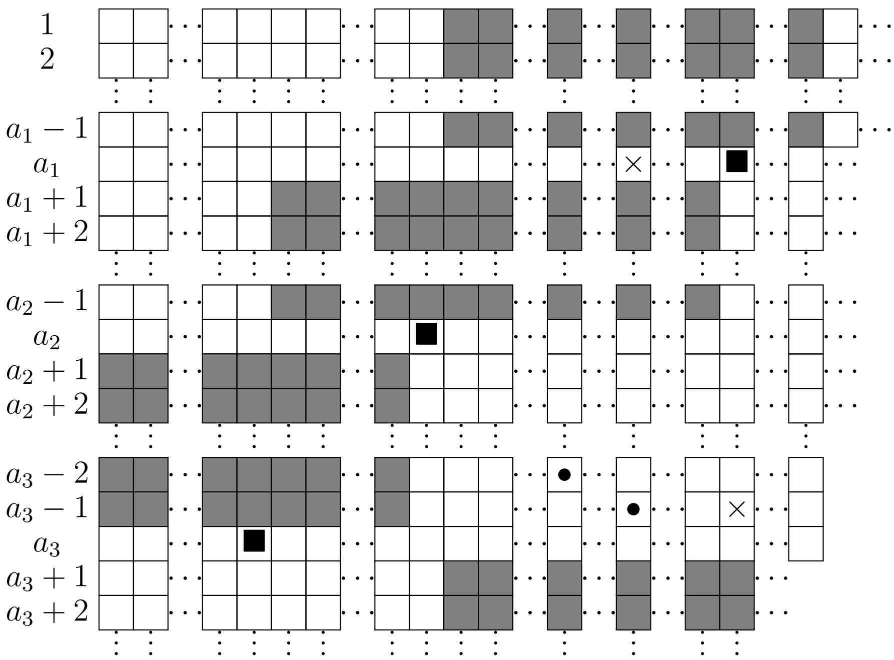

Figure 11: The squares marked with a solid black box are the elements of the chosen copy of J_3 for a separable, valid transversal T of J-type 2. The bullets mark other elements of T, and the crosses mark new elements of $\phi(T)$, i.e. elements of $\phi(T) \setminus T$. The gray squares are free of elements of T (and $\phi(T)$).

Case 1: $\{i, i+1\} \cap \{a_1, a_3 - 1\} = \emptyset$. We have $b_i = c_i$ and $b_{i+1} = c_{i+1}$, and thus i is an ascent (resp. descent) of T if and only if it is an ascent (resp. descent) of $\phi(T)$.

Case 2: $i = a_1 - 1, a_1$. By Lemma 6.1, we have $b_{a_1-1}, b_{a_1+1} \notin [b_{a_2}, b_{a_1}] \subseteq [b_{a_3-1}, b_{a_1}] = [c_{a_1}, b_{a_1}]$, where the subset relation holds by Lemma 6.2. By Lemma 6.2 again, we have $a_3 - 1 > a_1 + 1$, and it follows that $b_{a_1+1} = c_{a_1+1}$ and $b_{a_1-1} = c_{a_1-1}$. Therefore, i is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T, for $i = a_1 - 1, a_1$.

Case 3: $i = a_3 - 2, a_3 - 1$. We have $c_{a_3-1} = b_{a_1} > b_{a_3-1} > b_{a_3-2}, b_{a_3}$. By Lemma 6.2, we have that $a_3 - 2 > a_1$, which yields that $b_{a_3-2} = c_{a_3-2}$ and $b_{a_3} = c_{a_3}$. Therefore, $a_3 - 2$ is an ascent of $\phi(T)$ and $a_3 - 1$ a descent.

It follows that $\phi(T)$ is a valid transversal of $\mathcal{Y}$, as desired.

Next, we prove that $h_F(\phi(T)) = (a_1, a_3-2, a_3-1)$. It is clear that $(a_1, a_3-2, a_3-1) \in V(\phi(T))$, and we have $S(a_1, a_3-2, a_3-1) = (a_3, a_1, 0)$. Suppose for sake of deriving a contradiction that $d = (d_1, d_2, d_3) \in V(T)$ with $S(d) > (a_3, a_1, 0)$ in the lexicographic order. If $d_3 > a_3$, then by Lemma 6.1, we have $c_{d_3} = b_{d_3} < b_{a_2}$. This implies that $b_{d_i} = c_{d_i}$ for $i = 1, 2$ and it follows that $(d_1, d_2, d_3) \in V(T)$. If $d_1 < a_1$, then the second component of $S(d)$ is less than a_1, which implies that first component of $S(d)$ is greater than a_3. It follows that $S(d) > (a_3, a_1, a_2)$ in the lexicographic order, which contradicts the separability of T. It is clear that $d_1 \neq a_1$. If $d_1 > a_1$, then $(a_1, e_1, e_2) \in U(T)$, which contradicts the definition of h_J. Thus, we have $h_F(\phi(T)) = (a_1, a_3-2, a_3-1)$. It follows that $\phi(T)$ is of F-type 2, and it is clear that $\psi(\phi(T)) = T$.

We prove that if $e = (e_1, e_2, e_3) \in U(\phi(T))$, then $\#(e) \geqslant (a_3, a_1, 0)$ in the lexicographic order. Let $e \in U(\phi(T))$. First, we claim that $e_2 \neq a_1$. Assume for sake of deriving a contradiction that $e_2 = a_1$. The fact that $c_{a_1} < c_{a_3-1}$ implies that $e_3 \neq a_3 - 1$. By Lemma 6.1 we have $b_{e_1} > b_{a_1}$, which implies that $(e_1, e_2, e_3) \in U(T)$. This contradicts the choice of a. Hence, we have $e_2 \neq a_1$. Assume for sake of deriving a contradiction that $e_3 < a_3$. We divide into cases based on the values of e_1, e_2, e_3 to derive contradictions.

Case 1: $e_3 < a_3 - 1$. We have $(e_1, e_2, e_3) \in U(T)$ because $b_{e_i} = c_{e_i}$ for $i = 2, 3$ and $b_{e_1} \geqslant c_{e_1}$, which contradicts the choice of a.

Case 2: $e_3 = a_3 - 1$. The fact that $c_{a_1} < c_{a_3}$ implies that $e_1, e_2 \neq a_1$ and $b_{e_i} = c_{e_i}$ for $i = 1, 2$. Because $b_{a_3-1} < c_{a_3-1}$, we have $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a.

Hence, we may assume that $e_3 = a_3$. If $\{e_1, e_2\}$ and $\{a_1, a_3 - 1\}$ are disjoint, then clearly we have $(e_1, e_2, e_3) \in U(T)$, which implies that $\#(e) \geqslant \#(a)$ by the choice of a. It is impossible for e_1 to equal $a_3 - 1$, and if $e_2 = a_3 - 1$, then we have $e_1 \neq a_1$ and hence $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a. If $e_1 = a_1$, then the fact that $b_{a_1} > c_{a_1}$ implies that $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a. We have already dealt with the case of $e_2 = a_1$. Hence, the separability of T follows.

We have $b_i = c_i$ for all $i < a_1$, and $b_{a_1} > b_{a_3-1} = c_{a_1}$. Therefore, $(b_1, b_2, \ldots, b_n) > (c_1, c_2, \ldots, c_n)$ in the lexicographic order.

6.3 T is of J-type 3

See Figure 12. We first prove that $\phi(T)$ is a valid transversal of $\mathcal{Y}$. Because $Y_{a_3+1} = Y_{a_3} \geqslant b_{a_1}$, the set $\phi(T)$ is a transversal of Y. To verify that if $i \in A$ (resp. $i \in D$) then $i \in \mathrm{Asc}(\phi(T))$ (resp. $i \in \mathrm{Des}(\phi(T))$), we divide into cases based on the value of i.

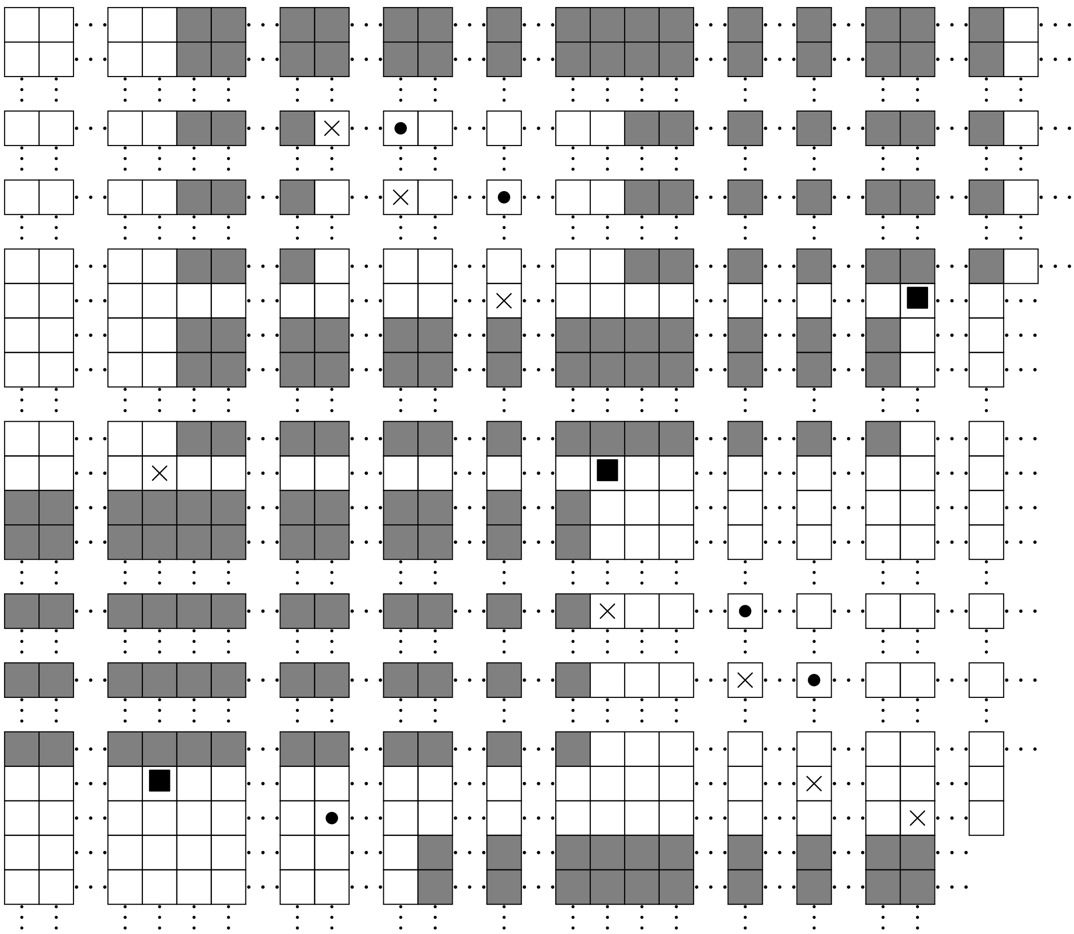

Figure 12: The squares marked with a solid black box are the elements of the chosen copy of J_3 for a separable, valid transversal T of J-type 3. The bullets mark some other elements of T, while the crosses mark new elements of $\phi(T)$, i.e. elements of $\phi(T)\setminus T$. The gray squares are free of elements of T (and $\phi(T)$). We suppose that $\Gamma_{[1,a_1)}^{[b_{a_3},b_{a_1}]}(T)=\{i_1,i_2\}$ and $\Gamma_{(a_2,a_3)}^{[b_{a_3},b_{a_1}]}(T)=\{i'_1,i'_2\}$.

Case 1: $\{i,i+1\}$ and $\Gamma_{[1,a_1]\cup[a_2,a_3+1]}^{[b_{a_3},b_{a_1}]}(T)$ are disjoint. We have $b_i=c_i$ and $b_{i+1}=c_{i+1}$, and thus i is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T.

Case 2: $i,i+1\in\Gamma_{[1,a_1]\cup[a_2,a_3]}^{[b_{a_3},b_{a_1}]}(T)$ with $i\neq a_1,a_3-1$. By Lemma 6.3, we have $b_i<b_{i+1}$ and $c_i<c_{i+1}$.

Case 3: $i\in K=\Gamma_{[1,a_1]\cup[a_2,a_3]}^{[b_{a_3},b_{a_1}]}(T)$, $i+1\notin K$, and $i\neq a_1,a_3$. We have $b_{a_3}<b_i,c_i<b_{a_1}$. Furthermore, we have $b_{i+1}\notin[b_{a_3},b_{a_1}]$, and therefore $c_i\notin[b_{a_3},b_{a_1}]$. It follows that i is an ascent (resp. descent) of $\phi(T)$ if and only if it is (resp. descent) of T.

Case 4: $i \notin K = \Gamma^{[b_{a_3}, b_{a_1}]}_{[1,a_1] \cup [a_2,a_3]}(T)$, $i+1 \in K$, and $i \neq a_1, a_2-1, a_3$. The argument is similar to the preceding case.

Case 5: $i = a_1, a_2-1$. Because $b_{a_1+1}, b_{a_2-1} \notin [b_{a_3}, b_{a_1}]$ by Lemma 6.1, we have $b_{a_1+1} < b_{a_1}$ if and only if $b_{a_1+1} < c_{a_1}$, and $b_{a_2-1} < b_{a_2}$ if and only if $b_{a_2-1} < c_{a_2}$. If $a_2 \neq a_1+1$, then we have $c_{a_1+1} = b_{a_1+1}$ and $c_{a_2-1} = b_{a_2-1}$, which implies that that a_1 is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T, and similarly for a_2-1. If $a_2 = a_1+1$, then $a_1 = a_2-1$ is a descent of both T and $\phi(T)$.

Case 6: $i = a_3, a_3+1$. We have $a_3 \in A$, and because $\mathcal{Y}$ is 1-alternating, we have $a_3+1 \in D$ and $a_3-1 \notin A$. By the definition of F-type we have $a_3-1 \notin D$ or $(b_{a_3-1} > b_{a_1} > c_{a_3})$. Furthermore, we have $c_{a_3} < b_{a_1} = c_{a_3+1}$ and $c_{a_3+1} = b_{a_1} > b_{a_3+1} > b_{a_3+2} = c_{a_3+2}$ by the definition of J-type and Lemma 6.1.

It follows that $\phi(T)$ is a valid transversal of $\mathcal{Y}$.

Next, we prove that $h_F(\phi(T)) = (a_1, a_2, a_3+1)$. It is clear that $(a_1, a_2, a_3+1) \in V(\phi(T))$ and $S(a_1, a_2, a_3+1) = (a_3, a_1, a_2) = \#(a)$. Let $(d_1, d_2, d_3) \in V(\phi(T))$ and suppose for sake of deriving a contradiction that $S(d) > \#(a)$ in the lexicographic order. We divide into cases based on the values of d_2, d_3, b_{d_3} to prove that $d_3 \leqslant a_3+1$.

Case 1: $d_3 > a_3+1$ and $b_{d_3} > b_{a_3+1}$. We have $(a_3+1, a_3+2, d_3) \in V(T)$ because $a_3+1 \in D$. This contradicts the separability of T because $S(a_3+1, a_3+2, d_3) \geqslant (a_3+2, a_1, 0)$ in the lexicographic order.

Case 2: $d_3 > a_3+1$, $b_{d_3} < b_{a_3+1}$, and $d_1, d_2 \neq a_2$. We have $b_{d_i} = c_{d_i}$ for all i. It follows that $(d_1, d_2, d_3) \in V(T)$, which contradicts the separability of T.

Case 3: $d_3 > a_3+1$, $b_{d_3} < b_{a_3+1}$, and $d_1 = a_2$. Lemma 6.1 implies that $d_2 > a_3$. Therefore, we have $(a_3, d_2, d_3) \in V(T)$,which contradicts the separability of T.

Case 4: $d_3 > a_3+1$, $b_{d_3} < b_{a_3+1}$, and $d_2 = a_2$. We have $(d_1, a_3, d_2) \in V(T)$, contradiction, which contradicts the separability of T.

Hence, we may assume that $d_3 \leqslant a_3+1$. Because $S(d) > \#(a)$ in the lexicographic order, we have (d is of F-type 2, $a_3-1 \in D$, $d_2 = a_3-2$, and $d_3 = a_3-1$) or $d_3 = a_3+1$.

Case 1: $d_3 < a_3+1$ (the former case). Lemma 6.3 implies that $b_{d_3} \notin [b_{a_3}, b_{a_1}]$, and the fact that $a_3-1 > a_2$ then yields that $b_{d_3} > b_{a_1}$ by Lemma 6.1. Because $d_2 = a_3-2 > a_2$, we have $b_{d_2} \geqslant b_{a_2}$ by Lemma 6.1. Regardless, if $b_{d_1} < b_{a_1}$, then we have $d_1 > a_2$, and it follows that $c_{d_1} < c_{d_2}$ by Lemma 6.3, which contradicts the assumption that $d \in V(\phi(T))$. Hence, we have $b_{d_1} > b_{a_1}$, which implies that $b_{d_1} = c_{d_1}$. It is clear that $b_{d_2} \leqslant \max\{c_{d_2}, b_{a_1}\}$, and therefore, we have $(d_1, d_2, d_3) \in V(T)$, which contradicts the separability of T.

Case 2: $d_3 = a_3 + 1$. If $d_2 = a_3$, then by Lemma 6.1 we have $d_1 \geqslant a_2$, but this implies that $d_1 \in \Gamma^{[b_{a_3}, b_{a_1}]}_{[a_2, a_3)}(T)$, which contradicts Lemma 6.3. We may assume that $d_2 \neq a_3$, which implies that d is of J-type 3 and $S(d) = (a_3, d_1, d_2)$. The fact that $S(d) > \#(a)$ in the lexicographic order implies that $d_1 \geqslant a_1$. We divide into subcases based on the relative values of c_{d_1} and b_{a_3}.

Subcase 2.1: $c_{d_1} > b_{a_3}$. By Lemma 6.1, we have that $d_1 < a_1$, which contradicts the assumption that $S(d) > \#(a)$ in the lexicographic order.

Subcase 2.2: $c_{d_1} = b_{a_3}$. We have $b_{d_2} = c_{d_2} < b_{a_3}$ and $a_2 < d_2 < a_3$, but the existence of such a d_2 contradicts Lemma 6.1.

Subcase 2.3: $c_{d_1} < b_{a_3}$. We have $c_{d_i} = b_{d_i}$ for $i = 1, 2$. Hence, we have $(d_1, d_2, a_3 + 1) \in V(T)$, which contradicts the separability of T.

Thus, we can conclude that $h_F(\phi(T)) = (a_1, a_2, a_3 + 1)$. It is clear that $\phi(T)$ is of F-type 3. We have $c_{a_2} = b_{a_3}$ and $c_{a_3+1} = b_{a_1}$, which implies that

$$\psi(\phi(T)) = \theta^{[b_{a_3}, b_{a_1}]}_{[a_2, a_3]}\left(\theta^{[b_{a_3}, b_{a_1}]}_{[1, a_1] \cup \{a_3+1\}}\left(\omega^{[b_{a_3}, b_{a_1}]}_{[1, a_1] \cup \{a_3+1\}}\left(\omega^{[b_{a_3}, b_{a_1}]}_{[a_2, a_3]}(T)\right)\right)\right) = T,$$

as desired.

We prove that if $e = (e_1, e_2, e_3) \in U(\phi(T))$, then $e_3 > a_3$. Let $e \in U(\phi(T))$. Suppose for sake of contradiction that $e_3 \leqslant a_3$. We divide into cases based on the value of c_{e_3} to derive a contradiction.

Case 1: $c_{e_3} > b_{a_1}$. We have $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a.

Case 2: $b_{a_2} \leqslant c_{e_3} < b_{a_1}$. By Lemma 6.1, we have $a_2 < e_3 < a_3$. By Lemma 6.3 and because $e_2 > e_3$ with $c_{e_2} > c_{e_3}$, we have $b_{e_2} > b_{a_1}$ or $e_2 < a_2$. However, the latter case implies that $b_{e_2} > b_{a_1}$ by Lemma 6.1 again. Then, $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a.

Case 3: $b_{a_3} < c_{e_3} < b_{a_2}$. By Lemma 6.1 we have $e_3 \leqslant a_1$. By Lemma 6.3, we have $b_{e_2} > b_{a_1}$. It follows that $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a.

Case 4: $c_{e_3} = b_{a_3}$. By Lemma 6.3, we have $b_{e_1} > b_{a_1}$ (because if $b_{e_1} < b_{a_1}$, then $c_{e_1} < c_{e_2}$ by Lemma 6.3, which contradicts the assumption that $e \in U(\phi(T))$). This implies that $(e_1, a_1, a_2) \in U(T)$, which contradicts the choice of a.

Case 5: $c_{e_3} < b_{a_3}$. By Lemma 6.1, we have $e_3 < a_2$. By Lemma 6.3 and because $e_1, e_2 < a_2$, at most one of c_{e_1}, c_{e_2} can be an element of $[b_{a_3}, b_{a_1}]$. It follows that $(e_1, e_2, e_3) \in U(T)$, which contradicts the choice of a.

The separability of $\phi(T)$ follows.

Let $m = \min \Gamma^{[b_{a_3}, b_{a_1}]}_{[1, a_1]}(T)$. We have $b_i = c_i$ for all $i < m$. Because $b_{a_1} > b_{a_2} > b_{a_3+1}$ by Lemma 6.1, Lemma 6.3 implies that $b_m > b_{a_3+1} = c_{i_1}$. Therefore, $(b_1, b_2, \ldots, b_n)$ is greater than $(c_1, c_2, \ldots, c_n)$ in the lexicographic order, as desired. □

7 Proof of Proposition 4.2

The proof of Proposition 4.2 is similar to the proof of Proposition 4.1. First, we define $E_\psi(T) \subseteq Y$, which is the analogue $E_\phi(T)$. Let $T = \{(i, b_i)\}$ be a separable, valid transversal of $\mathcal{Y}$ that contains F_3, and let $h_F(T) = (a_1, a_2, a_3)$. Then, let

$$E_\psi(T) = \left(\begin{array}{c} ([1, a_1) \times [b_{a_1}, Y_{a_3}]) \cup ((a_1, a_2) \times [b_{a_2}, b_{a_3}]) \\ \cup ((a_2, a_3) \times [1, b_{a_1}]) \cup ((a_3, \infty) \times (b_{a_1}, \infty)) \end{array} \right) \cap Y.$$

Once again, the critical property of $E_\psi(T)$ is the following lemma.

Lemma 7.1. *If T is a separable valid transversal of $\mathcal{Y}$ that contains F_3, then $E_\psi(T)$ does not contain any element of T.*

Proof. If $(i, b_i) \in [1, a_1) \times [b_{a_1}, Y_{a_3}]$, then $(i, a_1, a_2) \in U(T)$, which contradicts the separability of T. If $(i, b_i) \in (a_1, a_2) \times [b_{a_2}, b_{a_3}]$, then $(i, a_2, a_3) \in V(T)$ and $S(i, a_2, a_3) > S(a_1, a_2, a_3)$ in the lexicographic order. If $(i, b_i) \in (a_2, a_3) \times [1, b_{a_1}]$, then $(a_1, i, a_3) \in V(T)$ and $S(a_1, i, a_3) > S(a_1, a_2, a_3)$ in the lexicographic order. Both contradict the definition of h_F.

If $(i, b_i) \in (a_3, \infty) \times (b_{a_1}, \infty)$, then $v = (a_2, a_3, i)$ is a copy of F_3 in T. If $i \in A$, replace v by $(a_2, a_3, i + 1)$. Then, we have $v \in V(T)$, and $S(v) > S(h_F(T))$ in the lexicographic order, which contradicts the definition of h_F. □

The analogue of Lemma 6.3 is the following lemma, which will be used repeatedly in the proof of Proposition 4.2 for the case in which T has F-type 3.

Lemma 7.2. *Let $T = \{(i, b_i)\}$ be a separable, valid transversal of $\mathcal{Y}$ of F-type 3, and let $\psi(T) = \{(i, c_i)\}$.*

(a) Let $\Gamma^{[b_{a_3}, b_{a_1}]}_{[1,a_1]}(T) = \{i_1 < i_2 < \cdots < i_k\}$. Then, $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ and $c_{i_1} < c_{i_2} < \cdots < c_{i_k}$. In particular, if $i \in \Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1]}(T)$, then $b_i \leqslant b_{a_1}$.

(b) Let $\Gamma^{[b_{a_2}, b_{a_3})}_{[a_2,a_3)}(T) = \{i_1 < i_2 < \cdots < i_k\}$, then $b_{a_1} < b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ and $c_{i_1} < c_{i_2} < \cdots < c_{i_k}$. Furthermore, we have $b_{a_1} < c_{i_2} < \cdots < c_{i_k} < c_{a_3}$.

Proof. First, we prove part (a). If $j < j'$ with $b_{i_j} > b_{i'_j}$, then $(i_j, i_{j'}, a_3 - 1) \in U(T)$, which contradicts the separability of T. Because $c_{i_j} = b_{i_{j-1}}$, to prove that $c_{i_1} < c_{i_2} < \cdots < c_{i_k}$ it suffices to prove that $c_{i_1} < c_{i_2}$. But, if $c_{i_1} > c_{i_2}$, we have $b_{a_3} = c_{i_1} > c_{i_2} = b_{i_1}$, and therefore $(i_1, a_3 - 1, a_3) \in V(T)$. However, then we have $S(i_1, a_3 - 1, a_3) = (a_3 + 1, i_1 + 1, 0)$, which contradicts the definition of h_F. The last sentence follows because $b_{i_k} = b_{a_1}$.

The proof of part (b) is similar. Let $i_0 = a_1$. If $j < j'$ with $b_{i_j} > b_{i_{j'}}$, then $(i_j, i_{j'}, a_3 - 1) \in U(T)$, which contradicts the separability of T. To prove that $c_{i_1} < c_{i_2} < \cdots < c_{i_k}$, it suffices to prove that $c_{i_1} < c_{i_2}$, but this is clear because $c_{i_1} = b_{a_3} < b_{i_1} = c_{i_2}$. Let $i_{k+1} = a_3$. The last sentence follows because $c_{i_{j+1}} = b_{i_j}$ for $j \in [k]$. □

The following additional lemma will be also used in proof of Proposition 4.2 for T of F-type 3.

Lemma 7.3. *Let T be a separable, valid transversal of $\mathcal{Y}$ of F-type 3, let $h_F(T) = (a_1, a_2, a_3)$, and let $m = \min \Gamma_{[1,a_1]}^{[b_{a_2}, b_{a_3}]}(T)$. Then, the set $(a_3, \infty) \times (b_m, \infty) \cap Y$ does not contain any element of T.*

Proof. Suppose for sake of deriving a contradiction that $(i, b_i) \in (a_3, \infty) \times (b_m, \infty) \cap Y$. Then, $(m, a_2, i) \in V(T)$ and $S(m, a_2, i) \geqslant (a_3, m, 0) > (a_3 - 1, a_1, a_2) = S(h_F(T))$ in the lexicographic order, which contradicts the definition of h_F. □

Proof of Proposition 4.2. If $x = (x_1, x_2, x_3)$ is a copy of F_3 in T, then either $x_3 \notin A$ and $x \in V(T)$ or $x_3 \in A$ and $(x_1, x_2, x_3 + 1) \in V(T)$. Thus, if T contains F_3, then $\psi(T)$ is defined. We divide into cases based on the F-type of T. Let $h_F(T) = a = (a_1, a_2, a_3)$.

7.1 T is of F-type 1

See Figure 13. First, we prove that $\psi(T)$ is a valid transversal of $\mathcal{Y}$. Because $Y_{a_3} \geqslant b_{a_1}$, the set $\psi(T)$ is a transversal of Y. To verify that if $i \in A$ (resp. $i \in D$) then $i \in \mathrm{Asc}(\psi(T))$ (resp. $i \in \mathrm{Des}(\psi(T))$), we divide into cases on the value of i.

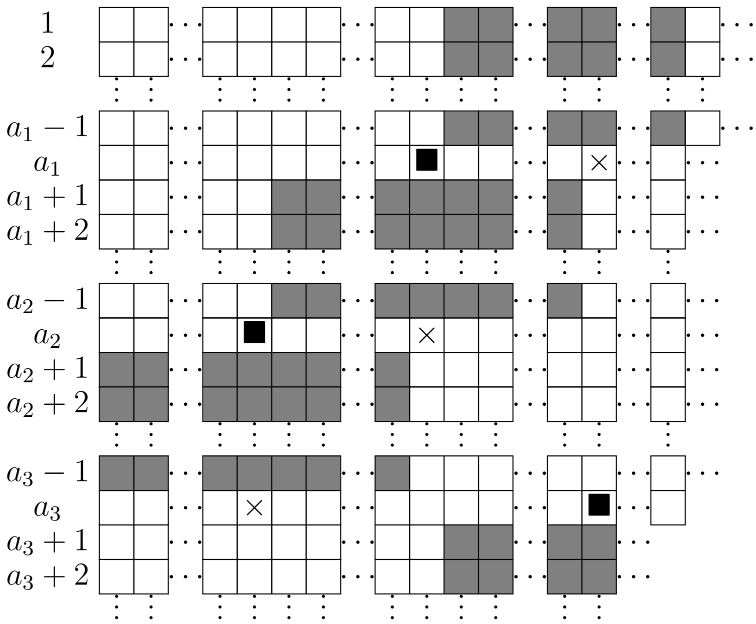

Figure 13: The squares marked with a solid black box are the elements of the chosen copy of F_3 for a separable, valid transversal T of F-type 1. The crosses mark new elements of $\psi(T)$, i.e. elements of $\psi(T) \setminus T$, and the gray squares are free of elements of T (and $\psi(T)$).

Case 1: $\{i, i+1\} \cap \{a_1, a_2, a_3\} = \emptyset$. Then, $b_i = c_i$ and $b_{i+1} = c_{i+1}$, which implies that i is an ascent (resp. descent) of $\psi(T)$ if and only if it is an ascent (resp. descent) of T.

Case 2: $i = a_1 - 1, a_1, a_2 - 1$. By Lemma 7.1, we have $b_{a_1-1}, b_{a_1+1} \notin (b_{a_1}, b_{a_3}) = (b_{a_1}, c_{a_1}))$, and $b_{a_2-1}, b_{a_2+1} \notin (b_{a_2}, c_{a_2})$. Thus, $t = a_1 - 1$ is an ascent (resp. descent) of $\psi(T)$ if and only if it is an ascent (resp. descent) of T. If $a_2 \neq a_1 + 1$, then the same holds for $t = a_1$ and $t = a_2 - 1$, and if $a_2 = a_1 + 1$, then $a_1 = a_2 - 1$ is a descent of both T and $\psi(T)$.

Case 3: $i = a_2 \neq a_3 - 1$. If $a_2 \neq a_3 - 1$, then a_2 is an ascent (resp. descent) of $\psi(T)$ if and only if it is an ascent (resp. descent) of T.

Case 4: $i = a_3 - 1, a_3$. By the definition of F-type and $V(T)$, we have $a_3 - 1, a_3 \notin A$, and the fact that $\mathcal{Y}$ is 1-alternating implies that $a_3 - 1, a_3 \notin D$. Furthermore, if $a_2 = a_3 - 1$, then $a_2 \notin A, D$.

It follows that $\psi(T)$ is a valid transversal of $\mathcal{Y}$.

Next, we prove that $h_J(\psi(T)) = (a_1, a_2, a_3)$. It is clear that $(a_1, a_2, a_3) \in U(\psi(T))$. Suppose for sake of deriving a contradiction that $d \in U(\psi(T))$ with $\#(d) < \#(a)$ in the lexicographic order. We divide into cases based on the values of d_3 and b_{d_3}.

Case 1: $d_3 < a_1$ or $b_{d_3} > b_{a_3}$. Then, $b_{d_i} = c_{d_i}$ for all $i \in [3]$ and $d \in U(T)$, which contradicts the separability of T.

Case 2: $d_3 = a_1$. We have $c_{d_3} > b_{d_3}$ and $b_{d_i} = c_{d_i}$ for all $i \in [2]$. Therefore, $d \in U(T)$, which contradicts the separability of T.

Case 3: $d_3 = a_2$. We divide into subcases based on the value of d_1.

Subcase 3.1: $d_1 = a_1$. We have $b_{a_1} < b_{d_2} = c_{d_2} < b_{a_3}$ with $a_1 < d_2 < a_2$, which contradicts Lemma 7.1.

Subcase 3.2: $d_1 \neq a_1$. We have $b_{d_1} = c_{d_1}$, as well as $c_{d_2} \geqslant b_{d_2} > b_{a_2}$. Therefore, $d \in U(T)$, which contradicts the separability of T.

Case 4: $a_1 < d_3 < a_2$ and $b_{d_3} < b_{a_3}$. Lemma 7.1 yields that $b_{d_3} < b_{a_2}$. We now divide into subcases based on the value of d_1.

Subcase 4.1: $d_1 = a_1$. Lemma 7.1 yields that $b_{d_2} < b_{a_2}$. Therefore $d \in U(T)$, which contradicts the separability of T.

Subcase 4.2: $d_1 \neq a_1$. We have $b_{d_1} = c_{d_1}$ and

$$c_{b_2} \geqslant b_{d_2} \geqslant \min\{b_{a_1}, c_{d_2}\} > c_{d_3} = b_{d_3}.$$

It follows that $d \in U(T)$, which contradicts the separability of T.

Case 5: $a_2 < d_3 < a_3$ and $b_{d_3} < b_{a_3}$. We divide into subcases based on $\{a_1\} \cap \{d_1, d_2\}$.

Subcase 5.1: $d_1 = a_1$. By Lemma 7.1 we have $c_{d_3} = b_{d_3} > b_{a_1} = c_{d_2}$, which yields that $d_2 \neq a_2$. Furthermore, by Lemma 7.1 again and because $b_{a_1} < b_{d_2} < c_{a_1} = b_{a_3}$, we have $d_2 > a_2$, but the fact that $(a_1, d_2, a_3) \in V(T)$ contradicts the definition of h_F.

Subcase 5.2: $d_2 = a_1$. We have $(d_1, a_1, a_2) \in U(T)$, which contradicts the separability of T.

Subcase 5.3: $a_1 \notin \{d_1, d_2\}$. We have $d \in U(T)$, which contradicts the separability of T.

It follows that $h_J(T) = (a_1, a_2, a_3)$, and it is clear that $\psi(T)$ is of J-type 1. Therefore, we have $\phi(\psi(T)) = T$, as desired.

We prove that if $e = (e_1, e_2, e_3) \in V(\psi(T))$, then $S(e) \leqslant S(a)$ in the lexicographic order. Suppose for sake of deriving a contradiction that $S(e) > S(a)$ in the lexicographic order. First, suppose that $e_3 > a_3$, hence that $b_{e_3} = c_{e_3}$. Furthermore, because $a_3 \notin A$, the first component of $S(e)$ must be greater than a_3. By Lemma 7.1, we have $b_{e_3} < b_{a_1}$. We divide into cases based on the value of $\{e_1, e_2\} \cap \{a_3\}$.

Case 1: $e_1 = a_3$. We have $(a_2, e_2, e_3) \in V(T)$, but the first component of $S(a_2, e_2, e_3)$ is greater than a_3, which contradicts the definition of h_F.

Case 2: $e_2 = a_3$. We have $b_{a_2} < c_{e_1} < b_{a_1}$, which yields that $b_{e_1} = c_{e_1}$. By Lemma 7.1, we have $e_1 < a_1$, and hence $(e_1, a_2, e_3) \in V(T)$, but the first component of $S(e_1, e_2, e_3)$ is greater than a_3, which contradicts the definition of h_F.

Case 3: $a_3 \notin \{e_1, e_2\}$. It follows from Lemma 7.1 that $b_{e_i} = c_{e_i}$ for all i and thus $e \in V(T)$, which contradicts the definition of h_F.

Hence, we may assume that $e_3 \leqslant a_3$. Because $S(e) > S(a)$ in the lexicographic order, either (e is of F-type 2, $a_3 - 2 \in A$, $e_3 = a_3 - 1$ and $e_2 = a_3 - 2$) or (e is of F-type 1 and $e_3 = a_3$).

Case 1: e is of F-type 2. Because $a_3 - 2 \geqslant a_2$, we have $a_3 - 1 \in D$, we have $b_{a_3} < b_{a_3-1}$, which implies that $b_{a_3-1} = c_{a_3-1}$. Additionally, by Lemma 7.1 and because $c_{a_2} = b_{a_1}$, we have $c_{a_3-2} \geqslant b_{a_1}$, and because $S(e) > S(a)$ in the lexicographic order, we have $e_1 > a_1$. Therefore, we have $c_{e_1} > b_{a_1}$ and thus $c_{e_1} = b_{e_1}$. It is also clear that $b_{e_2} \leqslant c_{e_2}$. It follows that $e \in U(T)$, which contradicts the separability of T.

Case 2: e is of F-type 1. Because $b_{e_i} < c_{e_3} = b_{a_2}$ for $i \in [2]$, we have $b_{e_i} = c_{e_i}$ for $i \in [2]$. Therefore, we have $e \in U(T)$, which contradicts the separability of T.

The separability of $\psi(T)$ follows.

For $i < a_1$, we have $b_i = c_i$ and $c_{a_1} = b_{a_3} > b_{a_1}$. Thus, $(b_1, b_2, \ldots, b_n)$ is less than $(c_1, c_2, \ldots, c_n)$ in the lexicographic order, as desired.

7.2 T is of F-type 2

See Figure 14. First, we prove that $\psi(T)$ is a valid transversal of $\mathcal{Y}$. It is clear that $\psi(T)$ is a transversal of Y. To verify that if $i \in A$ (resp. $i \in D$) then $i \in \mathrm{Asc}(\psi(T))$ (resp. $i \in \mathrm{Des}(\psi(T))$), we divide into cases based on the value of i.

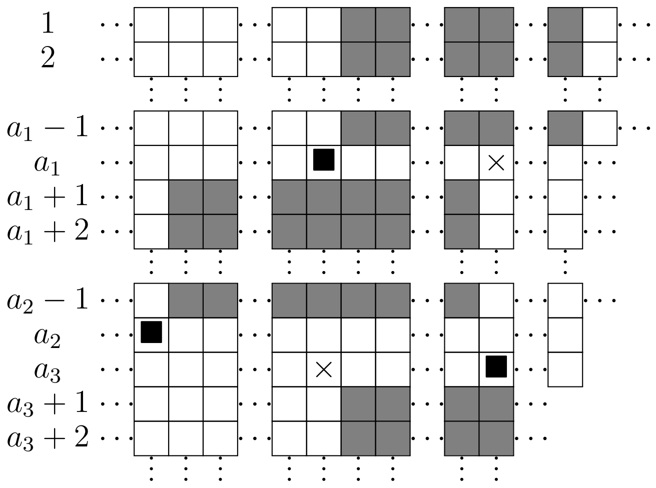

Figure 14: The squares marked with a solid black box are the elements of the chosen copy of F_3 for a separable, valid transversal T of F-type 2, and the crosses mark new elements of $\psi(T)$, i.e. elements of $\psi(T) \setminus T$. The gray squares are free of elements of T (and $\psi(T)$).

Case 1: $\{i, i+1\} \cap \{a_1, a_3\} = \emptyset$. Then, $b_i = c_i$ and $b_{i+1} = c_{i+1}$, which implies that i is an ascent (resp. descent) of $\psi(T)$ if and only if it is an ascent (resp. descent) of T.

Case 2: $i = a_1 - 1, a_1$. By Lemma 7.1, we have $b_{a_1-1}, b_{a_1+1} \notin (b_{a_2}, b_{a_3}) \supseteq (b_{a_1}, c_{a_1})$. It follows that $a_1 - 1$ is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T, and the same for a_1.

Case 3: $i = a_3 - 1$. We have $a_3 - 1 \in A$, but we also have $c_{a_3-1} = b_{a_2} < b_{a_1} = c_{a_3}$ and thus $a_3 - 1$ is an ascent of $\psi(T)$.

Case 4: $i = a_3$. Because $\mathcal{Y}$ is 1-alternating, we have $a_3 \in D$. However, by Lemma 7.1, we have $c_{a_3+1} = b_{a_3+1} < b_{a_1} = c_{a_3}$, and thus a_3 is a descent of $\psi(T)$.

It follows that $\psi(T)$ is a valid transversal of $\mathcal{Y}$, as desired.

Next, we prove that there is an integer y such that $h_J(T) = (a_1, y, a_3 + 1)$. First, because $Y_{a_3+1} = Y_{a_3} \geqslant b_{a_3} = c_{a_1}$ and $c_{a_3} = b_{a_1} > c_{a_3+1}$ (which follows from Lemma 7.1), we have $(a_1, a_3, a_3 + 1) \in U(\psi(T))$. Suppose for sake of deriving a contradiction that $d = (d_1, d_2, d_3) \in U(\psi(T))$ with $\#(d) < (a_3 + 1, a_1, 0)$ in the lexicographic order. We divide into cases based on the values of d_1, d_2, d_3 to derive contradictions.

Case 1: $d_3 < a_1$ or ($a_1 < d_3 < a_3 - 1$ and $a_1 \notin \{d_1, d_2\}$). We have $b_{d_i} = c_{d_i}$ for all $i \in [3]$, and thus $d \in U(T)$, which contradicts the separability of T.

Case 2: $d_3 = a_1$. Because $b_{a_1} < b_{a_3} = c_{a_1}$, we have $d \in U(T)$, contradiction.

Case 3: $d_i = a_1$ for some $i \in [2]$ and $a_3 \notin \{d_1, d_2, d_3\}$. By Lemma 7.1, we have $b_{d_{i+1}} \notin (b_{a_3}, b_{a_1})$. Therefore, we have $(d_1, d_2, d_3) \in U(T)$, which contradicts the separability of T.

Case 4: $d_3 = a_3$ and $a_1 \notin \{d_1, d_2\}$. It is clear that $d_2 \neq a_3 - 1$. Thus, we have $(d_1, d_2, a_3 - 1) \in U(T)$ because $b_{d_i} = c_{d_i}$ for $i \in [2]$ and $c_{a_3} > c_{a_3-1} = b_{a_3-1}$. This contradicts the separability of T.

Case 5: $d_3 = a_3$ and $d_1 = a_1$. We have $a_1 < d_2 < a_3$ with $b_{d_2} \in (b_{a_2}, b_{a_3})$, which contradicts Lemma 7.1.

Case 6: $d_3 = a_3$ and $d_2 = a_1$, then we have

$$b_{d_1} = c_{d_1} > c_{a_1} = b_{a_3} > b_{a_1} = c_{a_3} > c_{a_3-1} = b_{a_3-1},$$

where the last inequality follows from Lemma 7.1. Hence, we have $(d_1, d_2, a_3 - 1) \in U(T)$, which contradicts the separability of T.

Hence, we may assume that $d_3 = a_3 + 1$. Because $\#(d) < (a_3 + 1, a_1, 0)$, we may also assume that $d_1 < a_1$, which implies that $b_{d_1} = c_{d_1}$. Lemma 7.1 yields that $b_{d_1} < b_{a_1}$, and thus $b_{d_i} = c_{d_i}$ for all $i \in [3]$. It follows that $d \in U(T)$, which contradicts the separability of T. The fact that $h_J(T) = (a_1, y, a_3 + 1)$ for some y follows. It is clear that $\psi(T)$ is of J-type 2 and that $\phi(\psi(T)) = T$.

We prove that if $e = (e_1, e_2, e_3) \in V(\psi(T))$, then $S(e) \geqslant S(h_F(T))$ in the lexicographic order. If $e_3 = a_3$, then $b_{e_i} = c_{e_i}$ for all $i \in [2]$, and $b_{e_3} > c_{e_3}$. It follows that $e \in V(T)$, which contradicts the definition of h_F. Hence, we may assume that $e_3 > a_3$, and it follows that $b_{e_3} = c_{e_3}$. Lemma 7.1 yields that $b_{e_3} < b_{a_1}$, and thus $c_{e_i} < b_{a_1}$ for all i, which yields that $b_{e_i} = c_{e_i}$ for all i. This implies that $e \in V(T)$, which contradicts the definition of h_F. The separability of $\psi(T)$ follows.

For $i < a_1$, we have $b_i = c_i$. Because $b_{a_1} < b_{a_3} = c_{a_1}$, we have $(b_1, b_2, \ldots, b_n) < (c_1, c_2, \ldots, c_n)$ in the lexicographic order, as desired.

7.3 T is of F-type 3

See Figure 15. First, we prove that $\psi(T)$ is a valid transversal of $\mathcal{Y}$. This paragraph is similar to the first paragraph of the proof of Proposition 4.1 for the case in which T is of J-type 3. If $b_i > b_{a_3}$ or $i > a_3$, it is clear that $b_i = c_i$, and therefore T is a transversal of Y. To verify that if $i \in A$ (resp. $i \in D$) then $i \in \mathrm{Asc}(\psi(T))$ (resp. $i \in \mathrm{Des}(\psi(T))$), we divide into cases based on the value of i.

Case 1: $\{i, i + 1\}$ and $\Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1] \cup [a_2,a_3]}(T)$ are disjoint. Then, $b_i = c_i$ and $b_{i+1} = c_{i+1}$, and thus i is an ascent (resp. descent) of $\phi(T)$ if and only if it is an ascent (resp. descent) of T.

Case 2: $i, i + 1 \in \Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1] \cup [a_2,a_3-1]}(T)$ with $i \neq a_1, a_3 - 2$. By Lemma 7.2, we have $b_i < b_{i+1}$ and $c_i < c_{i+1}$.

Case 3: $i \in K = \Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1] \cup [a_2,a_3-1]}(T)$, $i+1 \notin K$, and $i \neq a_1, a_3-1$. We have $b_{a_2} < b_i, c_i < b_{a_1}$ and $b_{i+1} \notin [b_{a_2}, b_{a_3}]$, which implies that i is an ascent (resp. descent) of $\phi(T)$ if and only if it is (resp. descent) of T.

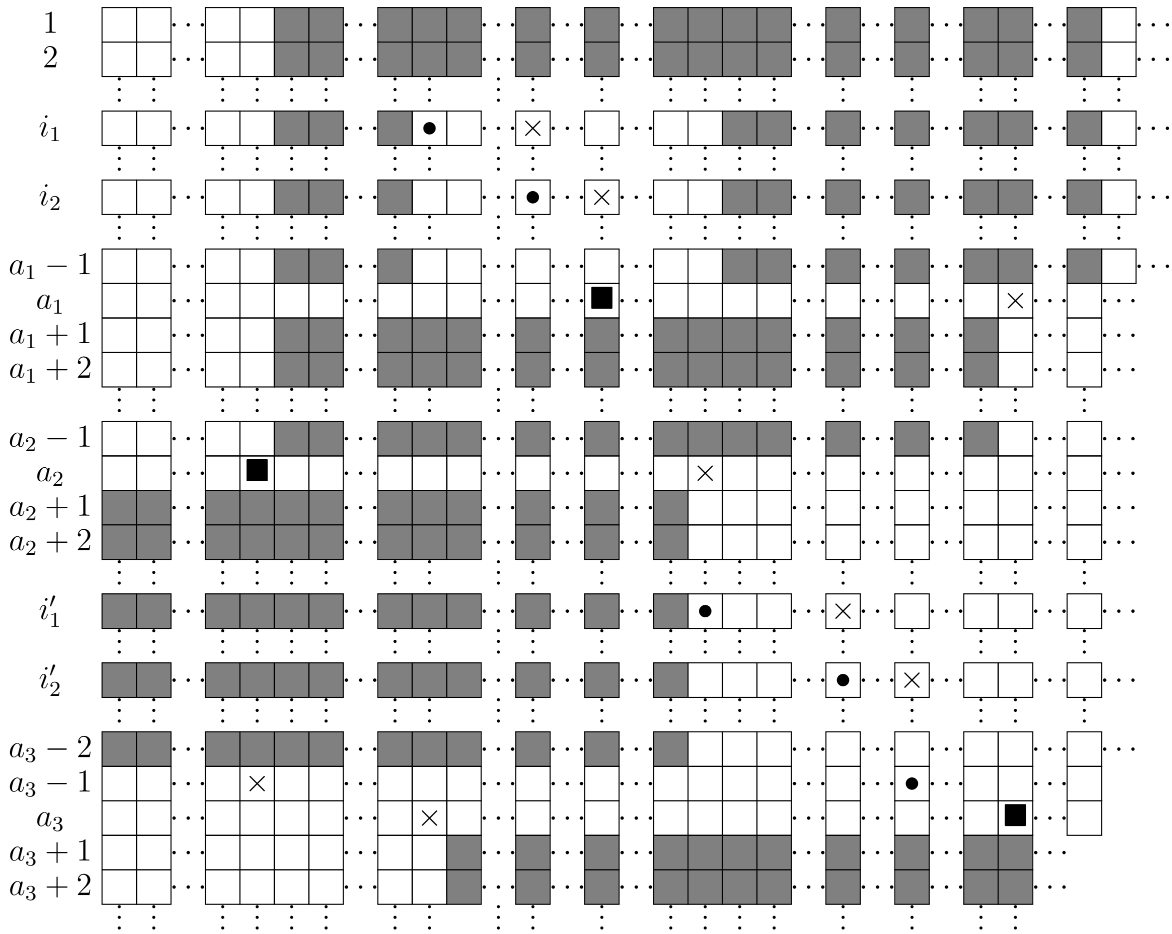

Figure 15: The squares marked with a solid black box are the elements of the chosen copy of F_3 for a separable, valid transversal $T = \{(i, b_i)\}$ of F-type 3. The bullets mark some other elements of T, while the crosses mark new elements of $\psi(T)$, i.e. elements of $\phi(T) \setminus T$. The gray squares are free of elements of T (and $\psi(T)$). We suppose that $\Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1)}(T) = \{i_1, i_2\}$ and $\Gamma^{[b_{a_2}, b_{a_3}]}_{(a_2,a_3-1)}(T) = \{i'_1, i'_2\}$.

Case 4: $i \notin K = \Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1]\cup[a_2,a_3-1]}(T)$, $i+1 \in K$, and $i \neq a_1, a_2-1, a_3-1$. The argument is similar to that of the preceding case.

Case 5: $i = a_1, a_2 - 1$. Because $b_{a_1+1}, b_{a_2-1} \notin [b_{a_2}, b_{a_3}]$ by Lemma 7.1, we have $b_{a_1+1} < b_{a_1}$ if and only if $b_{a_1+1} < c_{a_1}$, and $b_{a_2-1} < b_{a_2}$ if and only if $b_{a_2-1} < c_{a_2}$. If $a_2 \neq a_1 + 1$, then we have $c_{a_1+1} = b_{a_1+1}$ and $c_{a_2-1} = b_{a_2-1}$, which implies that that a_1 is an ascent (resp. descent) of $\psi(T)$ if and only if it is an ascent (resp. descent) of T, and similarly for $a_2 - 1$. If $a_2 = a_1 + 1$, then $a_1 = a_2 - 1$ is a descent of both T and $\psi(T)$.

Case 6: $i = a_3 - 2, a_3 - 1, a_3$. We have $a_3 - 1 \in A$, and because $\mathcal{Y}$ is 1-alternating, we have $a_3 \in D$ and $a_3 - 2 \notin A$. If $a_3 - 2 \in D$, we have $b_{a_3-2} > b_{a_3-1}$, and Lemma 7.2 implies that $a_3 - 2 \notin \Gamma^{[b_{a_2}, b_{a_3}]}_{[a_2, a_3-1]}$. Therefore, we have $c_{a_3-2} = b_{a_3-2} > b_{a_3-1} > c_{a_3-1}$. Regardless, we have $c_{a_3-1} = b_{a_2} < c_{a_3}$ by the definition of ψ. By Lemma 7.3 and the definition of ψ, we have $c_{a_3+1} = b_{a_3+1} < c_{a_3}$.

It follows that $\phi(T)$ is a valid transversal of $\mathcal{Y}$.

Next, we prove that $h_J(\psi(T)) = (a_1, a_2, a_3 - 1)$. It is clear that $(a_1, a_2, a_3 - 1) \in U(\psi(T))$. Suppose for sake of deriving a contradiction that $(d_1, d_2, d_3) \in U(\psi(T))$ with $\#(d) < (a_3 - 1, a_1, a_2)$ in the lexicographic order. We divide into cases based on the values of d_3 and b_{d_3} to prove that $d_3 = a_3 - 1$.

Case 1: $b_{d_3} > b_{a_3}$. We have $b_{d_i} = c_{d_i}$ for all $i \in [3]$, and thus $d \in U(T)$, which contradicts the separability of T.

Case 2: ($d_3 < a_3 - 1$ and $b_{a_1} < b_{d_3} < b_{a_3}$) or $d_3 = a_2$. By Lemma 7.1, we have $d_3 \geqslant a_2$, and Lemma 7.2 yields that that $c_{d_3} > b_{a_1}$. Then, by Lemma 7.2, we have $d_2 < a_2$ or $b_{d_2} > b_{a_3}$. We treat the two subcases separately.

Subcase 2.1: $d_2 < a_2$. Because $c_{d_2} > c_{d_3} > b_{a_1}$, we have $d_2 = a_1$ or $b_{d_2} > b_{a_3}$. If $d_2 = a_1$, then we have $(d_1, a_1, a_2) \in U(T)$, which contradicts the separability of T. If $b_{d_2} > b_{a_3}$, then we can apply the following subcase.

Subcase 2.2: $b_{d_2} > b_{a_3}$. We have $(d_1, d_2, d_3) \in U(T)$, which contradicts the separability of T.

Case 3: $d_3 < a_3 - 1$ and $b_{a_2} < b_{d_3} \leqslant b_{a_1}$. By Lemma 7.1 we have $d_3 \leqslant a_1$. Lemma 7.2 implies that $b_{d_2} > b_{a_3}$, from which it follows that $c_{d_2} = b_{d_2} > b_{a_3} > b_{d_3}$. Therefore, we have $(d_1, d_2, d_3) \in U(T)$, which contradicts the separability of T.

Case 4: $d_3 < a_3 - 1$ and $b_{d_3} < b_{a_2}$. By Lemma 7.1 we have $d_3 < a_2$. By Lemma 7.2, at most one of d_1, d_2 can be in $\Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1]}(T)$, while by Lemma 7.1, any $d_i \notin \Gamma^{[b_{a_2}, b_{a_3}]}_{[1,a_1]}(T)$ must satisfy $b_{d_i} \notin [b_{a_2}, b_{a_3}]$. It follows that $(d_1, d_2, d_3) \in U(T)$, which contradicts the separability of T.

Hence, we may assume that $d_3 = a_3 - 1$. If $d_1 < a_1$, then it follows from Lemma 7.2 that $c_{d_1} \notin (b_{a_1}, b_{a_3})$. Lemma 7.1 yields that $c_{d_1} \notin (b_{a_1}, Y_{a_3})$. Therefore, we have $c_{d_1} \leqslant b_{a_1}$, which implies that $c_{d_2} \in (b_{a_2}, b_{a_1})$. By Lemma 7.2, we have $b_{d_2} \in (b_{a_2}, b_{a_1})$, and Lemma 7.1 yields that $d_2 < a_1$. Applying Lemma 7.2 again yields that $c_{d_2} > c_{d_1}$, which contradicts the assumption that $d \in U(\phi(T))$. Hence, we may assume that $d_1 = a_1$, in which case Lemma 7.1 implies that $d_2 \geqslant a_2$. It follows that $h_J(\psi(T)) = (a_1, a_2, a_3 - 1)$, and therefore $\psi(T)$ is of J-type 3. Because $c_{a_1} = b_{a_3+1}$ and $c_{a_3} = b_{a_2}$, we have

$$\phi(\psi(T)) = \omega^{[b_{a_2}, b_{a_3}]}_{[1,a_1]\cup\{a_3\}} \left(\omega^{[b_{a_2}, b_{a_3}]}_{[a_2, a_3-1]} \left(\theta^{[b_{a_2}, b_{a_3}]}_{[a_2, a_3-1]} \left(\theta^{[b_{a_2}, b_{a_3}]}_{[1,a_1]\cup\{a_3\}}(T) \right) \right) \right) = T,$$

as desired.

We prove that if $e = (e_1, e_2, e_3) \in V(\psi(T))$, then $S(e) \geqslant S(h_F(T))$ in the lexicographic order. Let $e \in V(\psi(T))$. Suppose for sake of deriving a contradiction that $S(e) < S(h_F(T))$ in the lexicographic order. First, we prove that $e_3 \leqslant a_3$. Suppose for sake of deriving a contradiction that $e_3 > a_3$. Let $m = \min \Gamma_{[1,a_1]}^{[b_{a_2}, b_{a_3}]}(T)$. By Lemma 7.3, we have $c_{e_3} = b_{e_3} < b_m = c_{a_3+1}$. If $e_1 > a_3$, then $e \in V(T)$, which contradicts the definition of h_F, and thus we may assume that $e_1 \leqslant a_3$. Lemmata 7.1 and 7.2 imply that $e_1 = a_3$ or $e_1 < a_2$, and we treat the two cases separately.

Case 1: $e_1 = a_3$. Then, we have $(a_2, e_2, e_3) \in V(T)$, which contradicts the definition of h_F.

Case 2: $e_1 < a_2$. By Lemma 7.1 and because $b_m < c_x$ for all $x \in \Gamma_{[1,a_1]}^{[b_{a_2}, b_{a_1}]}$, we have $b_{e_1} < b_{a_2}$ and hence $b_{e_i} = c_{e_i}$ for all i. Thus, $(e_1, e_2, e_3) \in U(T)$, which contradicts the definition of h_F.

Hence, we may assume that $e_3 \leqslant a_3$. Because $e_3 \neq a_3 - 1$ and the first component of $S(e)$ is at least $a_3 - 1$, either (e is of F-type 2, $a_3 - 2 \in D$, $e_2 = a_3 - 3$ and $e_3 = a_3 - 2$) or $e_3 = a_3$, and we treat the cases separately.

Case 1: e is of F-type 2. Because $b_{a_3-2} > b_{a_3-1}$, Lemma 7.2 implies that $b_{a_3-2} > b_{a_3}$ and therefore $b_{a_3-2} = c_{a_3-2}$. By Lemma 7.1, we have $b_{a_3-3} \geqslant b_{a_1}$, and it follows that $b_{a_3-3} \leqslant \max\{b_{a_3}, c_{a_3-3} < c_{a_3-2} = b_{a_3-2}$. Lemma 7.1 implies that $b_{e_1} > b_{a_3}$ or $e_1 > d_2$. In the latter case, the fact that $c_{e_1} > c_{e_2}$ implies that $b_{e_1} > b_{a_3}$ as well. Thus, $b_{e_1} = c_{e_1}$ and $(e_1, e_2, e_3) \in V(T)$, which contradicts the definition of h_F.

Case 2: $e_3 = a_3$. By Lemma 7.1 and by the definition of ψ, there does not exist an index $i < a_3 - 1$ such that $c_{a_3-1} < c_i < c_{a_3}$. Therefore, we have $e_2 \neq a_3 - 1$. Because $a_3 - 1 \notin A$, it follows that e is of F-type 3. If $e_2 = a_3 - 1$, then we have $b_m = c_{a_3} > c_{e_1} > c_{e_2} = b_{a_2}$, and by Lemma 7.1 and the definition of ψ, we have $e_1 < a_1$. Thus, $e_1 \in \Gamma_{[1,a_1]}^{[b_{a_2}, b_{a_3}]}$ with $c_{e_1} < b_m < c_m$, which contradicts Lemma 7.2. Therefore, we may assume that $e_2 < a_3 - 1$ and e is of F-type 3. The fact that $S(e) > S(h_F(T))$ in the lexicographic order implies that $e_1 \geqslant a_1$, but $c_{a_1} > c_{a_3}$ and thus we may in fact assume that $e_1 > a_1$. By Lemma 7.1 and the definition of ψ, we have $c_{e_1} < b_{a_2}$, which implies that $b_{e_i} = c_{e_i}$ for $i \in [2]$. Therefore,$(e_1, e_2, e_3) \in V(T)$, which contradicts the definition of h_F.

The separability of $\psi(T)$ follows.

For all $i < m$, we have $b_i = c_i$. If $m < a_1$, then let $m' = \min \Gamma_{(m,a_1]}^{[b_{a_3}, b_{a_1}]}(T)$. By Lemma 7.2, we have $c_m = b_{m'} > b_m$. If $m = a_1$, then we have $c_m = b_{a_3} > b_m$. It follows that $(b_1, b_2, \ldots, b_n) > (c_1, c_2, \ldots, c_n)$ in the lexicographic order, as desired. $\square$

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