The optimal drawings of $K_{5,n}$

César Hernández-Vélez *
Instituto de Matemática e Estatística
Universidade de São Paulo
São Paulo, Brazil 05508-090
israel@ime.usp.br

Carolina Medina   Gelasio Salazar†
Instituto de Física
Universidad Autónoma de San Luis Potosí
San Luis Potosí, Mexico
{cmedina,gsalazar}@ifisica.uaslp.mx

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Abstract

Zarankiewicz's Conjecture (ZC) states that the crossing number $\mathrm{cr}(K_{m,n})$ equals $Z(m,n) := \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. Since Kleitman's verification of ZC for $K_{5,n}$ (from which ZC for $K_{6,n}$ easily follows), very little progress has been made around ZC; the most notable exceptions involve computer-aided results. With the aim of gaining a more profound understanding of this notoriously difficult conjecture, we investigate the *optimal* (that is, crossing-minimal) drawings of $K_{5,n}$. The widely known natural drawings of $K_{m,n}$ (the so-called *Zarankiewicz drawings*) with $Z(m,n)$ crossings contain *antipodal* vertices, that is, pairs of degree-m vertices such that their induced drawing of $K_{m,2}$ has no crossings. Antipodal vertices also play a major role in Kleitman's inductive proof that $\mathrm{cr}(K_{5,n}) = Z(5,n)$. We explore in depth the role of antipodal vertices in optimal drawings of $K_{5,n}$, for n even. We prove that if $n \equiv 2 \pmod 4$, then every optimal drawing of $K_{5,n}$ has antipodal vertices. We also exhibit a two-parameter family of optimal drawings $D_{r,s}$ of $K_{5,4(r+s)}$ (for $r, s \geqslant 0$), with no antipodal vertices, and show that if $n \equiv 0 \pmod 4$, then every optimal drawing of $K_{5,n}$ without antipodal vertices is (vertex rotation) isomorphic to $D_{r,s}$ for some integers r, s. As a corollary, we show that if n is even, then every optimal drawing of $K_{5,n}$ is the superimposition of Zarankiewicz drawings with a drawing isomorphic to $D_{r,s}$ for some nonnegative integers r, s.

Keywords: Crossing number; Turán's Brickyard Problem; Zarankiewicz Conjecture; optimal drawings; antipodal vertices

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1 Introduction

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane. A drawing of a graph is *good* if no adjacent edges cross, and no two edges cross each other more than once. It is trivial to show that every *optimal* (that is, crossing-minimal) drawing of a graph is good.

One of the most tantalizingly open crossing number questions was raised by Turán in 1944: what is the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$? Zarankiewicz [8] described how to draw $K_{m,n}$ with exactly $Z(m,n)$ crossings, where

$$Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

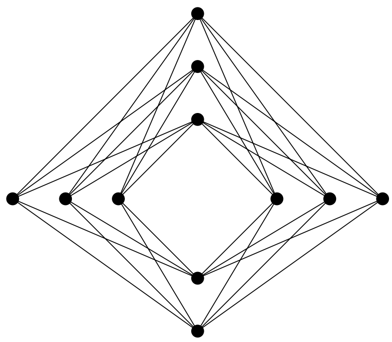

Figure 1: Drawing of $K_{5,6}$ with $Z(5,6) = 24$ crossings.

Zarankiewicz's construction is shown in Figure 1 for the case $m = 5, n = 6$. It is straightforward to generalize this drawing to a drawing of $K_{m,n}$ with $Z(m,n)$ crossings, for all positive integers m and n, and so $\mathrm{cr}(K_{m,n}) \leqslant Z(m,n)$. The drawings thus obtained are the *Zarankiewicz drawings* of $K_{m,n}$.

In [8], Zarankiewicz claimed to have proved that $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for all positive integers m, n. However, Kainen and Ringel independently found a flaw in Zarankiewicz's argument (see [5]), and the statement $\mathrm{cr}(K_{m,n}) = Z(m,n)$ is now known as *Zarankiewicz's Conjecture*.

Very little of substance is known about $\mathrm{cr}(K_{m,n})$. An elegant argument using purely combinatorial arguments (namely, Turán's theorem on the maximum number of edges in a triangle-free graph) plus $\mathrm{cr}(K_{3,3}) = 1$ shows that $\mathrm{cr}(K_{3,n}) = Z(3,n)$. An easy counting argument shows that $\mathrm{cr}(K_{2s-1,n}) = Z(2s-1,n)$ (for any $s \geqslant 1$) implies that $\mathrm{cr}(K_{2s,n}) = Z(2s,n)$. Thus it follows that $\mathrm{cr}(K_{4,n}) = Z(4,n)$. Kleitman [6] proved that $\mathrm{cr}(K_{5,n}) = Z(5,n)$. By our previous remark, this implies that $\mathrm{cr}(K_{6,n}) = Z(6,n)$.

After Kleitman's theorem, most progress around Zarankiewicz's Conjecture consists of computer-aided results. Woodall [7] verified Zarankiewicz's Conjecture for $K_{7,7}$ and $K_{7,9}$. De Klerk et al. [3] showed that $\lim_{n\to\infty} \operatorname{cr}(K_{7,n})/Z(7,n) \geqslant 0.968$ using semidefinite programming techniques. Also using semidefinite programming and deeper algebraic techniques, De Klerk et al. [4] proved that $\lim_{n\to\infty} \operatorname{cr}(K_{9,n})/Z(9,n) \geqslant 0.966$. In a related result, De Klerk and Pasechnik [2] recently showed that the 2-page crossing number $\nu_2(K_{7,n})$ of $K_{7,n}$ satisfies $\lim_{n\to\infty} \operatorname{cr}(K_{7,n})/Z(7,n) = 1$.

We finally mention that Christian et al. [1] proved that deciding Zarankiewicz's Conjecture is a finite problem for each fixed m.

To give a brief description of our results, let us color the 5 degree-n vertices of $K_{5,n}$ *black*, and color the n degree-5 vertices *white*. Two white vertices are *antipodal* in a drawing D of $K_{5,n}$ if the drawing of the $K_{5,2}$ they induce has no crossings. A drawing is *antipodal-free* if it has no antipodal vertices.

Antipodal pairs are evident in Zarankiewicz's drawings (moreover, the set of white vertices can be decomposed into two classes, such that any two white vertices in distinct classes are antipodal). Antipodal pairs are also crucial in the inductive step of Kleitman's proof, which does not concern itself with the different ways (if more than one) to achieve $Z(5,n)$ crossings with a drawing of $K_{5,n}$.

Given their preeminence in Zarankiewicz's Conjecture, we set out to investigate the role of antipodal pairs in the optimal drawings of $K_{5,n}$. Our main result (Theorem 1) characterizes optimal drawings of $K_{5,n}$, for even n, as follows. First, if $n \equiv 2 \pmod 4$, then all optimal drawings of $K_{5,n}$ have antipodal pairs. Second, if $n \equiv 0 \pmod 4$, then every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic (we review vertex rotation isomorphism in Section 2) to a drawing in a two-parameter family $D_{r,s}$ of drawings we have fully characterized. As a consequence of these facts, we show (Theorem 2) that if n is even, then every optimal drawing of $K_{5,n}$ can be obtained by starting with $D_{r,s}$, for some nonnegative (possibly zero) integers r and s, and then superimposing Zarankiewicz drawings.

The rest of this paper is organized as follows. In Section 2 we review the concept of vertex rotation, which is central to the criterion to decide when two drawings are isomorphic. In Section 3 we describe the two-parameter family of optimal, antipodal-free drawings $D_{r,s}$ (for integers $r, s \geqslant 0$) of $K_{5,4(r+s)}$. In Section 4 we state our main results. Theorem 1 claims that (i) if $n \equiv 2 \pmod 4$, then every optimal drawing of $K_{5,n}$ has antipodal vertices; and that (ii) if $n \equiv 0 \pmod 4$, then every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic to $D_{r,s}$ for some integers r, s such that $4(r+s) = n$. In Theorem 2 we state the decomposition of optimal drawings of $K_{5,n}$, along the lines of the previous paragraph. The proof of Theorem 2 is also given in this section; the rest of the paper is devoted to the proof of Theorem 1. In Section 5 we introduce the concept of a *clean* drawing. Loosely speaking, a drawing is clean if its white vertices can be naturally partitioned into *bags*, so that vertices in the same bag have the same (crossing number wise) properties. In Section 6 we introduce *keys*, which are labelled graphs that capture the essential (crossing number wise) information of a clean drawing. This abstraction (and the related concept of *core*) will prove to be extremely useful for the proof of Theorem 1. In Section 7 we investigate which labelled graphs can be the key of a relevant (clean, optimal, antipodal-free) drawing. Cores are certain more manageable subgraphs of keys, that retain all the (crossing number wise) useful information of a key. We devote Sections 8, 9, 10, and 11 to the task of completely characterizing which graphs can be the core of an antipodal-free optimal drawing. The information in these sections is then put together in Section 12, where we show that the core of every optimal drawing is isomorphic either to the 4-cycle or to the graph $\overline{C}_6$ obtained by adding to the 6-cycle a diametral edge. The proof of Theorem 1, given in Section 13, is an easy consequence of this full characterization of cores.

2 Rotations and isomorphic drawings

To help comprehension, throughout this paper we color the 5 degree-n vertices in $K_{5,n}$ *black*, and the n degree-5 vertices *white*. We label the black vertices $0, 1, 2, 3, 4$. Unless otherwise stated, we label the white vertices $a_0, a_1, \ldots, a_{n-1}$. We adopt the notation $[n] := \{0, 1, \ldots, n-1\}$.

Given vertices a_i, a_j with $i, j \in [n]$, we let $S(a_i)$ denote the *star* centered at a_i, that is, the subgraph (isomorphic to $K_{5,1}$) induced by a_i and the vertices $0, 1, 2, 3, 4$. If D is a drawing of $K_{5,n}$, we let $\mathrm{cr}_D(a_i, a_j)$ denote the number of crossings in D that involve an edge of $S(a_i)$ and an edge of $S(a_j)$, and we let $\mathrm{cr}_D(a_i) := \sum_{k\in[n],k\neq i} \mathrm{cr}_D(a_i, a_k)$. Formalizing the definition from Section 1, a_i and a_j are *antipodal* (*in* D) if $\mathrm{cr}_D(a_i, a_j) = 0$.

The *rotation* $\mathrm{rot}_D(a_i)$ of a white vertex a_i in a drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave a_i. We use the notation 01234 for permutations, and (01234) for cyclic permutations. For instance, the rotation $\mathrm{rot}_D(a_3)$ of the vertex a_3 in the drawing D in Figure 2 is (02431): following a counterclockwise order, if we start with the edge leaving from a_3 to 0, then we encounter the edge leaving to 2, then the edge leaving to 4, then the edge leaving to 3, and then the edge leaving to 1. We emphasize that a rotation is a cyclic permutation; that is, $(02431), (24310), (43102), (31024)$, and (10243) denote (are) the same rotation. We let Π denote the set of all cyclic permutations of $0, 1, 2, 3, 4$. Clearly, $|\Pi| = 5!/5 = 4! = 24$. The *rotation* $\mathrm{rot}_D(i)$ of a black vertex i is defined analogously: for each $i \in [5]$, $\mathrm{rot}_D(i)$ is a cyclic permutation of $a_0, a_1, \ldots, a_{n-1}$.

The *rotation multiset* $\mathrm{Rot}_M(D)$ *of* D is the multiset (that is, repetitions are allowed) containing the n rotations $\mathrm{rot}_D(a_i)$, for $i = 0, 1, \ldots, n-1$. The *rotation set* $\mathrm{Rot}(D)$ of D is the underlying set (that is, no repetitions allowed) of $\mathrm{Rot}_M(D)$. Thus, in the example of Figure 2, $\mathrm{Rot}_M(D) = [(04321), (04321), (01234), (02431)]$ (we use square brackets for multisets), and $\mathrm{Rot}(D) = \{(04321), (01234), (02431)\}$.

Two multisets M, M' of rotations are *equivalent* (we write $M \cong M'$) if one of them can be obtained from the other by a relabelling (formally, a self-bijection) of $0, 1, 2, 3, 4$. Two drawings D, D' of $K_{5,n}$ are *isomorphic* if $\mathrm{Rot}_M(D) \cong \mathrm{Rot}_M(D')$. Loosely speaking, two drawings D, D' of $K_{5,n}$ are isomorphic if $0, 1, 2, 3, 4$ and $a_0, a_1, \ldots, a_{n-1}$ can be relabelled (say in D'), if necessary, so that $\mathrm{rot}_D(a_i) = \mathrm{rot}_{D'}(a_i)$ for every $i \in [n]$.

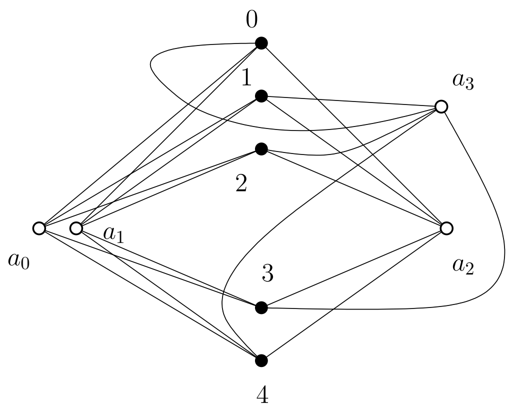

Figure 2: A drawing D of $K_{5,4}$ with $\mathrm{rot}_D(a_0) = \mathrm{rot}_D(a_1) = (04321), \mathrm{rot}_D(a_2) = (01234)$, and $\mathrm{rot}_D(a_3) = (02431)$. Thus the pair a_0, a_2 (as well as the pair a_1, a_2) is antipodal.

Our ultimate interest lies in optimal drawings (of $K_{5,n}$). It is not difficult to see (we will prove this later) that if D is an optimal drawing and a_i, a_j, a_k, a_ℓ are vertices such that $\mathrm{rot}_D(a_i) = \mathrm{rot}_D(a_j)$ and $\mathrm{rot}_D(a_k) = \mathrm{rot}_D(a_\ell)$, then $\mathrm{cr}_D(a_i, a_k) = \mathrm{cr}_D(a_j, a_\ell)$. Thus an optimal drawing of $K_{5,n}$ is adequately described by choosing a representative vertex of each rotation, and giving the information of how many vertices there are for each rotation. This supports the pertinence of focusing on the rotations as the criteria for isomorphism.

3 An antipodal-free drawing of $K_{5,4(r+s)}$

In this section we describe an antipodal-free drawing $D_{r,s}$ of $K_{5,4(r+s)}$, for each pair r, s of nonnegative integers.

The construction is based on the drawing D^* of $K_{5,6}$ in Figure 3. As shown, the rotations in D^* of the white vertices are $\mathrm{rot}_{D^*}(a_0) = (01234)$, $\mathrm{rot}_{D^*}(a_1) = (04231)$, $\mathrm{rot}_{D^*}(a_2) = (01342)$, $\mathrm{rot}_{D^*}(a_3) = (04312)$, $\mathrm{rot}_{D^*}(a_4) = (01432)$, $\mathrm{rot}_{D^*}(a_5) = (02314)$.

It is immediately checked that D^* is antipodal-free. Note that D^* itself is not optimal, as it has $25 = Z(5,6) + 1$ crossings.

Suppose first that both r and s are positive. To obtain $D_{r,s}$, we add $4(r+s) - 6$ white vertices to D^*. Now $r - 1$ of these vertices are drawn very close to a_1, and $r - 1$ are drawn very close to a_2; $s - 1$ vertices are drawn very close to a_4, and $s - 1$ are drawn very close to a_5; finally, $r + s - 1$ vertices are drawn very close to a_0, and $r + s - 1$ are drawn very close to a_3. It is intuitively clear what is meant by having a_i drawn "very close" to a_j. Formally, we require that: (i) a_i and a_j have the same rotation; (ii) $\mathrm{cr}_{D_{r,s}}(a_i, a_j) = 4$; and (iii) for any other vertex a_k, $\mathrm{cr}_{D_{r,s}}(a_i, a_k) = \mathrm{cr}_{D_{r,s}}(a_j, a_k)$. These properties are easily satisfied by having the added vertex a_i drawn sufficiently close to a_j, so that the edges incident with a_i follow very closely the edges incident with a_j.

If one of r or s is 0, then we make the obvious adjustments. That is, (i) if $r = 0$, then we remove a_1 and a_2, and for each $i = 0, 3, 4, 5$, we draw $s - 1$ new vertices very close to

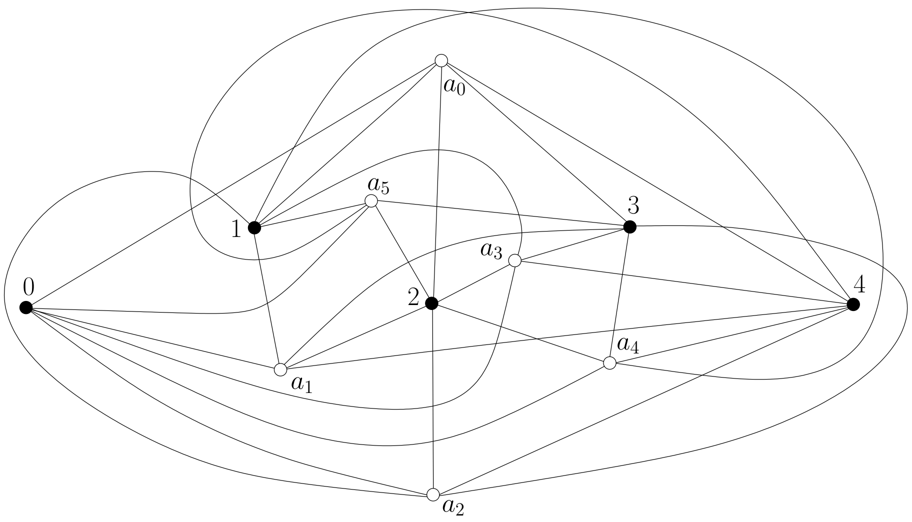

Figure 3: This antipodal-free drawing D^* of $K_{5,6}$ is the base of the construction of the optimal antipodal-free drawing $D_{r,s}$ of $K_{5,4(r+s)}$ for all r, s. It is easily verified that $\mathrm{rot}_{D^*}(a_0) = (01234)$, $\mathrm{rot}_{D^*}(a_1) = (04231)$, $\mathrm{rot}_{D^*}(a_2) = (01342)$, $\mathrm{rot}_{D^*}(a_3) = (04312)$, $\mathrm{rot}_{D^*}(a_4) = (01432)$, $\mathrm{rot}_{D^*}(a_5) = (02314)$.

a_i; and (ii) if $s = 0$, then we remove a_4 and a_5, and for each $i = 0, 1, 2, 3$, we draw $r - 1$ new vertices very close to a_i. (In the extreme case $r = s = 0$, we remove all the white vertices from D^*, and are left with an obviously optimal drawing of $K_{5,0}$).

For each $i = 0, 1, 2, 3, 4, 5$, the *bag* $[a_i]$ of a_i is the set that consists of the vertices drawn very close to a_i, plus a_i itself.

Note that each of $[a_0]$ and $[a_3]$ has $r + s$ vertices, each of $[a_1]$ and $[a_2]$ has r vertices, and each of $[a_4]$ and $[a_5]$ has s vertices.

An illustration of the construction for $r = 2$ and $s = 1$ is given in Figure 4, where the gray vertices are the ones added to D^*.

Claim. *For every pair r, s of nonnegative integers, $D_{r,s}$ is an antipodal-free optimal drawing of $K_{5,4(r+s)}$.*

Proof. First we note that since D^* is antipodal-free, it follows immediately that $D_{r,s}$ is also antipodal-free. Thus we only need to prove optimality.

An elementary calculation gives the number of crossings in $D_{r,s}$. For instance, take a vertex u in $[a_0]$. Now $\mathrm{cr}_{D_{r,s}}(u, v)$ equals (i) 4 if $v \in [a_0], v \neq u$; (ii) 1 if $v \in [a_1]$; (iii) 2 if $v \in [a_2]$; (iv) 1 if $v \in [a_3]$; (v) 1 if $v \in [a_4]$; and (vi) 2 if $v \in [a_5]$. Since $|[a_0]| = r + s$, $|[a_1]| = r$, $|[a_2]| = r$, $|[a_3]| = r + s$, $|[a_4]| = s$, and $|[a_5]| = s$, it follows that $\mathrm{cr}_{D_{r,s}}(u) = 4(r + s - 1) + r + 2r + (r + s) + s + 2s = 4(2r + 2s - 1)$.

A totally analogous argument shows that, actually, $\mathrm{cr}_{D_{r,s}}(w) = 4(2r + 2s - 1)$ for *every* white vertex w. Since there are $4(r + s)$ white vertices in total, it follows that $\mathrm{cr}(D_{r,s}) = (1/2)\Big(4(r+s)\Big)\Big(4(2r+2s-1)\Big) = \Big(4(r+s)\Big)\Big(4(r+s)-2\Big) = Z(5, 4(r+s))$. $\square$

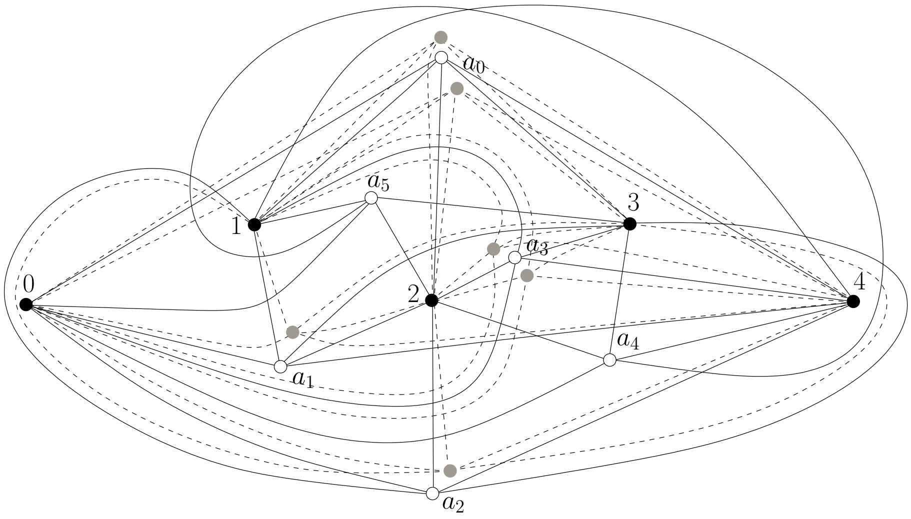

Figure 4: The antipodal-free drawing $D_{2,1}$. To obtain this optimal drawing of $K_{5,12} = K_{5,4(2+1)}$, we start with the drawing in Figure 3 and add two vertices very close to a_0, two vertices very close to a_3, one vertex very close to a_1, and one vertex very close to a_2. Since $s - 1 = 0$, no vertices are added very close to either a_4 or a_5. The added vertices are colored gray in this drawing.

4 Main results: the optimal drawings of $K_{5,n}$, for n even

We now state our main results.

Theorem 1. *Let n be a positive even integer.*

1. *If $n \equiv 2 \pmod 4$, then all optimal drawings of $K_{5,n}$ have antipodal vertices.*
2. *If $n \equiv 0 \pmod 4$, then every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic to $D_{r,s}$ (described in Section 3) for some integers r, s such that $4(r + s) = n$.*

Before moving on to the proof of Theorem 1 (the rest of the paper is devoted to this proof), we will show that it implies a decomposition of all the optimal drawings of $K_{5,n}$, for n even.

In Section 1 we defined, somewhat informally, a Zarankiewicz drawing. Let us now formally define these drawings using rotations (we focus on $K_{5,n}$, although the definition is obviously extended to $K_{m,n}$ for any m). For a nonnegative integer n, a drawing D of $K_{5,n}$ is a *Zarankiewicz drawing* if the white vertices can be partitioned into two sets, of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, so that vertices in different sets are antipodal in D, and vertices a_i, a_j in the same set satisfy $\mathrm{cr}_D(a_i, a_j) = 4$ (see Figure 1 for a Zarankiewicz drawing of $K_{5,6}$). A quick calculation shows that every Zarankiewicz drawing of $K_{5,n}$ is an optimal drawing.

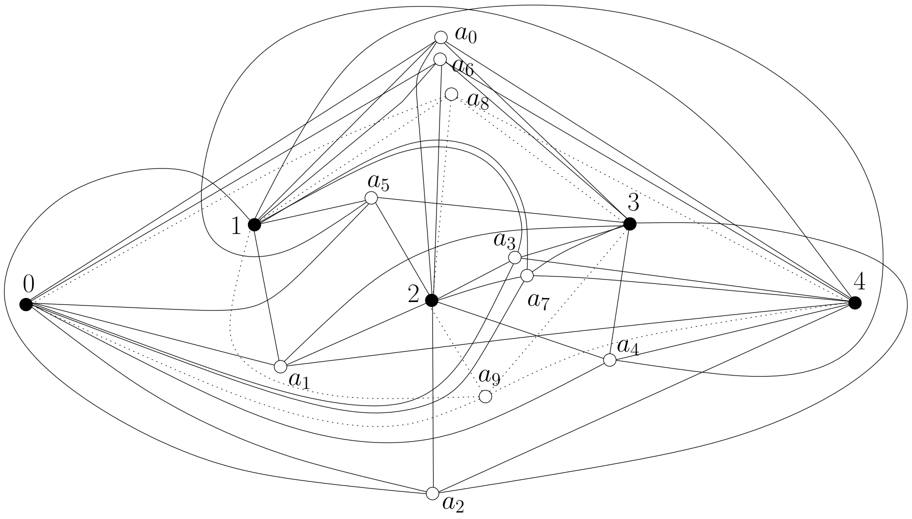

Figure 5: An optimal drawing of $K_{5,10}$ that is neither a Zarankiewicz drawing nor the superimposition of Zarankiewicz drawings. As predicted by Theorem 2, this is the superimposition of a Zarankiewicz drawing (the $K_{5,2}$ induced by a_8, a_9 and the five black vertices) plus a drawing $D_{r,s}$ (namely with $r = s = 1$).

Theorem 2 (Decomposition of optimal drawings of $\boldsymbol{K_{5,n}}$, for $\boldsymbol{n}$ even)**.** *Let D be an optimal drawing of $K_{5,n}$, with n even. Then the set of n white vertices can be partitioned into two sets A, B (one of which may be empty), with $|A| = 4t$ for some nonnegative integer t, such that: (i) the vertices in B can be decomposed into $|B|/2$ antipodal pairs; and (ii) the drawing of $K_{5,4t}$ induced by A is antipodal-free, and it is isomorphic to the drawing $D_{r,s}$ described in Section 3, for some integers r, s such that $r + s = t$. Equivalently, either D is the superimposition of Zarankiewicz drawings, or it can be obtained by superimposing Zarankiewicz drawings to the drawing $D_{r,s}$ described in Section 3, for some integers r, s (see Figure 5).*

Proof. We proceed by induction on n. It is trivial to check that the two white vertices of every optimal drawing of $K_{5,2}$ are an antipodal pair, and so the statement holds in the base case $n = 2$. For the inductive step, we consider an even integer n, and assume that the statement is true for all $k < n$.

Let D be an optimal drawing of $K_{5,n}$. If D has no antipodal pairs, then the statement follows immediately from Theorem 1 (without even using the induction hypothesis). Thus we may assume that D has at least one antipodal pair a_i, a_j. It suffices to show that the drawing D' that results by removing a_i and a_j from D is an optimal drawing of $K_{5,n-2}$, as then the result follows by the induction hypothesis. Clearly $\mathrm{cr}(D) = \mathrm{cr}(D') + \sum_{k \in [n]-\{i,j\}} (\mathrm{cr}_D(a_i, a_k) + \mathrm{cr}_D(a_j, a_k)) \geqslant \mathrm{cr}(D') + (n-2)Z(5,3) = \mathrm{cr}(D') + 4n - 8$. Thus $\mathrm{cr}(D') \leqslant \mathrm{cr}(D) - 4n + 8 = Z(5,n) - 4n + 8$. An elementary calculation shows that $Z(5,n) - 4n + 8 = Z(5, n-2)$, so we obtain $\mathrm{cr}(D') \leqslant Z(5, n-2)$. Since $\mathrm{cr}(K_{5,n-2}) = Z(5, n-2)$, it follows that $\mathrm{cr}(D') = Z(5, n-2)$, that is, D' is an optimal drawing of $K_{5,n-2}$. □

5 Clean drawings

A good drawing of $K_{5,n}$ is *clean* if for all distinct white vertices a_i, a_j, a_k, a_ℓ:

1. if $\mathrm{rot}_D(a_i) = \mathrm{rot}_D(a_j)$, we have $\mathrm{cr}_D(a_i, a_j) = 4$;
2. if $\mathrm{rot}_D(a_i) = \mathrm{rot}_D(a_j)$ and $\mathrm{rot}_D(a_k) = \mathrm{rot}_D(a_\ell)$, we have $\mathrm{cr}_D(a_i, a_k) = \mathrm{cr}_D(a_j, a_\ell)$; and
3. $\mathrm{cr}_D(a_i, a_k) \leqslant 4$.

Proposition 3. *Let D be an optimal drawing of $K_{5,n}$. Then there is an optimal drawing D', isomorphic to D, that is clean.*

Proof. For each white vertex a_i, define $d_i := \sum_{\{a_\ell \,|\, \mathrm{rot}_D(a_\ell) \neq \mathrm{rot}_D(a_i)\}} \mathrm{cr}_D(a_i, a_\ell)$. Let $\pi \in \mathrm{Rot}(D)$. Take a white vertex a_i with $\mathrm{rot}_D(a_i) = \pi$, such that for all j with $\mathrm{rot}_D(a_j) = \pi$ we have $d_i \leqslant d_j$. It is easy to see that we can move every vertex a_j with $\mathrm{rot}_D(a_j) = \pi$ very close to a_i, so that $\mathrm{cr}_D(a_i, a_k) = \mathrm{cr}_D(a_j, a_k)$ for every white vertex $a_k \notin \{a_i, a_j\}$, and so that $\mathrm{cr}_D(a_i, a_j) = 4$. If we perform this procedure for every rotation in $\mathrm{Rot}(D)$, the result is an optimal drawing D', isomorphic to D, that satisfies (1) and (2).

Now to prove that D' also satisfies (3) we suppose, by way of contradiction, that there exist a_i, a_k such that $\mathrm{cr}_D(a_i, a_k) > 4$. Define d_i, d_k as in the previous paragraph. We may assume without loss of generality that $d_i \leqslant d_k$. Now let D'' be the drawing that results from moving a_k very close to a_i, making it have the same rotation as a_i, and so that $\mathrm{cr}_{D''}(a_i, a_\ell) = \mathrm{cr}_{D''}(a_k, a_\ell)$ for every $\ell \notin \{i, k\}$, and $\mathrm{cr}_{D''}(a_i, a_k) = 4$. It is readily checked that D'' has fewer crossings than D', contradicting the optimality of D'. $\square$

Remark 4. We are interested in classifying optimal drawings up to isomorphism (Theorem 1). In view of Proposition 3, we may assume that all drawings of $K_{5,n}$ under consideration are clean. We will work under this assumption for the rest of the paper.

6 The key of a clean drawing

We now associate to every clean drawing of $K_{5,n}$ an edge-labeled graph that (as we will see) captures all its relevant crossing number information.

Let D be a clean drawing of $K_{5,n}$. The *key* $\Phi(D)$ of D is the (edge-labeled) complete graph whose vertices are the elements of $\mathrm{Rot}(D)$, and where each edge is labeled according to the following rule: if $\pi, \pi' \in \mathrm{Rot}(D)$, with $\mathrm{rot}_D(a_i) = \pi$ and $\mathrm{rot}_D(a_j) = \pi'$, then the label of the edge joining π and π' is $\mathrm{cr}_D(a_i, a_j)$. It follows from the cleanness of D that $\mathrm{cr}_D(a_i, a_j)$ does not depend on the choice of a_i and a_j, and so $\Phi(D)$ is well-defined for every clean drawing D. Moreover, it also follows that every edge label in $\Phi(D)$ is in $\{0, 1, 2, 3, 4\}$. The *core* of D is the subgraph $\Phi^1(D)$ of $\Phi(D)$ that consists of all the vertices of $\Phi(D)$ and the edges of $\Phi(D)$ with label 1. In Figure 6 we give a (clean and optimal) drawing D of $K_{5,3}$, and illustrate its key and its core.

Our main interest is in antipodal-free drawings, that is, those drawings in which every edge label in $\Phi(D)$ is in $\{1, 2, 3, 4\}$. A key is 0*–free* (respectively, 4-*free*) if none of its edges has 0 (respectively, 4) as a label. A key is $\{0, 4\}$-*free* if it is both 0- and 4-free.

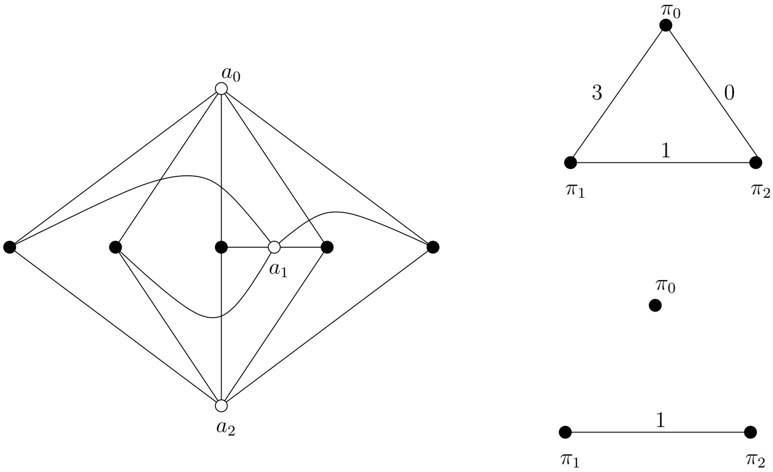

Figure 6: A drawing D of $K_{5,3}$. By letting $\mathrm{rot}_D(a_0) = \pi_0, \mathrm{rot}_D(a_1) = \pi_1$, and $\mathrm{rot}_D(a_2) = \pi_2$, we obtain the key $\Phi(D)$ (right, above) and the core $\Phi^1(D)$ (right, below) of D.

The main step in our strategy to understand optimal drawings is to characterize which labelled graphs are the key of some optimal drawing. To this end, we introduce a system of linear equations associated to each key, as follows.

Definition 5 (The system of linear equations of a key)**.** Let D be an optimal drawing of $K_{5,n}$, with n even. Let the vertices of $\Phi(D)$ (that is, the elements of $\mathrm{Rot}(D)$) be labelled $\pi_0, \pi_1, \ldots, \pi_{m-1}$, and let λ_{ij} denote the label of the edge $\pi_i\pi_j$, for all $i \neq j$. For each $i \in [m]$, the *linear equation* $E(\pi_i, \Phi(D))$ *for* π_i *in* $\Phi(D)$ is the linear equation on the variables $t_0, t_1, \ldots, t_{m-1}$ given by

$$E(\pi_i, \Phi(D)) \quad : \quad 2t_i + \sum_{j \in [m],\, j \neq i} (\lambda_{ij} - 2)t_j = 0.$$

The set $\{E(\pi_i, \Phi(D))\}_{i \in [m]}$ is the *system of linear equations associated to* $\Phi(D)$, and is denoted $\mathcal{L}(\Phi(D))$.

The characterization of when a labelled graph is the key of an optimal drawing is mainly based on the following crucial fact.

Proposition 6. *Let D be an optimal drawing of $K_{5,n}$, with n even. Then the system of linear equations $\mathcal{L}(\Phi(D))$ associated to $\Phi(D)$ has a positive integral solution $(t_0, t_1, \ldots, t_{m-1})$ such that $t_0 + t_1 + \cdots + t_{m-1} = n$.*

Proof. First we show that if D is an optimal drawing of $K_{5,n}$ with n even, then for every $i = 0, 1, \ldots, n-1$, we have $\mathrm{cr}_D(a_i) = 2n - 4$. To this end, suppose that $\mathrm{cr}_D(a_i) > 2n - 4$

for some i. Since D is optimal, $\mathrm{cr}(D) = Z(5,n) = n(n-2)$, and so the drawing D' of $K_{5,n-1}$ that results by removing a_i from D has fewer than $n(n-2) - (2n-4) = n^2 - 4n + 4 = (n-2)^2 = Z(5,n-1)$ crossings, contradicting that $\mathrm{cr}(K_{5,n-1}) = Z(5,n-1)$. Thus $\mathrm{cr}_D(a_i) \leqslant 2n-4$ for every i. Now suppose that $\mathrm{cr}_D(a_i) < 2n-4$ for some i. Then $\mathrm{cr}(D) = (1/2)\sum_{j\in[n]} \mathrm{cr}_D(a_j) < (1/2)(2n-4)n = n(n-2)$, contradicting that $\mathrm{cr}(K_{5,n}) = Z(5,n) = n(n-2)$. Thus for every $i \in [n]$ we have $\mathrm{cr}_D(a_i) = 2n-4$, as claimed.

Now let $\pi_0, \pi_1, \ldots, \pi_{m-1}$ be the elements of $\mathrm{Rot}(D)$ (that is, the vertices of $\Phi(D)$), and for each $i,j \in [m], i \neq j$, let λ_{ij} denote the label of the edge $\pi_i\pi_j$ in $\Phi(D)$. For each $i \in [m]$, let t_i be the number of vertices with rotation π_i in D. Then (using that D is clean) for every $i \in [m]$ and every white vertex a_k with $\mathrm{rot}_D(a_k) = \pi_i$ we have $\mathrm{cr}_D(a_k) = 4(t_i - 1) + \sum_{j\in[m],j\neq i} \lambda_{ij}t_j$. Now from the previous paragraph for each a_k we have $\mathrm{cr}_D(a_k) = 2n-4$. Using that $n = \sum_{j\in[m]} t_j$, we obtain $4(t_i-1) + \sum_{j\in[m],j\neq i} \lambda_{ij}t_j = 2\sum_{j\in[m]} t_j - 4$. Equivalently, $2t_i + \sum_{j\in[m],j\neq i}(\lambda_{ij} - 2)t_j = 0$, for every $i \in [m]$. Thus $(t_0, t_1, \ldots, t_{m-1})$ is a positive integral solution of $\mathcal{L}(\Phi(D))$. □

7 Properties of the key of a clean drawing

We start with an easy, yet crucial, observation.

Proposition 7. *Let D be an optimal drawing of $K_{5,n}$. Then, for any three distinct white vertices a_i, a_j, a_k, $\mathrm{cr}_D(a_i,a_j) + \mathrm{cr}_D(a_j,a_k) + \mathrm{cr}_D(a_i,a_k)$ is an even number greater than or equal to 4.*

Proof. This follows since $\mathrm{cr}(K_{5,3}) = Z(5,3) = 4$ and (see for instance [6]) every good drawing of $K_{5,3}$ has an even number of crossings. □

The following is an equivalent form of this statement, in the setting of keys.

Proposition 8. *Let D be a clean drawing of $K_{5,n}$, and let π_0, π_1, π_2 be vertices of $\Phi(D)$. Let λ_{ij} be the label of the edge $\pi_i\pi_j$, for $i,j \in \{0,1,2\}, i \neq j$. Then $\lambda_{01} + \lambda_{12} + \lambda_{02}$ is an even number greater than or equal to 4.* □

Let γ, κ be cyclic permutations on the same set of symbols. A *route* from γ to κ is a set of distinct transpositions, which may be ordered into some sequence such that the successive application of (all) the transpositions in this sequence takes γ to κ. For instance, if $\gamma = (abcd)$ and $\kappa = (acdb)$, then $\{(bd),(bc)\}$ is a route from γ to κ: if we apply first (bc) to γ, and then (bd) to the resulting cyclic permutation, we obtain κ.

The *size* $|P|$ of a route P is its number of transpositions. An *antiroute* from γ to κ is a route from γ to the reverse cyclic permutation $\overline{\kappa}$ of κ. Note that if P is a route (respectively, antiroute) from γ to κ, then P is also a route (respectively, antiroute) from κ to γ. The *antidistance* between two cyclic permutations is the smallest size of an antiroute between them.

The following is an easy consequence of (the proof of) Theorem 5 in [7].

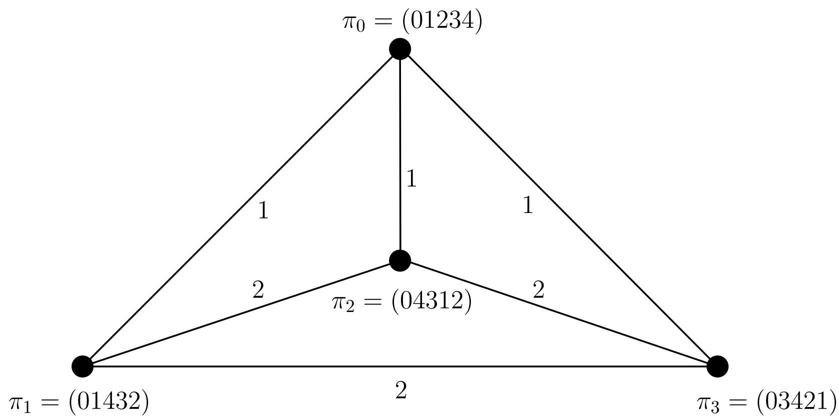

Figure 7: This cannot be the key of a clean drawing of $K_{5,n}$.

Lemma 9. *Let D be a good drawing of $K_{5,2}$, with white vertices a_0, a_1. Then there is an antiroute from $\mathrm{rot}_D(a_0)$ to $\mathrm{rot}_D(a_1)$ of size $\mathrm{cr}_D(a_0, a_1)$.* □

The following statement is implicitly proved in the discussion after the proof of [7, Theorem 5].

Lemma 10. *Let D be a clean drawing of $K_{5,r}$ with white vertices $a_0, a_1, \ldots, a_{r-1}$, and let $\pi_i := \mathrm{rot}_D(a_i)$. Suppose that $\pi_i \neq \pi_j$ whenever $i \neq j$, and for all $i \neq j$ let $\lambda_{ij} := \mathrm{cr}_D(a_i, a_j)$. For $k = 0, 1, 2, 3, 4$, let $\gamma_k := \mathrm{rot}_D(k)$. Then there exist:*

1. *for all $i, j \in [r]$ with $i \neq j$, an antiroute P_{ij} from π_i to π_j of size λ_{ij};*
2. *for all $k, \ell \in [5]$ with $k \neq \ell$, an antiroute $Q_{k\ell}$ from γ_k to γ_ℓ;*

such that the transposition $(a_i\, a_j)$ is in $Q_{k\ell}$ if and only if the transposition $(k\, \ell)$ is in P_{ij}.
□

We now use these powerful statements to prove that certain graphs cannot be the subgraphs of the key of a clean drawing.

Proposition 11. *The graph in Figure 7 does not occur as an induced subgraph of the key of any clean drawing of $K_{5,n}$.*

Proof. Since an induced subgraph of a key is a key, it suffices to prove that this graph cannot be the key of a clean drawing of $K_{5,n}$.

Suppose by way of contradiction that the graph in Figure 7 is the key of some clean drawing of $K_{5,n}$. This implies in particular that there exists a drawing D of $K_{5,4}$ with white vertices a_0, a_1, a_2, a_3 such that $\mathrm{rot}_D(a_i) = \pi_i$ for $i = 0, 1, 2, 3$, with $\pi_0 = (01234), \pi_1 = (01432), \pi_2 = (04312)$, and $\pi_3 = (03421)$, and $\mathrm{cr}_D(a_0, a_1) = \mathrm{cr}_D(a_0, a_2) = \mathrm{cr}_D(a_0, a_3) = 1$, and $\mathrm{cr}_D(a_1, a_2) = \mathrm{cr}_D(a_1, a_3) = \mathrm{cr}_D(a_2, a_3) = 2$.

The required contradiction is obtained by showing that there do not exist rotations $\mathrm{rot}_D(0), \mathrm{rot}_D(1), \mathrm{rot}_D(2), \mathrm{rot}_D(3), \mathrm{rot}_D(4)$, and antiroutes $P_{ij}, Q_{k\ell}$ that satisfy Lemma 10 (with the given values of $\mathrm{cr}_D(a_i, a_j)$ for $i, j \in \{0, 1, 2, 3\}$, $i \neq j$). We start by determining the possible antiroutes P_{ij} (these depend only on the information we already have). Then we investigate the possible antiroutes $Q_{k\ell}$ consistent with each choice of the antiroutes P_{ij}, and prove that, in all cases, every possible choice of $\mathrm{rot}_D(0), \mathrm{rot}_D(1), \mathrm{rot}_D(2)$, $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ leads to an inconsistency.

The following facts are easily verified: (i) the only antiroute from π_0 to π_1 of size 1 is $\{(01)\}$; (ii) the only antiroute from π_0 to π_2 of size 1 is $\{(12)\}$; (iii) the only antiroute from π_0 to π_3 of size 1 is $\{(34)\}$; (iv) the only antiroute of size 2 from π_1 to π_2 is $\{(02), (34)\}$; (v) there are two distinct antiroutes of size 2 from π_2 to π_3, namely $\{(01), (02)\}$ and $\{(03), (04)\}$; and (vi) there are two distinct antiroutes of size 2 from π_1 to π_3, namely $\{(02), (12)\}$ and $\{(23), (24)\}$.

Now for $i, j \in \{0, 1, 2, 3\}, i \neq j$, let P_{ij} be the antiroute guaranteed by Lemma 10. By the previous observations it follows that necessarily $P_{01} = \{(01)\}$, $P_{02} = \{(12)\}$, $P_{03} = \{(34)\}$, and $P_{12} = \{(02), (34)\}$. Also by the previous observations there are two choices for P_{23}, namely $\{(01), (02)\}$ and $\{(03), (04)\}$; and there are two choices for P_{13}, namely $\{(02), (12)\}$ and $\{(23), (24)\}$.

Thus $P_{01}, P_{02}, P_{03}, P_{12}$ are all determined:

$$P_{01} = \{(01)\}, P_{02} = \{(12)\}, P_{03} = \{(34)\}, P_{12} = \{(02), (34)\},$$

and there are four possible combinations of P_{13} and P_{23}:

(a) $P_{23} = \{(01), (02)\}$ and $P_{13} = \{(02), (12)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1), (a_2a_3)\}$, $Q_{02} = \{(a_1a_2), (a_2a_3), (a_1a_3)\}$, $Q_{03} = \emptyset$, $Q_{04} = \emptyset$, $Q_{12} = \{(a_0a_2), (a_1a_3)\}$, $Q_{13} = \emptyset$, $Q_{14} = \emptyset$, $Q_{23} = \emptyset$, $Q_{24} = \emptyset$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

(b) $P_{23} = \{(01), (02)\}$ and $P_{13} = \{(23), (24)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1), (a_2a_3)\}$, $Q_{02} = \{(a_1a_2), (a_2a_3)\}$, $Q_{03} = \emptyset$, $Q_{04} = \emptyset$, $Q_{12} = \{(a_0a_2)\}, Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \{(a_1a_3)\}, Q_{24} = \{(a_1a_3)\}$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

(c) $P_{23} = \{(03), (04)\}$ and $P_{13} = \{(02), (12)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1)\}, Q_{02} = \{(a_1a_2), (a_1a_3)\}, Q_{03} = \{(a_2a_3)\}, Q_{04} = \{(a_2a_3)\}, Q_{12} = \{(a_0a_2), (a_1a_3)\}$, $Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \emptyset, Q_{24} = \emptyset$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

(d) $P_{23} = \{(03), (04)\}$ and $P_{13} = \{(23), (24)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1)\}, Q_{02} = \{(a_1a_2)\}$, $Q_{03} = \{(a_2a_3)\}, Q_{04} = \{(a_2a_3)\}, Q_{12} = \{(a_0a_2)\}, Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \{(a_1a_3)\}, Q_{24} = \{(a_1a_3)\}$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

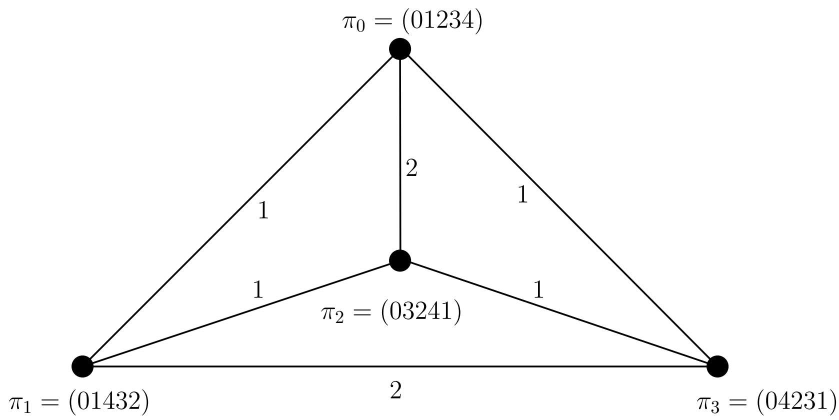

Figure 8: This cannot be the key of a clean drawing of $K_{5,n}$.

We only analyze (that is, derive a contradiction from) (a). The cases (b), (c), and (d) are handled in a totally analogous manner.

Since $Q_{13} = Q_{14} = \emptyset$, it follows that $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ are both equal to the reverse of $\mathrm{rot}_D(1)$; in particular, $\mathrm{rot}_D(3) = \mathrm{rot}_D(4)$. Since $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ and $Q_{12} = \{(a_0a_2), (a_1a_3)\}$, it follows that in $\mathrm{rot}_D(1)$: (i) a_0 and a_1 must be adjacent; (ii) a_2 and a_3 must be adjacent; (iii) a_0 and a_2 must be adjacent; and (iv) a_1 and a_3 must be adjacent. It follows immediately that $\mathrm{rot}_D(1)$ is either $(a_0a_2a_3a_1)$ or $(a_0a_1a_3a_2)$. Since $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ are both the reverse of $\mathrm{rot}_D(1)$, then each of $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ is either $(a_0a_1a_3a_2)$ or $(a_0a_2a_3a_1)$. However, since $Q_{34} = \{(a_0a_3), (a_1a_2)\}$, then one must reach the reverse of $\mathrm{rot}_D(4)$ from $\mathrm{rot}_D(3)$ by applying the transpositions (a_0a_3) and (a_1a_2) (in some order). Since neither of these transpositions may be applied to $(a_0a_1a_3a_2)$ or $(a_0a_2a_3a_1)$, we obtain the required contradiction. □

Proposition 12. *The graph in Figure 8 does not occur as an induced subgraph of the key of any clean drawing of* $K_{5,n}$.

Proof. Since an induced subgraph of a key is a key, it suffices to prove that this graph cannot be the key of a clean drawing of $K_{5,n}$.

Suppose by way of contradiction that the graph in Figure 8 is the key of some clean drawing of $K_{5,n}$. Thus there exists a drawing D of $K_{5,4}$ with white vertices a_0, a_1, a_2, a_3 such that $\mathrm{rot}_D(a_i) = \pi_i$ for $i = 0, 1, 2, 3$, with $\pi_0 = (01234), \pi_1 = (01432), \pi_2 = (03241)$, and $\pi_3 = (04231)$, and $\mathrm{cr}_D(a_0, a_1) = \mathrm{cr}_D(a_1, a_2) = \mathrm{cr}_D(a_2, a_3) = \mathrm{cr}_D(a_0, a_3) = 1$, and $\mathrm{cr}_D(a_0, a_2) = \mathrm{cr}_D(a_1a_3) = 2$. For $i, j \in \{0, 1, 2, 3\}, i \neq j$, let P_{ij} be the antiroute guaranteed by Lemma 10. It is easy to verify that the only antiroute of size 1 from π_0 to π_1 is $\{(01)\}$, and so necessarily $P_{01} = \{(01)\}$. Analogous arguments show that necessarily $P_{23} = \{(01)\}$ and that $P_{12} = P_{03} = \{(23)\}$. It is also readily checked that there are two antiroutes of size 2 from π_0 to π_2, namely $\{(04), (14)\}$ and $\{(24), (34)\}$ (moreover, these are also the two antiroutes of size 2 from π_1 to π_3). Thus each of P_{02} and P_{13} is either $\{(04), (14)\}$ or $\{(24), (34)\}$.

Thus P_{01}, P_{03}, P_{12}, and P_{23} are all determined:

$$P_{01} = P_{23} = \{(01)\}, P_{03} = P_{12} = \{(23)\},$$

and there are four possible combinations of P_{02} and P_{13}:

(a) $P_{02} = P_{13} = \{(04), (14)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{04} = \{(a_0a_2), (a_1a_3)\}$, $Q_{14} = \{(a_0a_2), (a_1a_3)\}, Q_{23} = \{(a_0a_3), (a_1a_2)\}$, and $Q_{02} = Q_{03} = Q_{12} = Q_{13} = Q_{24} = Q_{34} = \emptyset$.

(b) $P_{02} = \{(04), (14)\}$ and $P_{13} = \{(24), (34)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{04} = Q_{14} = \{(a_0a_2)\}$, $Q_{23} = \{(a_0a_3), (a_1a_2)\}, Q_{24} = Q_{34} = \{(a_1a_3)\}$, and $Q_{02} = Q_{03} = Q_{12} = Q_{13} = \emptyset$.

(c) $P_{02} = \{(24), (34)\}$ and $P_{13} = \{(04), (14)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}$, $Q_{04} = Q_{14} = \{(a_1a_3)\}$, $Q_{23} = \{(a_0a_3), (a_1a_2)\}$, $Q_{24} = Q_{34} = \{(a_0a_2)\}$, and $Q_{02} = Q_{03} = Q_{12} = Q_{13} = \emptyset$.

(d) $P_{02} = P_{13} = \{(24), (34)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{23} = \{(a_0a_3), (a_1a_2)\}, Q_{24} = Q_{34} = \{(a_0a_2), (a_1a_3)\}$, and $Q_{02} = Q_{03} = Q_{04} = Q_{12} = Q_{13} = Q_{14} = \emptyset$.

We only analyze (that is, derive a contradiction from) (a). The cases (b), (c), and (d) are handled analogously.

Since $Q_{02} = Q_{03} = Q_{12} = Q_{13} = Q_{24} = Q_{34} = \emptyset$, it follows that $\mathrm{rot}_D(2)$ and $\mathrm{rot}_D(3)$ are equal to each other, and equal to the reverse of each of $\mathrm{rot}_D(0)$, $\mathrm{rot}_D(1)$, and $\mathrm{rot}_D(4)$. Thus $\mathrm{rot}_D(0) = \mathrm{rot}_D(1) = \mathrm{rot}_D(4)$. Since $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ and $Q_{04} = \{(a_0a_2), (a_1a_3)\}$, it follows that in $\mathrm{rot}_D(0)$: (i) a_0 and a_1 must be adjacent; (ii) a_2 and a_3 must be adjacent; (iii) a_0 and a_2 must be adjacent; and (iv) a_1 and a_3 must be adjacent. Thus $\mathrm{rot}_D(0)$ is either $(a_0a_2a_3a_1)$ or $(a_0a_1a_3a_2)$. Now since $Q_{23} = \{(a_0a_3), (a_1a_2)\}$, it follows that in $\mathrm{rot}_D(2)$ (and hence in its reverse $\mathrm{rot}_D(0)$) we have that a_0 is adjacent to a_3, and that a_1 is adjacent to a_2. But this is impossible, since in neither $(a_0a_2a_3a_1)$ nor $(a_0a_1a_3a_2)$ any of these adjacencies occurs. □

8 Properties of cores. I. Forbidden subgraphs

We recall that the *core* of a clean drawing D of $K_{5,n}$ is the subgraph $\Phi^1(D)$ of $\Phi(D)$ that consists of all the vertices of $\Phi(D)$ and the edges of $\Phi(D)$ with label 1. Note that while $\Phi(D)$ is obviously connected, $\Phi^1(D)$ may be disconnected. As all edges of a core are labelled 1, we sometimes omit the reference to the edge labels altogether when working with $\Phi^1(D)$.

Our first result on the structure of cores is a workhorse for the next few sections.

Claim 13. *If π_1, π_2 and π_3 are distinct rotations for white vertices in a drawing of $K_{5,n}$, then there exists at most one rotation π_0 such that there is an antiroute of size 1 from π_0 to each of π_1, π_2, and π_3.*

Proof. By way of contradiction, suppose that there exist distinct vertices $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ and antiroutes of size 1 from π_i to π_1, π_2, and π_3, for $i = 0$ and 4. For $j = 1, 2, 3$ the antiroutes from π_0 and π_4 to π_j induce a route $P_{04}(j)$ of size two from π_0 to π_4. Assume without loss of generality that $\pi_0 = (01234)$.

Suppose that for some j, the transpositions in $P_{04}(j)$ involve (in total) four distinct elements in $\{0, 1, 2, 3, 4\}$. It is immediately checked that this implies that $P_{04}(j)$ is the only route of size 2 from π_0 to π_4, and that this in turn implies that at least two of π_1, π_2, and π_3 are equal to each other, a contradiction. Thus each of $P_{04}(1), P_{04}(2)$, and $P_{04}(3)$ involve fewer than four elements in $\{0, 1, 2, 3, 4\}$.

None of these routes can involve only two elements (since they have size 2, and $\pi_0 \neq \pi_4$), and so we conclude that each of $P_{04}(1), P_{04}(2)$, and $P_{04}(3)$ involve exactly three elements in $\{0, 1, 2, 3, 4\}$. In particular, $P_{04}(1)$ must equal either $\{(k, k+1), (k, k+2)\}$ or $\{(k+1, k+2), (k, k+2)\}$, for some $k \in \{0, 1, 2, 3, 4\}$ (operations are modulo 5; we note that we deviate from the usual notation and separate the elements of a transposition with a comma, for readability purposes).

We derive a contradiction assuming that the first possibility holds; the other possibility is handled analogously. Relabelling $0, 1, 2, 3$, and 4, if needed, we may assume that $P_{04}(1) = \{(01), (02)\}$. Thus π_4 is (03412). It is readily verified that the only routes of size 2 from $\pi_0 = (01234)$ to $\pi_4 = (03412)$ are $P_{04}(1) = \{(01), (02)\}$ and $\{(03), (04)\}$. This in turn immediately implies that the antiroutes of size 1 from π_0 to π_1, π_2, and π_3 are either $\{(01)\}$ or $\{(04)\}$, since the transpositions (02) and (03) cannot be applied to π_0. But then we arrive from π_0 to two elements in $\{\overline{\pi_1}, \overline{\pi_2}, \overline{\pi_3}\}$ by applying the same transposition; that is, $\pi_i = \pi_j$ for some $i, j \in \{1, 2, 3\}$, $i \neq j$, a contradiction. □

Proposition 14. *Let D be an optimal drawing of $K_{5,n}$. Suppose that $\Phi(D)$ is $\{0, 4\}$-free. Then:*

1. *$\Phi^1(D)$ does not contain $K_{2,3}$ as a subgraph.*
2. *$\Phi^1(D)$ has maximum degree at most 3.*
3. *$\Phi^1(D)$ does not contain as a subgraph the graph obtained from K_4 by subdividing exactly once each of the edges in a 3-cycle (see Fig. 9).*

Proof. We start by noting that (1) follows immediately by Claim 13 and Lemma 9.

Suppose now by way of contradiction that $\Phi^1(D)$ has a vertex π_0 of degree at least 4. Thus $\Phi^1(D)$ has distinct vertices $\pi_1, \pi_2, \pi_3, \pi_4$ such that the edge joining π_0 to π_i has label

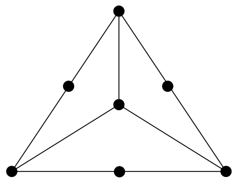

Figure 9: The graph obtained by subdividing exactly once each of the edges in a 3-cycle of K_4.

1, for $i = 1, 2, 3, 4$. Thus, for $i = 1, 2, 3, 4$, there exists an antiroute from π_0 to π_i of size 1. Without loss of generality we may assume $\pi_0 = (01234)$. The five cyclic rotations that have an antiroute of size 1 to π_0 are $(01432), (03214), (03421), (04312)$, and (04231). By performing a relabelling $j \to j+1$ on $\{0, 1, 2, 3, 4\}$ for some $j \in \{0, 1, 2, 3, 4\}$ (with operations modulo 5) if needed (note that the cyclic permutation $\pi_0 = (01234)$ is left unchanged in such a relabelling), we may assume without loss of generality that $\{\pi_1, \pi_2, \pi_3, \pi_4\} = \{(01432), (03214), (03421), (04312)\}$. By exchanging $\pi_1, \pi_2, \pi_3, \pi_4$ if needed, we may assume that $\pi_1 = (01432), \pi_2 = (04312)$, and $\pi_3 = (03421)$.

Since $\Phi(D)$ is $\{0, 4\}$-free, it follows by Proposition 8 that the edge joining π_i to π_j has label 2, for $i, j \in \{1, 2, 3\}, i \neq j$. Thus, for $i, j = 1, 2, 3, i \neq j$, there exists an antiroute from π_i to π_j of size 2. Thus $\Phi(D)$ contains as a subgraph the graph in Figure 7, contradicting Proposition 11. This proves (2).

We finally prove (3). Suppose by way of contradiction that $\Phi^1(D)$ contains as a subgraph the graph obtained from K_4 by subdividing once each of the edges in a 3-cycle (Fig. 9). Let ρ_0 be the "central vertex" in Fig. 9, that is, the only vertex in $\Phi^1(D)$ adjacent to three degree-3 vertices, and let ρ_1, ρ_3, ρ_4 denote these three vertices. An argument similar to the one in the second paragraph of this proof shows the following: if $\rho_0 = (01234)$ is a vertex adjacent to vertices ρ_1, ρ_3, ρ_4 in $\Phi^1(D)$, then we may assume (that is, perhaps after a relabelling of $0, 1, 2, 3, 4$), that $\rho_1 = (01432), \rho_3 = (04231)$, and $\rho_4 = (04312)$. Now let ρ_2 be the vertex adjacent to ρ_1 and ρ_3 in $\Phi^1(D)$. Thus it follows that in $\Phi(D)$, the edges joining ρ_0 and ρ_1, ρ_0 and ρ_3, ρ_1 and ρ_2, and ρ_2 and ρ_3 are labelled 1. By Proposition 8, the edge joining ρ_1 and ρ_3, as well as the edge joining ρ_0 and ρ_2 have even labels, which must be 2 since $\Phi(D)$ is $\{0, 4\}$-free. Now it is easy to verify that the only cyclic permutation other than ρ_0 which has antiroutes of size 1 to both ρ_1 and ρ_3 is (03241). Thus ρ_2 must be (03241). But then the subgraph of $\Phi(D)$ induced by ρ_0, ρ_1, ρ_2, and ρ_3 is isomorphic to the graph in Figure 8, contradicting Proposition 12. $\square$

9 Properties of cores. II. Structural properties

Proposition 15. *Let D be an optimal drawing of $K_{5,n}$, with n even. Suppose that $\Phi(D)$ is $\{0, 4\}$-free. Then:*

1. *$\Phi^1(D)$ is bipartite.*

2. $\Phi^1(D)$ *is connected.*

Proof. Suppose that $C = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{r-1}, \pi_r, \pi_0)$ is an odd cycle in $\Phi^1(D)$. It follows from Proposition 8 that $\pi_0\pi_2$ must have an even label in $\Phi(D)$, since $\pi_0\pi_1$ and $\pi_1\pi_2$ are both labelled 1 in $\Phi(D)$; now this even label must be 2, since $\Phi(D)$ is $\{0,4\}$-free. Similarly, since $\pi_2\pi_3$ and $\pi_3\pi_4$ are also labelled 1 in $\Phi(D)$, then $\pi_2\pi_4$ must also be labelled 2 in $\Phi(D)$. Now since both $\pi_0\pi_2$ and $\pi_2\pi_4$ have label 2 in $\Phi(D)$, it follows that $\pi_0\pi_4$ also has label 2 in $\Phi(D)$. By repeating this argument we find that $\pi_0\pi_j$ must have label 2 in $\Phi(D)$ for every even j. In particular, $\pi_0\pi_r$ must have label 2, contradicting that $\pi_0\pi_r$ is in $\Phi^1(D)$ (that is, that the label of $\pi_0\pi_r$ in $\Phi(D)$ is 1). Thus $\Phi^1(D)$ cannot have an odd cycle. This proves (1).

To prove (2) we assume, by way of contradiction, that $\Phi^1(D)$ is not connected.

We start by observing that $\Phi(D)$ must have at least one edge labelled 1. Indeed, otherwise every edge $\Phi(D)$ has label of at least 2, and so $\mathrm{cr}(D) \geqslant 2\binom{n}{2} = n(n-1) > Z(5,n)$, contradicting the optimality of D.

Thus there exists a component H of $\Phi^1(D)$ with at least 2 vertices. Let U be the set of white vertices whose rotation is a vertex in H, and let V be all the other white vertices. Let $r := |U|$ and $s := |V|$. Note that

$$\begin{aligned} \mathrm{cr}(D) &= \sum_{\substack{a_i,a_j\in U,\\ a_i\neq a_j}} \mathrm{cr}_D(a_i,a_j) + \sum_{\substack{a_i,a_j\in V,\\ a_i\neq a_j}} \mathrm{cr}_D(a_i,a_j) + \sum_{a_i\in U, a_j\in V} \mathrm{cr}_D(a_i,a_j) \\ &\geqslant Z(5,r) + Z(5,s) + 2rs, \end{aligned} \tag{1}$$

since every vertex of U is joined to every vertex of V by an edge with a label 2 or greater.

We claim that, moreover, strict inequality must hold in (1). To see this, first we note that, since H has at least 2 vertices, it follows that there exist white vertices a_k, a_ℓ whose rotations are in H and such that $\mathrm{cr}_D(a_k, a_\ell) = 1$. Since by assumption $\Phi^1(D)$ is not connected, there is a vertex π in $\Phi^1(D)$ not in H. Let a_i be a white vertex such that $\mathrm{rot}_D(a_i) = \pi$. Now $\mathrm{cr}_D(a_k, a_i)$ and $\mathrm{cr}_D(a_\ell, a_i)$ are both at least 2. However, we cannot have $\mathrm{cr}_D(a_k, a_i)$ and $\mathrm{cr}_D(a_\ell, a_i)$ both equal to 2, since then $\mathrm{cr}_D(a_k, a_\ell) = 1$ would contradict Proposition 7. Thus either $\mathrm{cr}_D(a_k, a_i)$ or $\mathrm{cr}_D(a_\ell, a_i)$ is at least 3. This proves that Inequality (1) must be strict, that is,

$$\mathrm{cr}(D) > Z(5,r) + Z(5,s) + 2rs. \tag{2}$$

Independently of the parity of r and s, $\mathrm{cr}(K_{5,r}) \geqslant r(r-2)$ and $\mathrm{cr}(K_{5,s}) \geqslant s(s-2)$. Using (2), we obtain $\mathrm{cr}(D) > r(r-2) + s(s-2) + 2rs = (r+s)(r+s-2) = Z(5, r+s) = Z(5,n)$, contradicting the optimality of D. This finishes the proof of Proposition 15(2). □

10 Properties of cores. III. Minimum degree

Proposition 16. *Let D be an optimal drawing of $K_{5,n}$, with n even. Suppose that $\Phi(D)$ is $\{0,4\}$-free. Let $\pi_0, \pi_1, \pi_2, \pi_3$ be a path in $\Phi^1(D)$. Suppose that in $\Phi^1(D)$, π_1 is the only* *vertex adjacent to both π_0 and π_2, and π_2 is the only vertex adjacent to both π_1 and π_3. Then:*

1. *every vertex in $\Phi^1(D)$ is adjacent (in $\Phi^1(D)$) to a vertex in $\{\pi_0, \pi_1, \pi_2, \pi_3\}$; and*

2. *π_0 and π_3 are adjacent in $\Phi^1(D)$.*

Proof. Let $\pi_0, \pi_1, \ldots, \pi_{r-1}$ be the vertices of $\Phi^1(D)$ (and of $\Phi(D)$ as well). For $i, j \in [r], i \neq j$, let λ_{ij} denote the label of the edge that joins π_i to π_j in $\Phi(D)$. Recall that $\Phi^1(D)$ is bipartite (Proposition 15(1)). Since $\pi_0, \pi_1, \pi_2, \pi_3$ is a path in $\Phi(D)$, it follows that π_0 and π_2 are in the same chromatic class A, and π_1 and π_3 are in the same chromatic class B. Moreover, since $\Phi(D)$ is $\{0,4\}$-free, it follows from Proposition 8 that $\lambda_{ij} = 2$ whenever π_i and π_j belong to the same chromatic class. Thus we have $\lambda_{02} = \lambda_{13} = 2$ and (since $\pi_0, \pi_1, \pi_2, \pi_3$ is a path in $\Phi^1(D)$) $\lambda_{01} = \lambda_{12} = \lambda_{23} = 1$. It follows that the equations of $\mathcal{L}(\Phi(D))$ corresponding to π_0, π_1, π_2, and π_3 are:

$$\begin{array}{rrcrcrcrcrcl}
E_0: & 2t_0 & - & t_1 & & & + & (\lambda_{03}-2)t_3 & + & \sum\limits_{\substack{j\in[r]\\ j>3}} (\lambda_{0j}-2)t_j & = & 0,\\
E_1: & -t_0 & + & 2t_1 & - & t_2 & & & + & \sum\limits_{\substack{j\in[r]\\ j>3}} (\lambda_{1j}-2)t_j & = & 0,\\
E_2: & & - & t_1 & + & 2t_2 & - & t_3 & + & \sum\limits_{\substack{j\in[r]\\ j>3}} (\lambda_{2j}-2)t_j & = & 0,\\
E_3: & (\lambda_{03}-2)t_0 & & & - & t_2 & + & 2t_3 & + & \sum\limits_{\substack{j\in[r]\\ j>3}} (\lambda_{3j}-2)t_j & = & 0,
\end{array}$$

where for simplicity we define $E_i := E(\pi_i, \Phi(D))$ for $i \in \{0,1,2,3\}$. Summing up these four linear equations we obtain

$$(\lambda_{03}-1)t_0 + (\lambda_{03}-1)t_3 + \sum_{j\in[r], j>3} (\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}-8)t_j = 0 \tag{3}$$

We claim all the coefficients in (3) are nonnegative. Since $\lambda_{03} \geqslant 1$, this is true for the coefficients of t_0 and t_3. Since $\Phi(D)$ is $\{0\}$-free, each $\lambda_{ij} \geqslant 1$, for $i, j \in \{0,1,2,3\}$.

Now fix $j > 3$. For the sake of definiteness, suppose π_j is in the same chromatic class of $\Phi^1(D)$ as π_0 and π_2. (The other possibility is completely analogous.) Then π_j is not adjacent to either π_0 and π_2, so $\lambda_{0j} \geqslant 2$ and $\lambda_{2j} \geqslant 2$. The hypothesis about π_2 implies π_j is not adjacent to both π_1 and π_3. Thus, $\lambda_{1j} + \lambda_{3j} \geqslant 1 + 3 = 4$, so $\sum_{i=0}^{3} \lambda_{ij} \geqslant 8$, as required.

Since each t_j is positive and each coefficient is nonnegative, (3) implies that $\lambda_{03} = 1$, and, for each $j > 3$, $\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{3j} = 8$. The former yields Item (2).

For $j > 3$, in the case π_j is in the same chromatic class as π_0 and π_2, we conclude from the above remarks that $\lambda_{0j} = 2, \lambda_{2j} = 2$, and $\lambda_{1j} + \lambda_{3j} = 4$, whence π_j is adjacent to precisely one of π_1 and π_3. If π_j is in the other chromatic class of $\Phi^1(D)$, then π_j is adjacent to precisely one of π_0 and π_2. This proves (1). ☐

Proposition 17. *Let D be an optimal drawing of $K_{5,n}$, with n even. Suppose that $\Phi(D)$ is $\{0,4\}$-free. Then $\Phi^1(D)$ has minimum degree at least 2.*

Proof. By way of contradiction, suppose that $\Phi^1(D)$ has a vertex of degree 0 or 1.

Suppose first that $\Phi^1(D)$ has a vertex of degree 0. Then the connectedness of $\Phi^1(D)$ implies that this is the only vertex in $\Phi^1(D)$ (and, consequently, the only vertex in $\Phi(D)$). Thus all vertices of D have the same rotation. Since if a_i, a_j have the same rotation in a drawing D' then $\mathrm{cr}_{D'}(a_i, a_j) = 4$, it follows that $\mathrm{cr}(D) \geqslant 4\binom{n}{2} = 2n(n-1) > n(n-2) = Z(5,n)$, a contradiction.

Thus we may assume that $\Phi^1(D)$ has a vertex of degree 1.

Let $\pi_0, \pi_1, \ldots, \pi_{m-1}$ denote the vertices of $\Phi^1(D)$. Without any loss of generality we may assume that π_0 has degree 1 in $\Phi^1(D)$. For $i, j \in [m]$, let λ_{ij} denote the label of the edge $\pi_i\pi_j$.

We divide the rest of the proof into two cases.

Case 1. *$\Phi^1(D)$ has a path with 4 vertices starting at π_0.*

Without loss of generality, let $\pi_0, \pi_1, \pi_2, \pi_3$ be this path. Since π_0 is a leaf, it follows that π_1 is the only vertex of $\Phi^1(D)$ adjacent to both π_0 and π_2. We note that then there must be a vertex in $\Phi^1(D)$ (say π_4, without loss of generality) adjacent to both π_1 and π_3, as otherwise it would follow by Proposition 16(2) that π_0 is adjacent to π_3, contradicting that π_0 is a leaf. Thus $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_1)$ is a cycle.

For $i, j \in [5]$, let λ_{ij} denote the label of $\pi_i\pi_j$ in $\Phi(D)$. Since the edges $\pi_0\pi_1$, $\pi_1\pi_2$, $\pi_2\pi_3$, $\pi_3\pi_4$ and $\pi_1\pi_4$ are all in $\Phi^1(D)$, it follows that $\lambda_{01} = \lambda_{12} = \lambda_{23} = \lambda_{34} = \lambda_{14} = 1$. Now since $\Phi(D)$ is $\{0,4\}$-free, using Proposition 8 it follows that $\lambda_{02} = \lambda_{04} = \lambda_{24} = \lambda_{13} = 2$ and (since $\pi_0\pi_3$ is not in $\Phi^1(D)$) that $\lambda_{03} = 3$.

Subcase 1.1. *$\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ are all the vertices in $\Phi^1(D)$.*

In this case the linear system $\mathcal{L}(\Phi(D))$ reads:

$$\begin{array}{rcrcrcrcrcrcl}
E_0 & : & 2t_0 & - & t_1 & & & + & t_3 & & & = & 0,\\
E_1 & : & -t_0 & + & 2t_1 & - & t_2 & & & - & t_4 & = & 0,\\
E_2 & : & & - & t_1 & + & 2t_2 & - & t_3 & & & = & 0,\\
E_3 & : & t_0 & & & - & t_2 & + & 2t_3 & - & t_4 & = & 0,\\
E_4 & : & & - & t_1 & & & - & t_3 & + & 2t_4 & = & 0,
\end{array}$$

where for brevity we let $E_i := E(\pi_i, \Phi(D))$ for $i \in [5]$.

Subtracting E_4 from E_2, we obtain that $t_2 = t_4$. Adding the equations E_0, E_1, E_2, and using $t_2 = t_4$, we obtain $t_0 = 0$. Thus the system $\mathcal{L}(\Phi(D))$ has no positive integral solution, contradicting (by Proposition 6) the optimality of D.

Subcase 1.2. *$\Phi^1(D)$ has a vertex not in $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$.*

The connectedness of $\Phi^1(D)$ implies there is a vertex π_5 not in $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ but adjacent to one of $\pi_0, \ldots, \pi_4$. Since π_0 is a leaf only adjacent to π_1, then $i \neq 0$. Since

π_1 already has degree 3 in $\Phi^1(D)$, it follows from Proposition 14(2) that $i \neq 1$. Thus i is either $2, 3$ or 4. Since the roles of 2 and 4 are symmetric, we may conclude that π_5 is adjacent to either π_2 or to π_3.

Suppose first that π_5 is adjacent to π_3 in $\Phi^1(D)$.

In this case $\lambda_{35} = 1$. Using Proposition 8, that $\Phi(D)$ is $\{0, 4\}$-free, that π_0 is only adjacent to π_1, and Claim 13, we obtain $\lambda_{05} = \lambda_{25} = \lambda_{45} = 2$ and that $\lambda_{15} = 3$. Thus in this case the 0-th and the 5-th equations of the system $\mathcal{L}(\Phi(D))$ read:

$$\begin{array}{ccccccccccccc} E_0 & : & 2t_0 & - & t_1 & + & t_3 & & & + & \sum\limits_{j\in[m],j>5} (\lambda_{0j} - 2)t_j & = & 0. \\ E_5 & : & & + & t_1 & - & t_3 & + & 2t_5 & + & \sum\limits_{j\in[m],j>5} (\lambda_{5j} - 2)t_j & = & 0. \end{array}$$

where for brevity we let $E_i := E(\pi_i, \Phi(D))$ for $i = 0$ and 5.

Adding these equations, we get

$$2t_0 + 2t_5 + \sum_{j\in[m],j>5} (\lambda_{0j} + \lambda_{5j} - 4)t_j = 0. \tag{4}$$

We now argue that $\lambda_{0j} + \lambda_{5j} - 4 \geqslant 0$ whenever $j > 5$. To see this, note that π_0 and π_5 are in the same chromatic class. If π_j is in the same chromatic class, then, since $\Phi(D)$ is $\{0, 4\}$-free, it follows that λ_{0j} and λ_{5j} are both 2, and so $\lambda_{0j} + \lambda_{5j} - 4 \geqslant 0$, as claimed. If π_j is in the other chromatic class, then both λ_{0j} and λ_{5j} are odd. Since π_0 is a leaf whose only adjacent vertex is π_1, it follows that $\lambda_{0j} = 3$. On the other hand, λ_{5j} is either 1 or 3. In particular, $\lambda_{5j} \geqslant 1$, and thus also in this case $\lambda_{0j} + \lambda_{5j} - 4 \geqslant 0$, as claimed. It follows from this observation and (4) that

$$2t_0 + 2t_5 \leqslant 0,$$

and so the system $\mathcal{L}(\Phi(D))$ has no positive integral solution, contradicting Proposition 6.

Suppose finally that π_5 is adjacent to π_2 in $\Phi^1(D)$.

Consider then the path $\pi_0, \pi_1, \pi_2, \pi_5$. Since π_0 is a leaf, it follows that π_1 is the only vertex adjacent to both π_0 and π_2. Now note that π_2 is the only vertex adjacent to both π_1 and π_5, since by Proposition 14(2) π_1 cannot be adjacent to any vertex other than π_0, π_2, and π_4. Thus Proposition 16 applies, and so we must have that π_0 and π_5 are adjacent in $\Phi^1(D)$. But this is impossible, since the only vertex in $\Phi^1(D)$ adjacent to the leaf π_0 is π_1.

CASE 2. *$\Phi^1(D)$ has no path with 4 vertices starting at π_0.*

We recall that π_0 is a leaf in $\Phi^1(D)$. Let π_1 be the vertex adjacent to π_0.

Suppose first that π_0 and π_1 are the only vertices in $\Phi^1(D)$. Then $\mathcal{L}(\Phi(D))$ consists of only two equations, namely $2t_1 - t_0 = 0$ and $2t_0 - t_1 = 0$. This system obviously has no positive integral solutions, contradicting Proposition 6. We may then assume that there is an additional vertex π_2 in $\Phi^1(D)$.

We recall that π_0 is a leaf in $\Phi^1(D)$. Let π_1 be the vertex adjacent to π_0. Since $\Phi^1(D)$ is connected and has no path of four vertices containing π_0, it follows that every other vertex of $\Phi^1(D)$ is adjacent to π_1; that is, $\Phi^1(D)$ is $K_{1,r}$ for some $r \geqslant 1$. Proposition 14(2) implies $r \leqslant 3$.

The equation for π_1 is: $2t_1 - \sum_{j \neq 1} t_j = 0$, while, for $j \neq 1$, the equation for π_j is $2t_j - t_1 = 0$. Using only the last equations, all the t_j other than t_1 are the same value t and $t_1 = 2t$. Now the t_1-equation is $4t - rt = 0$. Since $r \leqslant 3$, this implies the contradiction that $t = 0$. □

11 Properties of cores. IV. Girth and maximum size

Proposition 18. *Let D be an optimal drawing of $K_{5,n}$, with n even. Suppose that $\Phi(D)$ is $\{0,4\}$-free. Then:*

1. *$\Phi^1(D)$ has girth 4.*

2. *If v is a degree-2 vertex in $\Phi^1(D)$, then v is in a 4-cycle in $\Phi^1(D)$.*

3. *$\Phi^1(D)$ has at most 7 vertices.*

Proof. By Proposition 17, the minimum degree of $\Phi^1(D)$ is at least 2. Since $\Phi^1(D)$ is simple and bipartite, it immediately follows that the girth of $\Phi^1(D)$ is a positive number greater than or equal to 4. Let $\pi_0, \pi_1, \pi_2, \pi_3$ be a path in $\Phi^1(D)$. If there is a vertex other than π_1 adjacent to both π_0 and π_2, or a vertex other than π_2 adjacent to both π_1 and π_3, then $\Phi^1(D)$ clearly has a 4-cycle, and we are done. Otherwise, it follows from Proposition 16(2) that π_0 is adjacent to π_3, and so $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ is a 4-cycle. Thus (1) follows.

Now let π_1 be a degree-2 vertex in $\Phi^1(D)$. Since $\Phi^1(D)$ has minimum degree at least 2, using (1) it obviously follows that there exists a path $\pi_0, \pi_1, \pi_2, \pi_3$ in $\Phi^1(D)$. If there is a vertex adjacent to both π_0 and π_2 other than π_1, then π_1 is obviously contained in a 4-cycle. In such a case we are done, so suppose that this is not the case. Since π_1 is only adjacent to π_0 and π_2, using that the degree of π_1 is 2 it follows that no vertex other than π_2 is adjacent to both π_1 and π_3. Thus it follows from Proposition 16(2) that π_0 and π_3 are adjacent in $\Phi^1(D)$. Thus π_1 is contained in the 4-cycle $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$, and (2) follows.

Let $C = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ be a 4-cycle in $\Phi^1(D)$; the existence of C is guaranteed from (1). By Proposition 14(1) $\Phi^1(D)$ contains no subgraph isomorphic to $K_{2,3}$, and so, in $\Phi^1(D)$, no vertex other than π_1 or π_3 is adjacent to both π_0 and π_2, and no vertex other than π_2 or π_0 is adjacent to both π_1 and π_3. Thus Proposition 16 applies. Using Proposition 14(2) and Proposition 16(1), we obtain that $\Phi^1(D)$ has at most 4 vertices other than π_0, π_1, π_2, and π_3; that is, $\Phi^1(D)$ has at most 8 vertices in total; moreover, if $\Phi^1(D)$ has exactly 8 vertices, then every vertex of C has degree 3. Since C was an arbitrary 4-cycle, we have actually proved that if $\Phi^1(D)$ has 8 vertices, then every vertex contained in a 4-cycle must have degree 3. In view of (2), this implies that if $\Phi^1(D)$ has 8 vertices, then it must be cubic.

Now the unique (up to isomorphism) cubic connected bipartite graph on 8 vertices is the 3-cube. Since the 3-cube contains as an induced subgraph the graph in Figure 9, it follows that $\Phi^1(D)$ cannot have exactly 8 vertices. $\square$

12 The possible cores of an antipodal-free optimal drawing

Our goal in this section is to establish Lemma 21, which states that the core of every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic to either a 4-cycle or to the graph $\overline{C}_6$ obtained from the 6-cycle by adding an edge joining two diametrically oposed vertices (see Figure 10).

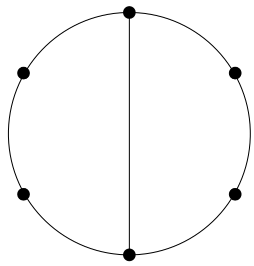

Figure 10: The graph $\overline{C}_6$.

We first show this for the particular case in which $\Phi(D)$ is not only antipodal-free (that is, 0-free), but also 4-free:

Proposition 19. *Let D be an optimal drawing of $K_{5,n}$, with n even. If $\Phi(D)$ is $\{0,4\}$-free, then $\Phi^1(D)$ is isomorphic to the 4-cycle or to $\overline{C}_6$.*

Proof. By Proposition 14(2) and Proposition 17, every vertex of $\Phi^1(D)$ has degree 2 or 3. If $\Phi^1(D)$ has no degree 3 vertices, then $\Phi^1(D)$ is a cycle. Proposition 18(1) implies $\Phi^1(D)$ is a 4-cycle, as required.

Thus, we may assume $\Phi^1(D)$ has a vertex of degree 3. Since all vertices of $\Phi^1(D)$ have degree 2 or 3, there is an even number of vertices of degree 3. Thus, $\Phi^1(D)$ is a subdivision of a cubic graph H and the degree-3 vertices of $\Phi^1(D)$ are its *nodes*.

Since by Proposition 18(3) $\Phi^1(D)$ has at most seven vertices, it has either $2, 4$, or 6 nodes.

Claim 1. $\Phi^1(D)$ *does not have six nodes.*

Proof. Suppose the contrary. Then $\Phi^1(D)$ has at most one vertex of degree 2, so $\Phi^1(D)$ is H with at most one edge subdivided at most once. Since by construction $\Phi^1(D)$ is simple and by Proposition 15(1) it is bipartite, it follows that H is simple. There are only two simple cubic graphs with six vertices, namely $K_{3,3}$ and the triangular prism T_3 (this is the simple cubic graph with a matching whose removal leaves two disjoint 3-cycles).

Because T_3 is not bipartite and no single subdivision makes T_3 into a bipartite graph, H must be $K_{3,3}$. In this case, $\Phi^1(D)$ has $K_{2,3}$ as an induced subgraph, contradicting Proposition 14(1). ■

Claim 2. $\Phi^1(D)$ *does not have four nodes.*

Proof. If H has a loop, then it must be subdivided three times to make $\Phi^1(D)$ simple and bipartite. Thus, H without this loop is a bipartite, simple graph with three vertices of degree 3 and one vertex of degree 1; this is impossible.

If H has a pair of parallel edges, then these must be subdivided in total at least twice to make $\Phi^1(D)$ simple and bipartite. Thus, there is only one such pair of parallel edges. Deleting one of these parallel edges produces a simple graph with two degree-3 vertices and two adjacent degree-2 vertices. This is also not possible.

Therefore, H is simple and so it is K_4. It is readily seen that there are only two ways to turn K_4 into a bipartite graph using at most three edge subdivisions. One way is to subdivide once each of the edges in a 3-cycle of K_4, and the other way is to subdivide (once) two nonadjacent edges (in the latter case, we obtain a graph that has a subgraph isomorphic to $K_{2,3}$). By Proposition 14, neither of these graphs can be the core of D. ■

Claims 1 and 2 show that $\Phi^1(D)$ has precisely two nodes. Thus, H is either an edge with each end incident with a loop or H is three parallel edges. If the former, then each loop must be subdivided at least three times to obtain the simple, bipartite graph $\Phi^1(D)$. But then $\Phi^1(D)$ has at least 8 vertices, contradicting Proposition 18(3). Thus, H is three parallel edges. Because $\Phi^1(D)$ is bipartite, each edge of H is subdivided the same number of times, modulo 2, to produce $\Phi^1(D)$.

If some edge is subdivided three or more times, then $\Phi^1(D)$ has a vertex of degree 2 that is not in any 4-cycle, contradicting Proposition 18(2). If no edge is subdivided twice, then each edge is subdivided exactly once. In this case, $\Phi^1(D)$ is $K_{2,3}$, contradicting Proposition 14(1).

Therefore, some edge of H is subdivided twice. It follows that each edge is subdivided either 0 or 2 times, so two are subdivided twice and one not at all. Thus, $\Phi^1(D)$ is $\overline{C}_6$, as required. □

Proposition 20. *Let D be an antipodal-free, optimal drawing of $K_{5,n}$, with n even. Then $\Phi(D)$ is 4-free.*

Proof. By way of contradiction, suppose that $\Phi(D)$ is not 4-free. Then there exist distinct rotations π, π', and white vertices a_i, a_j such that $\mathrm{rot}_D(a_i) = \pi$ and $\mathrm{rot}_D(a_j) = \pi'$, and $\mathrm{cr}_D(a_i, a_j) = 4$.

Without loss of generality, suppose that $\mathrm{cr}_D(a_i) \leqslant \mathrm{cr}_D(a_j)$. We move, one by one, every vertex a_j with rotation π' very close to a_i, so that in the resulting drawing D' we have $\mathrm{cr}_{D'}(a_j, a_k) = \mathrm{cr}_{D'}(a_i, a_k)$ for every vertex $k \notin \{i, j\}$. It is readily checked that the resulting drawing D' is also optimal, and $\Phi(D')$ has one fewer edge with label 4 than $\Phi(D)$. By repeating this process as many times as needed, we arrive to a drawing D^o such that $\Phi(D^o)$ has exactly one edge with label 4 (if $\Phi(D)$ has exactly one edge with label 4 to begin with, then we let $D^o = D$). Denote by π_0, π_1 the vertices of $\Phi(D^o)$ whose joining edge has label 4.

If we apply the described process one more time to D^o with $\pi = \pi_0$ and $\pi' = \pi_1$, we obtain a $\{0,4\}$-free optimal drawing E of $K_{5,n}$. By Proposition 19, $\Phi^1(E)$ contains a 4-cycle $(\pi_0, \pi_2, \pi_3, \pi_4, \pi_0)$. Now if we apply the process to D^o with $\pi = \pi_1$ and $\pi' = \pi_0$, then we obtain another $\{0,4\}$-free optimal drawing F of $K_{5,n}$. Note that π_2, π_3, π_4 are not affected in the process, and so $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_1)$ is a 4-cycle in $\Phi^1(F)$. Thus it follows that $\Phi^1(D^o)$ has two degree-3 vertices π_2 and π_4, plus the vertices π_0, π_1, π_3, each of which is joined to both π_2 and π_4 with an edge labelled 1. This contradicts Claim 13. □

Lemma 21. *Let D be an antipodal-free, optimal drawing of $K_{5,n}$, with n even. Then $\Phi^1(D)$ is isomorphic either to the 4-cycle or to $\overline{C}_6$.*

Proof. By Proposition 20, $\Phi(D)$ is 4-free. By hypothesis $\Phi(D)$ is also 0-free (since D is antipodal-free), and so $\Phi(D)$ is $\{0,4\}$-free. The lemma then follows by Proposition 19. □

13 Proof of Theorem 1

We need one final result before moving on to the proof of Theorem 1. In the following proposition and its proof, for clarity we sometimes add commas to present a cyclic permutation as (i, j, k, ℓ, m), rather than our usual $(ijk\ell m)$.

Proposition 22. *Let D be a drawing of $K_{5,n}$. Suppose that $\Phi(D)$ is $\{0,4\}$-free, and that $\Phi^1(D)$ is a 4-cycle $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$. Suppose that $\pi_0 = (01234)$. Then there exists an $m \in \{0,1,2,3,4\}$ and a relabelling of $\{0,1,2,3,4\}$ that leaves π_0 invariant, such that (operations are modulo 5):*

- $\pi_2 = (m, m+1, m+3, m+4,\ m+2)$*; and*
- $\{\pi_1, \pi_3\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$.

Proof. The reverse permutation $\overline{\pi_0}$ of π_0 is (43210). Since $\pi_0\pi_1$ and $\pi_0\pi_3$ have label 1 in $\Phi(D)$, it follows that each of π_1 and π_3 is obtained from $\overline{\pi_0}$ by performing one transposition. Thus there exist distinct $k, m \in \{0,1,2,3,4\}$ such that $\{\pi_1, \pi_3\} = \{(k, k+4, k+2, k+3, k+1), (m, m+4, m+2, m+3, m+1)\}$.

Suppose that $k = m+3$. Using a relabelling on $\{0,1,2,3,4\}$ that leaves (01234) invariant, we may assume that $m = 2$ and $k = 0$. Then $\{\pi_1, \pi_3\} = \{(04231), (03214)\}$. Now since the edge joining π_2 to each of π_1 and π_3 in $\Phi(D)$ has label 1, it follows that there are antiroutes of size 1 from π_2 to each of π_1 and π_3. It is easy to check that the only such possibility is that $\pi_2 = (04132)$. Using the relabelling $j \mapsto j-2$ on $\{0,1,2,3,4\}$, we get $\{\pi_0, \pi_1, \pi_2, \pi_3\} = \{(01234), (01432), (03241), (04231)\}$. But then $\Phi(D)$ is the labelled graph in Fig. 8, contradicting Proposition 12. An analogous contradiction is obtained under the assumption $k = m+2$. Thus $k = m+1$ or $k = m+4$.

Suppose that $k = m+1$. Thus $\{\pi_1, \pi_3\} = \{(m+1, m, m+3, m+4, m+2), (m, m+4, m+2, m+3, m+1)\}$. Using the relabelling $j \mapsto j-1$ on $\{0,1,2,3,4\}$ (which obviously leaves (01234) invariant), we obtain $\{\pi_1, \pi_3\} = \{(m, m+4, m+2, m+3, m+1), (m+4, m+3, m+1, m+2, m)\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$, as required. Finally, since the edge joining π_2 to each of π_1 and π_3 in $\Phi(D)$ has label 1, it follows that $\pi_2 = (m, m+1, m+3, m+4, m+2)$. The case $k = m+4$ is handled in a totally analogous manner. $\square$

Proposition 23. *Suppose that D is a drawing of $K_{5,n}$. Suppose that $\Phi(D)$ is $\{0,4\}$-free, and that $\Phi^1(D)$ is isomorphic to $\overline{C}_6$. Let the vertices of $\Phi^1(D)$ be labeled π_0, π_1, π_2, π_3, π_4, π_5, so that $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ and $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$ are 4-cycles. Suppose that $\pi_0 = (01234)$. Then there exists an $m \in \{0, 1, 2, 3, 4\}$ and a relabelling of $\{0, 1, 2, 3, 4\}$ that leaves π_0 invariant, such that (operations are modulo 5):*

- *$\pi_3 = (m, m+4, m+3, m+1, m+2)$;*
- *$\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((m, m+4, m+2, m+3, m+1), (m, m+1, m+3, m+4, m+2)), ((m, m+1, m+4, m+3, m+2), (m, m+2, m+3, m+1, m+4))\}$.*

Proof. By Proposition 22, there exists an $m \in \{0, 1, 2, 3, 4\}$ such that $\pi_2 = (m, m+1, m+3, m+4, m+2)$ and $\{\pi_1, \pi_3\} = A := \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$. By the same proposition, there exists a $k \in \{0, 1, 2, 3, 4\}$ such that $\pi_5 = (k, k+1, k+3, k+4, k+2)$ and $\{\pi_3, \pi_4\} = B := \{(k, k+4, k+2, k+3, k+1), (k, k+4, k+3, k+1, k+2)\}$.

Since $\pi_2 \neq \pi_5$, it follows that $m \neq k$. Thus k is either $m+1, m+2, m+3$, or $m+4$. Note that if $k = m+2$ or $k = m+3$ then $A \cap B = \emptyset$, which contradicts that $\{\pi_3\} = A \cap B$. Thus k is either $m+1$ or $m+4$.

We work out the details for the case $k = m+1$; the case $k = m+4$ is handled in a totally analogous manner. Since $\{\pi_3\} = A \cap B$, it follows that $\pi_3 = (m, m+4, m+2, m+3, m+1) = (m+1, m, m+4, m+2, m+3)$. Therefore $\pi_1 = (m, m+4, m+3, m+1, m+2) = (m+1, m+2, m, m+4, m+3)$, $\pi_2 = (m, m+1, m+3, m+4, m+2) = (m+1, m+3, m+4, m+2, m)$, $\pi_4 = (m+1, m, m+3, m+4, m+2)$, and $\pi_5 = (m+1, m+2, m+4, m, m+3)$. Using the relabelling $j \to j-1$ on $\{0, 1, 2, 3, 4\}$ (which leaves (01234) invariant), we obtain $\pi_1 = (m, m+1, m+4, m+3, m+2)$, $\pi_2 = (m, m+2, m+3, m+1, m+4)$, $\pi_3 = (m, m+4, m+3, m+1, m+2)$ $\pi_4 = (m, m+4, m+2, m+3, m+1)$, and $\pi_5 = (m, m+1, m+3, m+4, m+2)$. $\square$

Proof of Theorem 1. Let D be an antipodal-free drawing of $K_{5,n}$, with n even. In view of Proposition 3 (see Remark 4), we may assume that D is clean, so that $\Phi(D)$ and $\Phi^1(D)$ are well-defined.

In view of Lemma 21, $\Phi^1(D)$ is isomorphic either to the 4-cycle or to $\overline{C}_6$.

Case 1. *$\Phi(D)$ is isomorphic to $\overline{C}_6$.*

In this case $\Phi(D)$ has 6 vertices, which we label $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$, so that $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ and $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$ are 4-cycles. For $i, j \in \{0, 1, 2, 3, 4, 5\}$, $i \neq j$, let λ_{ij} be the label of the edge $\pi_i\pi_j$. Since $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ and $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$ are 4-cycles in $\Phi^1(D)$, it follows that all the edges in these 4-cycles have label 1 in $\Phi(D)$; that is, $\lambda_{01} = \lambda_{12} = \lambda_{23} = \lambda_{03} = \lambda_{04} = \lambda_{45} = \lambda_{35} = 1$. By Proposition 8, λ_{02} is even. Since

$\Phi(D)$ is antipodal-free, and (by Property (2) of a clean drawing) $\lambda_{ij} \leqslant 4$ for all i, j, it follows that λ_{02} is either 2 or 4. By Proposition 20 $\Phi(D)$ is 4-free, hence $\lambda_{02} = 2$. The same argument shows that $\lambda_{05} = \lambda_{13} = \lambda_{14} = \lambda_{25} = \lambda_{34} = 2$. Since $\lambda_{35} = 1$ and $\lambda_{13} = 2$, by Proposition 8, λ_{15} is odd. If $\lambda_{15} = 1$, then $\{\pi_0, \pi_5\} \cup \{\pi_1, \pi_2, \pi_4\}$ is a $K_{2,3}$ in $\Phi^1(D)$, contradicting Proposition 8; thus $\lambda_{15} = 3$. An analogous argument shows that $\lambda_{24} = 3$.

The linear system $\mathcal{L}(\Phi(D))$ associated to $\Phi(D)$ (see Definition 5) is then:

$$\begin{array}{lcrcrcrcrcrcrcl}
E_0 & : & 2t_0 & - & t_1 & & & - & t_3 & - & t_4 & & & = & 0. \\
E_1 & : & -t_0 & + & 2t_1 & - & t_2 & & & & & + & t_5 & = & 0. \\
E_2 & : & & - & t_1 & + & 2t_2 & - & t_3 & + & t_4 & & & = & 0. \\
E_3 & : & -t_0 & & & - & t_2 & + & 2t_3 & & & - & t_5 & = & 0. \\
E_4 & : & -t_0 & & & + & t_2 & & & + & 2t_4 & - & t_5 & = & 0. \\
E_5 & : & & & t_1 & & & - & t_3 & - & t_4 & + & 2t_5 & = & 0.
\end{array} \tag{5}$$

It is straightforward to check that if $(t_0, t_1, t_2, t_3, t_4, t_5)$ is a positive solution to this system, then $t_1 = t_2$, $t_4 = t_5$ and $t_0 = t_3 = t_1 + t_4$. By Proposition 6, this implies that $n \equiv 0 \pmod 4$. This proves (1).

We have thus proved that the white vertices of D are partitioned into 6 classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$, such that $|\mathcal{C}_1| = |\mathcal{C}_2|$, $|\mathcal{C}_4| = |\mathcal{C}_5|$, $|\mathcal{C}_0| = |\mathcal{C}_3| = |\mathcal{C}_1| + |\mathcal{C}_4|$, and such that for $i = 0, 1, 2, 3, 4, 5$, each vertex in $\mathcal{C}_i$ has rotation π_i. Let $r := |\mathcal{C}_1|$ and $s := |\mathcal{C}_4|$, so that $|\mathcal{C}_2| = r$, $|\mathcal{C}_5| = s$, and $|\mathcal{C}_0| = |\mathcal{C}_3| = r + s$. Note that $4(r + s) = n$.

If necessary, relabel $\{0, 1, 2, 3, 4\}$ so that $\pi_0 = (01234)$. By Proposition 23, perhaps after a further relabelling of $\{0, 1, 2, 3, 4\}$ (that leaves π_0 invariant), there exists an $m \in \{0, 1, 2, 3, 4\}$ such that $\pi_3 = (m, m+4, m+3, m+1, m+2)$, and $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((m, m+4, m+2, m+3, m+1), (m, m+1, m+3, m+4, m+2)), ((m, m+1, m+4, m+3, m+2), (m, m+2, m+3, m+1, m+4))\}$. Now perform the further relabelling $j \mapsto j - m$. After this relabelling (which again leaves π_0 invariant), we have $\pi_3 = (04312)$ and $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((04231), (01342)), ((01432), (02314))\}$.

We have thus proved that (perhaps after a relabelling of $\{0, 1, 2, 3, 4\}$) there exist integers r, s such that D has $r + s$ vertices with rotation $\pi_0 = (01234)$, r vertices with rotation $\pi_1 = (04231)$, r vertices with rotation $\pi_2 = (01342)$, $r + s$ vertices with rotation $\pi_3 = (04312)$, s vertices with rotation $\pi_4 = (01432)$, and s vertices with rotation $\pi_5 = (02314)$. That is, D is isomorphic to the drawing $D_{r,s}$ from Section 3.

Case 2. $\Phi(D)$ *is isomorphic to the* 4*-cycle.*

In this case $\Phi(D)$ has 4 vertices, which we label $\rho_0, \rho_1, \rho_2, \rho_3$, so that $(\rho_0, \rho_1, \rho_2, \rho_3, \rho_0)$ is a cycle. The linear system $\mathcal{L}(\Phi(D))$ associated to $\Phi(D)$ is the one that results by taking $t_4 = t_5 = 0$ in the linear system (5), and omitting the equations E_4 and E_5.

It is straightforward to check that if (t_0, t_1, t_2, t_3) is a solution to this system, then $t_0 = t_1 = t_2 = t_3$. By Proposition 6, this implies that $n \equiv 0 \pmod 4$. This proves (1).

Thus the white vertices of D are partitioned into 4 classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, each of size $n/4$, so that each vertex in class $\mathcal{C}_i$ has rotation ρ_i.

Label the vertices $0, 1, 2, 3, 4$ so that $\rho_0 = (01234)$. Then, by Proposition 22, possibly after a relabelling of $\{0, 1, 2, 3, 4\}$ that leaves ρ_0 invariant, there is an $m \in \{0, 1, 2, 3, 4\}$

such that $\rho_2 = (m, m+1, m+3, m+4,\ m+2)$, and $\{\rho_1, \rho_3\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$. Now we perform the relabelling $j \mapsto j - m$ on $\{0, 1, 2, 3, 4\}$ (which obviously leaves ρ_0 invariant), we obtain $\rho_2 = (01342)$ and $\{\rho_1, \rho_3\} = \{(04231), (04312)\}$.

We have thus proved that D has r vertices with rotation (01234), r vertices with rotation (01342), r vertices with rotation (04231), and r vertices with rotation (04312). That is, D is isomorphic to the drawing $D_{r,0}$ from Section 3, with $r = n/4$. $\square$

14 Concluding remarks

A reviewer of an earlier version of this paper asked the following:

Question 24. Under what conditions can we superimpose Zarankiewicz drawings of $K_{5,t_1}, K_{5,t_2}, \ldots, K_{5,t_k}$, and a $D_{r,s}$, to obtain an optimal drawing of $K_{5,n}$ with $n = t_1 + t_2 \cdots + t_k + 4(r+s)$?

A natural starting point to tackle this question is the following (in principle, weaker) problem:

Question 25. Under what conditions can we superimpose Zarankiewicz drawings of $K_{5,t_1}, K_{5,t_2}, \ldots, K_{5,t_k}$, to obtain an optimal drawing of $K_{5,n}$ with $n = t_1 + t_2 \cdots + t_k$?

Suppose we are given Zarankiewicz drawings $D_1, D_2, \ldots, D_k$ of $K_{5,t_1}, K_{5,t_2}, \ldots, K_{5,t_k}$, respectively. Since each D_i is a Zarankiewicz drawing, the rotation set of D_i consist of only two cyclic permutations π_i, ρ_i, which are reverse permutations of each other. We may assume without any loss of generality that for all distinct $i, j \in [k]$, $\{\pi_i, \rho_i\} \cap \{\pi_j, \rho_j\} = \emptyset$. Now in order to superimpose $D_1, D_2, \ldots, D_k$ successfully, there must exist a drawing D of $K_{5,k}$ with white vertices $a_1, a_2, \ldots, a_k$ such that:

(a) for each $i \in [k]$, the white vertex a_i has rotation π_i;

(b) for each $i, j \in [k], i \neq j$, the number of crossings in D between the star with center a_i and the star with center a_j equals the antidistance between π_i and π_j.

We do not have a characterization of which collections $\{\pi_1, \pi_2, \ldots, \pi_k\}$ have a drawing satisfying these conditions. So far we have found that whenever $k \leqslant 3$, such a drawing exists (although we do not have a proof of this apparently simple statement). On the other hand, there are examples with $k = 4$ for which no such drawing exists; this was first observed by Kleitman [6]. We are nowhere near a complete characterization which, as we have observed, would be required to give an answer to Questions 24 and 25.

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