On sets with few intersection numbers in finite projective and affine spaces

Nicola Durante

Dipartimento di Matematica e appl. "R. Caccioppoli"
Universitá di Napoli "Federico II"
Naples, Italy

ndurate@unina.it

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Abstract

In this paper we study sets X of points of both affine and projective spaces over the Galois field $\mathrm{GF}(q)$ such that every line of the geometry that is neither contained in X nor disjoint from X meets the set X in a constant number of points and we determine all such sets. This study has its main motivation in connection with a recent study of neighbour transitive codes in Johnson graphs by Liebler and Praeger [*Designs, Codes and Crypt.*, 2014]. We prove that, up to complements, in $\mathrm{PG}(n, q)$ such a set X is either a subspace or $n = 2, q$ is even and X is a maximal arc of degree m. In $\mathrm{AG}(n, q)$ we show that X is either the union of parallel hyperplanes or a cylinder with base a maximal arc of degree m (or the complement of a maximal arc) or a cylinder with base the projection of a quadric. Finally we show that in the affine case there are examples (different from subspaces or their complements) in $\mathrm{AG}(n, 4)$ and in $\mathrm{AG}(n, 16)$ giving new neighbour transitive codes in Johnson graphs.

1 Introduction

1.1 Preliminaries

Let $\mathrm{AG}(n, q)$, q power of a prime number, be the affine space of dimension n ($n \geqslant 2$) over the Galois field $\mathrm{GF}(q)$, let Σ_∞ be the $(n-1)$-dimensional projective space of its points at infinity and let $\mathrm{PG}(n, q) = \mathrm{AG}(n, q) \cup \Sigma_\infty$ its projective completion, the projective space of dimension n over $\mathrm{GF}(q)$.

Let $m_1 < m_2 < \ldots < m_t$ be non-negative integers. A set X of points of $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$) is of *class* $[m_1, m_2, \ldots, m_t]_1$ if it intersects every line of $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$) either in m_1 or m_2 or $\ldots$ or m_t points. If X is of class $[m_1, m_2, \ldots, m_t]_1$ and for every $i \in \{1, \ldots, t\}$ there exists a line L meeting X exactly in m_i points, then X is of *type*

$(m_1, m_2, \ldots, m_t)_1$. If a line L is either contained in X (*line of* X) or it is disjoint from X (*line external* to X) we will say that $|X \cap L|$ is an *improper* intersection number for both the affine and projective cases; all the others will be *proper* intersection numbers. If m_i is a proper intersection number in the sequel we will call m_i-*secant* a line meeting X exactly in m_i points.

We are interested in subsets X of points of $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$) with exactly one proper intersection number. Such sets either have an external line or contain a line (see Proposition 8). For such sets X we simply say that lines meeting X in the unique proper intersection number of points are *secant*. As usual with a -1-dimensional projective subspace of $\mathrm{PG}(n, q)$ we mean the empty set $\emptyset$.

In the last forty years many classes of pointsets, specially with few intersection numbers mainly in finite projective spaces (but sometimes also in finite affine spaces) have been studied (see e.g. [8] for a list of results).

Among the known sets with few intersection numbers some with external lines or containing lines in $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$) are: subspaces, quadrics, Hermitian varieties, subgeometries, maximal arcs in $\mathrm{PG}(2, q)$ (or $\mathrm{AG}(2, q)$), q even, internal points of conics in $\mathrm{PG}(2, q)$ (or $\mathrm{AG}(n, q)$), q odd or their complements (see e.g. [8],[14],[18], for the definitions of these sets). However almost all these examples have at least two proper intersection numbers. The only ones listed above with just one proper intersection number are subspaces and their complements or maximal arcs and their complements.

We give here the definition of maximal arcs since they will be used many times in the sequel.

Definition 1. A set X of type $(0, m)_1$ in $\mathrm{PG}(2, q)$ is known as a *maximal arc of degree* m or an m-*maximal arc* (*hyperoval* if $m = 2$) (see *[3], [27]*).

Let X be an m-maximal arc of $\mathrm{PG}(2, q)$. If $m = 1$, then X is a point and if $m = q$, then $X = \mathrm{AG}(2, q)$. If $1 < m < q$, then counting $|X|$ via the lines on a point of X and via the lines on a point not of X we get $|X| = (q + 1)(m - 1) + 1 = am$ for some integer a; hence $m|q$. In the sequel we will also use the term *proper maximal arc* for an m-maximal arc with $1 < m < q$. From a result of Ball, Blokhuis and Mazzocca [2] a proper maximal arc of $\mathrm{PG}(2, q)$ of degree m can exists only for q even. There are examples of proper maximal arcs in $\mathrm{PG}(2, q), q$ even (see [11], [35], [20]).

We mention here some classification results on sets with few intersection numbers either with an external line or containing a line. Before we give the definition of singular points for a set X of points of $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$).

Definition 2. Let X be a set of points of $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$). A point p of X is *singular* if every line through p meets X either just in p or it is contained in X. The set X is *singular* if it does contain singular points otherwise it is *nonsingular*.

The first result we recall is the celebrated Theorem by Beniamino Segre characterizing (nonsingular) conics of $\mathrm{PG}(2, q), q$ odd.

Theorem 3. [26] *Let* X *be a set of type* $(0, 1, 2)_1$ *in* $\mathrm{PG}(2, q), q$ *odd. If* $|X| = q + 1$*, then* X *is a* (*nonsingular*) *conic.*

Another characterization result, due to Ueberberg, regards Baer subplanes, unitals and m-maximal arcs in $\mathrm{PG}(2,q)$.

Theorem 4. [37] *Let X be a set of class $[0,1,m]_1$ in $\mathrm{PG}(2,q)$. If $m \geqslant \sqrt{q}+1$, then X is one of the following:*

- *a set of x collinear points with $x \in \{0,1,m\}$;*
- *a Baer subplane, a unital or an m-maximal arc.*

The hypothesis $m \geqslant \sqrt{q}+1$ is necessary in the previous theorem since there are counterexamples to the previous list for $m < \sqrt{q}+1$ (see [4]). The same paper also contains the following characterization of Baer subgeometries and affine subspaces in projective spaces.

Theorem 5. [37] *Let X be a set of class $[0,1,m]_1$ in $\mathrm{PG}(n,q)$, $n \geqslant 3$. If $m \geqslant \sqrt{q}+1$ and X spans the full space, then X is either a Baer subgeometry or an affine subspace.*

Finally we mention a characterization result for the set of internal points of a (nonsingular) conic of $\mathrm{PG}(2,q), q$ odd due to De Feyter and De Clerck:

Theorem 6. [9] *Let X be a set of type $(0, \frac{q-1}{2}, \frac{q+1}{2})_1$ in $\mathrm{PG}(2,q)$, q odd, then X is the set of internal points of a (nonsingular) conic.*

Note that none of the sets mentioned up to now, except subspaces and their complements, contain both a line and have an external line and all but m-maximal arcs and internal points of a conic have at least a 1-secant line. We will give examples of sets of points with one proper intersection number, different from 1, containing lines and having external lines in affine spaces.

Finally we give some definitions needed in the paper. Let Π be an affine plane contained in $\mathrm{AG}(n,q)$, $n \geqslant 3$ and denote by L_∞ its line at infinity that is contained in Σ_∞, the hyperplane at infinity of $\mathrm{AG}(n,q)$. Let $\mathcal{V}_\infty \subset \Sigma_\infty$ be an $(n-3)$-dimensional projective subspace skew with L_∞ and let M be a subset of points of Π. The *cone* with vertex $\mathcal{V}_\infty$ and base M is the subset of $\mathrm{PG}(n,q) = \mathrm{AG}(n,q) \cup \Sigma_\infty$ that is the union of the $(n-2)$-dimensional projective subspaces of $\mathrm{PG}(n,q)$ spanned by $\mathcal{V}_\infty$ and a point $p \in M$. In the sequel it will be denoted by $Cone(\mathcal{V}_\infty, M)$. The *cylinder* with vertex $\mathcal{V}_\infty$ and base M is the subset of $\mathrm{AG}(n,q)$ given by $\mathrm{AG}(n,q) \cap Cone(\mathcal{V}_\infty, M) = Cone(\mathcal{V}_\infty, M) \setminus \mathcal{V}_\infty$. In the sequel it will be denoted by $Cyl(\mathcal{V}_\infty, M)$.

If Q is a nonsingular quadric of $\mathrm{PG}(n,q)$ we call *rank* of Q one plus the maximum dimension of subspaces contained in Q. If n is odd, then Q is either an *elliptic* quadric or a *hyperbolic* quadric according that the rank of Q is either $\frac{n-1}{2}$ or $\frac{n+1}{2}$. If n is even, then the rank of Q is necessarily $\frac{n}{2}$, Q is a *parabolic* quadric. and there exists a special point $\mathcal{N}$ in $\mathrm{PG}(n,q) \setminus Q$, called the *nucleus* of Q, with the property that every line on $\mathcal{N}$ meets Q in exactly one point (see [14], [18]).

1.2 Motivations

In the sequel if X is a set of points of $\mathrm{PG}(n,q)$ (or $\mathrm{AG}(n,q)$), $n \geqslant 2$, we will denote by G the group $P\Gamma L(n,q)$ (or $A\Gamma L(n,q)$) and by G_X the stabiliser in G of the set X. We will denote by X^c the complement of X in $\mathrm{PG}(n,q)$ (or $\mathrm{AG}(n,q)$). We will be interested in point sets X of both $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$ satisfying the following:
Property $(*)$: *G_X is transitive on the set $X \times X^c$.*

Here is the main motivation for studying such sets of points: In connection with a study of neighbour transitive codes in Johnson graphs [19] by Liebler and Praeger, subsets X of points of $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$, for $n \geqslant 2$, with at most three possible line-intersection sizes arise, such that Property $(*)$ holds [19, Section 5.3]. With the notations established above these are sets of class $[0,m,q+1]_1$ for $\mathrm{PG}(n,q)$ and class $[0,m,q]_1$ for $\mathrm{AG}(n,q)$. The examples that were given in [19, Sections 6,7] are: subspaces and their complements in both (projective and affine) spaces and the (regular) hyperoval and its complement in $\mathrm{AG}(2,4)$. The analysis in [19] showed that, apart from these examples, the value of m was restricted to $m \in \{2, \sqrt{q}+1\}$ in the projective case, and in the affine case the pair (q,m) was restricted to one of the two: either $(4,2)$ or $(16,4)$. Classification in these exceptional cases was left open, and at her plenary lecture in Ferrara at the Conference of Finite Geometry in honor of Frank De Clerck, Praeger asked what was known about subsets of class $[0,m,q+1]_1$ for projective geometry and of class $[0,m,q]_1$ for affine geometry, regardless of the symmetry restrictions.

Question (C. Praeger 2012) *Is it possible, for $n \geqslant 2$, to characterise the subsets of $PG(n,q)$ of class $[0,m,q+1]_1$ and the subsets of $AG(n,q)$ of class $[0,m,q]_1$? In particular for what values of m do they arise?*

In this paper we answer this question and moreover solve the open cases in [19]. We will show that there are no examples for the open cases $m \in \{2, \sqrt{q}+1\}$ in projective spaces (see Theorem 21) while there are examples for affine spaces for both $(q,m) = (16,4)$ and $(q,m) = (4,2)$ (see Theorem 22). In the sequel we will make use of the following proposition whose easy proof is omitted here.

Proposition 7. *A set X of points of* $\mathrm{PG}(n,q)$(*or* $\mathrm{AG}(n,q)$) *satisfies Property $(*)$ if and only if*

(1) G_X is transitive on X;

(2) $G_{X,p}$, the stabiliser in G_X of a point p of X, is transitive on X^c.

We also note that:

Remark 2. *Every subset of* $\mathrm{AG}(n,q)$, *$q=2$, is of class* $[0,1,q]_1$ *and it is a subspace.*

Remark 3. *If X is a set of points of* $\mathrm{PG}(n,q)$ (*or* $\mathrm{AG}(n,q)$) *with one proper intersection number, then also X^c has this property. Moreover as explained in* [19, 1.4 (d)] *the set X satisfies Property $(*)$ if and only if X^c satisfies Property $(*)$.*

2 Sets with two intersection numbers in projective and affine spaces

In this section we will study sets with two intersection numbers in both projective and affine spaces. We will distinguish the two cases affine and projective. Before doing so we first mention the following easy:

Proposition 8. *Let X be a nonempty set of type $(m)_1$ in $\mathrm{PG}(n,q)$ (or $\mathrm{AG}(n,q)$), then $X = \mathrm{PG}(n,q)$ (or $\mathrm{AG}(n,q)$).*

Proof. If X is a nonempty set of type $(m)_1$ of $\mathrm{PG}(n,q)$ (or $\mathrm{AG}(n,q)$), then counting $|X|$ via the lines on a point of X we obtain $|X| = \frac{q^n-1}{q-1}(m-1)+1$ while counting $|X|$ via the lines on a point not on X we have $|X| = \frac{q^n-1}{q-1}m$, so a contradiction. Hence the only possibility is that X is the full point set of $\mathrm{PG}(n,q)$ (or $\mathrm{AG}(n,q)$). $\square$

2.1 The projective case

From Proposition 8 it follows that for a proper nonempty subset X of points of $\mathrm{PG}(n,q)$ two is the minimum number of intersection sizes w.r.t. lines. Sets with two intersection numbers in $\mathrm{PG}(n,q)$ have been studied by many different authors mainly for $n = 2$ because of the connection of such sets with strongly regular graphs, 2-weight codes etc. (see [5]).

For our purposes we need to study sets of type $(0,m)_1$ and sets of type $(m,q+1)_1$ in projective spaces of order q since there are no sets of type $(0,q+1)_1$. Indeed if X is a set of type $(0,q+1)_1$ in $\mathrm{PG}(n,q)$, then let L be a line contained in X and let p be a point of L. All lines on p meet X and hence are contained in X. It follows that $X = \mathrm{PG}(n,q)$, but then there is no line missing X, a contradiction.

Proposition 9. *If X is a set of type $(0,m)_1$ in $\mathrm{PG}(n,q)$, then one of the following occurs:*

- *X is a point $(m = 1)$;*
- *$X = \mathrm{AG}(n,q)$ $(m = q)$;*
- *X is a proper m-maximal arc in $\mathrm{PG}(2,q)$, q even, $m|q$ $(1 < m < q)$.*

Proof. Let X be a set of type $(0,m)_1$ in $\mathrm{PG}(n,q)$. If $n = 2$, then X is an m-maximal arc. Suppose next that $n \geqslant 3$. Every plane meets X either in the empty set or in a maximal arc of degree m. Let L be an m-secant line to X and let L_0 be an external line to X. Counting $|X|$ via the planes on L we get $|X| = \frac{q^{n-1}-1}{q-1}(qm-q)+m$ while counting $|X|$ via the planes on L_0 we get $|X| = b(qm-q+m)$ for some integer b. It follows that $qm-q+m$ divides $(\frac{q^{n-1}-1}{q-1}-1)m$. If $n = 3$, then $qm-q+m$ divides qm. Hence either $m = q$ and $|X| = (q+1)(q^2-q)+q = q^3$ so $X = \mathrm{AG}(3,q)$ or $m < q$ and hence we have that $q-m \geqslant \frac{qm}{2}$ so $m \leqslant \frac{2q}{q+2}$ hence $m = 1$ and X is a point. It follows that in $\mathrm{PG}(n,q), n \geqslant 3$ the only sets of type $(0,m)_1$ are either one point or $\mathrm{AG}(n,q)$. Indeed if X

is a set of type $(0,m)_1$ in $\mathrm{PG}(n,q), n \geqslant 4$, then in every threedimensional subspace S the set $X_S = X \cap S$ is a set of class $[0,m]_1$ and, from the previous part, we have either $m=1$ and X_S is a point or $m=q$ and X_S is $\mathrm{AG}(3,q)$ or $X_S = \emptyset$. Suppose X is not a point, hence $m \neq 1$, then there exists a threedimensional subspace S such that X_S is $\mathrm{AG}(3,q)$, so $m=q$ hence X^c is a set of type $(1,q+1)_1$ in $\mathrm{PG}(n,q)$, so X^c is a hyperplane. This gives that X is $\mathrm{AG}(n,q)$. □

A proof of the previous Proposition can be found (in italian) also in [29].

If X is a set of type $(m,q+1)_1$ in $\mathrm{PG}(n,q)$, then its complementary set $X^c = \mathrm{PG}(n,q) \setminus X$ is of type $(0,q+1-m)_1$ and hence, from the previous proposition we have:

Corollary 10. *If X is a set of type $(m,q+1)_1$ in $\mathrm{PG}(n,q)$, then one of the following occurs:*

- *X is the complement of $\mathrm{AG}(n,q)$, so a hyperplane $(m=1)$;*
- *X is the complement of a point $(m=q)$;*
- *X is the complement of a proper $(q+1-m)$-maximal arc in $\mathrm{PG}(2,q)$, q is even $(q+1-m)|q$ $(1<m<q)$.*

2.2 The affine case

From Proposition 8 it follows that for a proper nonempty subset X of points of $\mathrm{AG}(n,q)$ two is the minimum number of intersection sizes w.r.t. lines. Sets with two intersection numbers in $\mathrm{AG}(n,q)$ have been studied in [34] where it is proved that, if the two intersection numbers are both proper, then they can only occur for q an odd square, $|X| = \frac{q^n \pm \sqrt{q^n}}{2}$ and intersection sizes $\frac{q-\sqrt{q}}{2}$, $\frac{q+\sqrt{q}}{2}$. There are examples for such sets in $\mathrm{AG}(2,9)$. Indeed Penttila and Royle have found in [21] two different sets of type $(3,6)_1$ in $\mathrm{AG}(2,9)$ with $|X| = 36$ and automorphism groups of order 2 and 3 (plus the complement of such sets). In [24] Rodgers constructs a set of type $(36,45)$ in $\mathrm{AG}(2,81)$ (plus the complement).

For our purposes we need to study sets of type $(0,m)_1$ and sets of type $(m,q)_1$ in $\mathrm{AG}(n,q)$ since there are no sets of type $(0,q)_1$ in $\mathrm{AG}(n,q)$. Indeed if X is a set of type $(0,q)_1$ in $\mathrm{AG}(n,q)$, then let L be a line contained in X and let p be a point of L. All lines on p meet X and hence are contained in X. It follows that $X = \mathrm{AG}(n,q)$, but then there is no line missing X, a contradiction. A set of type $(0,m)_1$ in $\mathrm{AG}(2,q) = \mathrm{PG}(2,q) \setminus L_\infty$ is a maximal arc of degree m in $\mathrm{PG}(2,q)$ having L_∞ as external line. Hence, either $m=1$ and X is a point or $1<m<q$, $m|q$ and X is a maximal arc of degree m in $\mathrm{AG}(2,q)$, q even. In $\mathrm{AG}(3,q)$ if X is a set of type $(0,m)_1$, in the same way as in $\mathrm{PG}(3,q)$ we get that $m=1$ and X is a point. It follows that in $\mathrm{AG}(n,q), n \geqslant 3$ the only sets of type $(0,m)_1$ are the single points. Hence we have the following:

Proposition 11. *If X is a set of type $(0,m)_1$ in $\mathrm{AG}(n,q)$, then one of the following occurs:*

- *X is a point $(m=1)$;*

- *X is a proper m-maximal arc in* $\mathrm{AG}(2,q)$*, q even, $m|q$* $(1<m<q)$.

If X is a set of type $(m,q)_1$ in $\mathrm{AG}(n,q)$, then its complementary set $X^c = \mathrm{AG}(n,q) \setminus X$ is of type $(0, q-m)_1$ hence, from previous proposition, we have:

Corollary 12. *If X is a set of type $(m,q)_1$ in* $\mathrm{AG}(n,q)$*, then one of the following occurs:*

- *X is the complement of a point in* $\mathrm{AG}(n,q)$ $(m=q-1)$*;*
- *X is the complement of a proper $(q-m)$-maximal arc in* $\mathrm{AG}(2,q)$*, q even, $(q-m)|q$* $(1<m<q-1)$.

3 Sets with three intersection numbers in projective and affine spaces

3.1 The projective case

In this section we will study sets of type $(0,m,q+1)_1$ in $\mathrm{PG}(n,q)$.

Proposition 13. *If X is a set of type $(0,m,q+1)_1$ in* $\mathrm{PG}(n,q)$*, then:*

- *X is a subspace of dimension i with $1 \leqslant i \leqslant n-2$ $(m=1)$ or its complement* $(m=q)$.

Proof. Let L be a line of X and let L_0 be an external line to X. For every plane $\overline{\Pi}$ containing the line L the set $X_{\overline{\Pi}} = X \cap \overline{\Pi}$ is a set of class $[m, q+1]_1$ of $\overline{\Pi}$. Hence either $\overline{\Pi}$ is contained in X or $X_{\overline{\Pi}}$ is the complement of a $(q+1-m)$-maximal arc. So one of the following occurs for $X_{\overline{\Pi}}$

- $X_{\overline{\Pi}} = \overline{\Pi}$;
- $X_{\overline{\Pi}} = L$ $(m=1)$;
- $X_{\overline{\Pi}}$ is the complement of a point $(m=q)$;
- $X_{\overline{\Pi}}$ is the complement of a proper $(q+1-m)$-maximal arc of $\overline{\Pi}$, q is even, $(q+1-m)|q$ $(1<m<q)$.

On the other hand for every plane $\overline{\Pi}'$ containing L_0, the set $X_{\overline{\Pi}'} = X \cap \overline{\Pi}'$ is a set of class $[0,m]_1$ of $\overline{\Pi}'$ and hence it is either the empty set or an m-maximal arc of $\overline{\Pi}'$. So one of the following occurs for $X_{\overline{\Pi}'}$:

- $X_{\overline{\Pi}'} = \emptyset$;
- $X_{\overline{\Pi}'}$ is a point $(m=1)$;
- $X_{\overline{\Pi}'} = \mathrm{AG}(2,q)$ $(m=q)$;

- $X_{\overline{\Pi}'}$ is a proper m-maximal arc of $\overline{\Pi}'$, q is even, $m|q$ $(1 < m < q)$.

It follows that if $m = 1$, then X is a set of type $(0, 1, q+1)_1$ and hence it is a subspace of $\mathrm{PG}(n, q)$ of dimension i with $1 \leqslant i \leqslant n-2$. If $m = q$, then X is a set of type $(0, q, q+1)_1$ and hence its complement X^c is a set of type $(0, 1, q+1)_1$ so X is the complement of a subspace of dimension i with $1 \leqslant i \leqslant n-2$. If $1 < m < q$, then there exists a plane $\overline{\Pi}$ containing L such that $X_{\overline{\Pi}}$ is the complement of a proper maximal arc of degree $q+1-m$ (otherwise every plane on L is contained in X and $X = \mathrm{PG}(n, q)$, a contradiction). This gives that q is even and $(q+1-m)|q$. On the other hand there exists a plane $\overline{\Pi}_0$ containing L_0 such that $X_{\overline{\Pi}_0}$ is a maximal arc of degree m (otherwise every plane on L_0 has empty intersection with X and so $X = \emptyset$, a contradiction). So q is even and $m|q$. From $1 < m < q$, $(q+1-m)|q$ and $m|q$ we get a contradiction. $\square$

From the last proposition and the results in the previous section:

Theorem 14. *If X is a non empty set of class $[0, m, q+1]_1$ in $\mathrm{PG}(n, q)$, then one of the following occurs:*

- *either X or X^c is a subspace of dimension i with $0 \leqslant i \leqslant n-1$ ($m = 1$ or $m = q$);*
- *X is a proper m-maximal arc of $\mathrm{PG}(2, q)$, q is even, $m|q$ $(1 < m < q)$;*
- *X is the complement of a proper $(q+1-m)$-maximal arc of $\mathrm{PG}(2, q)$, q is even, $(q+1-m)|q$ $(1 < m < q)$.*

Regarding the possible examples X of sets of class $[0, m, q+1]_1$ in $\mathrm{PG}(n, q), n \geqslant 2$ whose stabiliser in $P\Gamma L(n+1, q)$ satisfies Property $(*)$ the following holds:

Theorem 15. *A set X of points of $\mathrm{PG}(n, q), n \geqslant 2$ satisfies Property $(*)$ if and only if either X or X^c is a subspace of any dimension.*

Proof. A subspace of any dimension of $\mathrm{PG}(n, q)$, as well as its complement, gives an example of a set of points X of $\mathrm{PG}(n, q)$ satisfying Property $(*)$ (see [19, Proposition 7.4]). Vice versa if X satisfies Property $(*)$ from [19, Proposition 7.4] we know that X must be a set of class $[0, m, q+1]_1$, hence X is one of the examples given in Theorem 14. Moreover, again from [19, Proposition 7.4], we know that if X is different from a subspace or its complement, then $m \in \{2, \sqrt{q}+1\}$. If X is a proper maximal arc, then $m|q$ so $m = 2$ and X is a hyperoval since $m = \sqrt{q}+1$ does not divide q. Moreover G_X must be transitive on the secant lines to X. In [10] it is shown that the only proper maximal arcs with stabiliser in $P\Gamma L(n+1, q)$ satisfying this property are: the hyperoval in $\mathrm{PG}(2, 4)$ and the dual of the regular hyperoval in $\mathrm{PG}(2, q)$, q even. So $q = 4$ and X is the hyperoval of $\mathrm{PG}(2, 4)$. But as shown in [19, Remark 7.5] the hyperoval of $\mathrm{PG}(2, 4)$ has stabiliser in $P\Gamma L(3, 4)$ that does not satisfy Property $(*)$ since the stabiliser of a point of X in G_X is not transitive on the points of X^c (see Proposition 7). So no proper maximal arc satisfies Property $(*)$. Since X satisfies Property $(*)$ if and only if X^c satisfies Property $(*)$ we get a contradiction also in the case that X is the complement of a proper maximal arc. Hence assertion follows. $\square$

Observation 16. *Regarding the open cases of Liebler and Praeger [19] we have proved that there are no examples in the projective spaces with* $m \in \{2, \sqrt{q}+1\}$.

3.2 The affine case

In this section we study sets of type $(0,m,q)_1$ in $\mathrm{AG}(n,q)$. Let L be a line of X and let L_0 be a line external to X.

If $n=2$, then a set X of type $(0,m,q)_1$ in $AG(2,q)$ is the union of m parallel lines. Indeed if X would contain two intersecting lines L and L', then there would be no lines in $AG(2,q)$ missing X, a contradiction. Hence:

Proposition 17. *Let* X *be a set of type* $(0,m,q)_1$ *in* $\mathrm{AG}(2,q)$*, then* X *is the union of* m *parallel lines.*

Together with the results from the previous sections we have:

Proposition 18. *Let* X *be a nonempty proper set of class* $[0,m,q]_1$ *in* $\mathrm{AG}(2,q)$*, then* X *is one of the following:*

- *a point* $(m=1)$ *or the complement of a point* $(m=q-1)$*;*
- *the union of* m *parallel lines* $(1 \leqslant m \leqslant q-1)$*;*
- *a proper* m*-maximal arc* (q *even,* $m|q$)*;*
- *the complement of a proper* $(q-m)$*-maximal arc* (q *even,* $(q-m)|q$)*.*

Let now $n \geqslant 3$ and let X be a set of type $(0,m,q)_1$ in $\mathrm{AG}(n,q)$.

Observation 19. *If* $m=1$*, then every line meeting* X *in at least two points is contained in* X*; so* X *is a subspace of dimension* i *of* $\mathrm{AG}(n,q)$ *and since* X *contains a line it is* $1 \leqslant i \leqslant n-1$*. If* $m=q-1$*, then* X^c *is of type* $(0,1,q)_1$ *and hence* X^c *is a subspace of* $\mathrm{AG}(n,q)$ *of dimension at least* 1 *and at most* $n-1$*.*

From now on we can assume that $1 < m < q-1$ and so that there is no plane of $\mathrm{AG}(n,q)$ meeting X either in a point or a line or in the complement of a point or a line.

Observation 20. *Let* X *be a set of type* $(0,m,q)_1$ *in* $\mathrm{AG}(n,q)$ *and let* $\mathrm{PG}(n,q) = \mathrm{AG}(n,q) \cup \Sigma_\infty$*. Put* $X^* = X \cup \Sigma_\infty$*. The set* X^* *is of type* $(1,m+1,q+1)_1$ *in* $\mathrm{PG}(n,q)$ *and it contains a hyperplane. As on every point of* X *there is at least a secant line to* X*, then also on every point of* $X^* \setminus \Sigma_\infty$ *there is at least a secant line to* X^**. Hence if* X^* *contains singular points those points are in* Σ_∞*. Vice versa if* K *is a set of type* $(1,m+1,q+1)_1$ *in* $\mathrm{PG}(n,q)$ *containing a hyperplane, say* Σ_∞*, then the set* $X = K \setminus \Sigma_\infty$ *is a set of type* $(0,m,q)_1$ *in* $\mathrm{AG}(n,q) = \mathrm{PG}(n,q) \setminus \Sigma_\infty$*.*

From the previous observation it is clear that if we determine all the sets of type $(1,m+1,q+1)_1$ in $\mathrm{PG}(n,q)$ containing a hyperplane, then also all the sets of type $(0,m,q)_1$ in $\mathrm{AG}(n,q)$ will be determined. Sets of type $(1,m+1,q+1)_1$ in $\mathrm{PG}(n,q)$ have been studied in [31] (case $m \neq \frac{q}{2}$), then carried on in [16] and [17] (case $m = \frac{q}{2}$, $n \neq 3$) and in [12] (case $n=3$). Putting all these papers together the following has been proved:

Theorem 21. *Let K be a set of type $(1, m+1, q+1)_1$ in $\mathrm{PG}(n,q), n \geqslant 3, q > 4, 1 < m < q-1$. Then K is one of the following sets:*

- *the union of $m+1$ hyperplanes from a pencil of hyperplanes;*
- *a cone with vertex an i-dimensional $(-1 \leqslant i \leqslant n-3)$ projective subspace $\mathcal{V}$ of $\mathrm{PG}(n,q)$ and base an Hermitian variety $H(n-i-1,q)$ of an $(n-i-1)$-dimensional subspace S skew with $\mathcal{V}$.*
- *a cone with vertex an $(n-3)$-dimensional subspace $\mathcal{V}$ and base either a unital or a Baer subplane or a proper m-maximal arc plus a line or the complement of a proper $(q-m)$-maximal arc in a plane $\overline{\Pi}$ skew with $\mathcal{V}$.*
- *a cone with vertex an i-dimensional $(-1 \leqslant i \leqslant n-4)$ projective subspace $\mathcal{V}$ of $\mathrm{PG}(n,q)$, q even and base the projection of a nonsingular quadric $\mathcal{Q}$ of an $(n-i)$-dimensional subspace S' from a point v (different from the nucleus of $\mathcal{Q}$ if $n-i$ is even) into a $(n-i-1)$-dimensional subspace S contained in S' with S skew with $\mathcal{V}$ and $v \notin S$.*

We are ready to prove a classification result for sets with one proper intersection number in affine spaces:

Theorem 22. *If X is a set of type $(0, m, q)_1$ in $\mathrm{AG}(n,q), n \geqslant 3, q > 4$, then it is one of the following:*

- *the union of m parallel hyperplanes;*
- *a cylinder with vertex an $(n-3)$-dimensional projective subspace $\mathcal{V}_\infty$ at infinity and base a proper maximal arc of degree m in a plane Π whose line at infinity L_∞ is skew with $\mathcal{V}_\infty$;*
- *a cylinder vertex an $(n-3)$-dimensional projective subspace $\mathcal{V}_\infty$ at infinity and base the complement of a proper maximal arc of degree $q-m$ in a plane Π whose line at infinity L_∞ is skew with $\mathcal{V}_\infty$;*
- *a cylinder with vertex an i-dimensional projective subspace at infinity $\mathcal{V}_\infty(-1 \leqslant i \leqslant n-4)$ and base a set $M = K \setminus \Sigma'_\infty$, where K is the projection of a nonsingular quadric Q of a $\mathrm{PG}(n-i,q)$ from a point v (different from the nucleus of Q if $n-i$ is even) into an $\mathrm{PG}(n-i-1,q)$ not on v contained in $\mathrm{PG}(n-i,q)$, skew with $\mathcal{V}_\infty$ and $\Sigma'_\infty = \mathrm{PG}(n-i-2,q) = K \cap \Sigma_\infty$ is the hyperplane of $\mathrm{PG}(n-i-1,q)$ contained in K $(m = q/2)$.*

Proof. If either $m = 1$ or $m = q-1$, from Observation 19 we have that X is either a hyperplane or the complement of a hyperplane, that is the union of $q-1$ parallel hyperplanes. From now on we may assume $1 < m < q-1$, $n \geqslant 3$ and $q > 4$. From Observation 20 it follows that if we can determine all sets of type $(1, m+1, q+1)_1$ in $\mathrm{PG}(n,q)$ containing a hyperplane, then we would have determined all sets of type $(0,m,q)_1$ in $\mathrm{AG}(n,q)$. In Theorem 21 all sets of type $(1, m+1, q+1)_1$ are described. In what follows we will determine which ones of these sets contain a hyperplane.

- The first example of set of type $(1, m+1, q+1)_1$ described in Theorem 21 contains $m+1$ hyperplanes from a pencil of hyperplanes. By removing one of these hyperplanes we get a set of type $(0, m, q)_1$ in $\mathrm{AG}(n, q)$ given by the union of m parallel hyperplanes.

- The second example in Theorem 21, the cone with base an Hermitian variety does not contain hyperplanes. Indeed if it would contain a hyperplane, then the Hermitian variety $H(n-i-1, q)$ should contain an $(n-i-2)$-dimensional subspace and that is a contradiction since the maximum dimension of subspaces contained in $H(n-i-1, q)$ is $\frac{n-i-2}{2}$ that is strictly less than $n-i-2$ for $i \leqslant n-3$.

- In the third class of examples the only ones containing a hyperplane are the cone with base an m-maximal arc plus a line and the cone with base the complement of a maximal arc.

- The last class of examples, a cone with base the projection of a nonsingular quadric of a subspace does contain a hyperplane. Indeed if $n-i$ is even, then Q is a parabolic quadric of $\mathrm{PG}(n-i, q)$. Let $\mathcal{N}$ be its nucleus and let v be a point of $\mathrm{PG}(n-i, q)$ not on Q and different from $\mathcal{N}$ and let $\mathrm{PG}(n-i-1, q)$ be a hyperplane of $\mathrm{PG}(n-i, q)$ not on v. The projection from v of the points of Q on tangent lines through v is a hyperplane $\Sigma'_\infty = \mathrm{PG}(n-i-2, q)$ of $\mathrm{PG}(n-i-1, q)$. It contains also the point $\mathcal{N}'$, projection of $\mathcal{N}$ from v into Σ'_∞. The $q^{n-i-1}/2$ secant lines on v to Q project into a set X' of $\mathrm{AG}(n-i-1, q) = \mathrm{PG}(n-i-1, q) \setminus \Sigma'_\infty$ and the $q^{n-i-1}/2$ external lines to Q on v project into X'^c, the complement of X' in $\mathrm{AG}(n-i-1, q)$. Both X' and X'^c are sets of type $(0, q/2, q)_1$ in $\mathrm{AG}(n-i-1, q)$ and are isomorphic. The cone with vertex an i-dimensional subspace $\mathcal{V}$ of $\mathrm{PG}(n, q)$ and base either X' or X'^c is a set of type $(0, q/2, q)_1$ containing a hyperplane, namely the hyperplane spanned by $\mathcal{V}$ and Σ'_∞. If $n-i$ is odd, then Q is either an hyperbolic quadrics Q^+ or an elliptic quadric Q^- of $\mathrm{PG}(n-i, q)$. Let v be any point of $\mathrm{PG}(n-i, q)$ not on Q and let $\mathrm{PG}(n-i-1, q)$ be a hyperplane of $\mathrm{PG}(n-i, q)$ not on v. The polar hyperplane of v w.r.t Q is in both cases a parabolic quadric given by $v^\perp \cap Q$. The projection from v of $v^\perp \cap Q$ is a hyperplane Σ'_∞ of $\mathrm{PG}(n-i-1, q)$. There are $q^{n-i-1}/2 \pm q^{\frac{n-i-1}{2}}/2$ secant lines from v to $Q^\pm$ and they project from v into a set X' of $q^{n-i-1}/2 \pm q^{\frac{n-i-1}{2}}/2$ points of $\mathrm{AG}(n-i-1, q) = \mathrm{PG}(n-i-1, q) \setminus \Sigma'_\infty$. The $q^{n-i-1}/2 \mp q^{\frac{n-i-1}{2}}/2$ external lines from v to $Q^\pm$ project from v into the set of X'^c of size $q^{n-i-1}/2 \mp q^{\frac{n-i-1}{2}}/2$, the complement of X' in $\mathrm{AG}(n-i-1, q)$. The two sets X' so obtained and their complements are all of type $(0, q/2, q)_1$. The cone with vertex an i-dimensional subspace $\mathcal{V}$ and base either the set X' or X'^c is a set of type $(0, q/2, q)_1$ of $\mathrm{AG}(n, q)$ containing the hyperplane spanned by $\mathcal{V}$ and Σ'_∞.

As already observed by removing a hyperplane from those of the above sets that contain a hyperplane we obtain sets of type $(0, m, q)_1$ in the affine space and no other possible set of type $(0, m, q)_1$ in $\mathrm{AG}(n, q)$ can exist. Hence the assertion follows. □

3.3 The cases $q \in \{2, 3, 4\}$

We recall that from Observation 19 we have that $1 < m < q-1$ and so that there is no plane of $\mathrm{AG}(n, q)$ meeting X either in a point or a line or in the complement of a point or a line. If $q = 2$, then every subset of $\mathrm{AG}(n, q)$ is of class $[0, 1, q]_1$ and it is a subspace. If $q = 3$, then for a set X of type $(0, m, 3)_1$ in $\mathrm{AG}(n, 3)$ we have that either $m = 1$ or $m = 2 = q - 1$ and from Observation 19 we have that X is either a hyperplane or the union of two parallel hyperplanes. Finally assume $q = 4$. If either $m = 1$ or $m = q-1 = 3$ again from Observation 19 we have that X is either a hyperplane or the union of three parallel hyperplanes. If $1 < m < q - 1$, then $m = 2$. Hence we have to study sets of class $[1, 3, 5]_1$ in $\mathrm{PG}(n, 4)$. These sets have been studied first in [3], [25], [31] and [17] were partial results were obtained. All sets of class $[1, 3, 5]_1$ of $\mathrm{PG}(3, 4)$ have been obtained in [15] and finally in [28] all such sets have been desribed in $\mathrm{PG}(n, 4), n \geqslant 3$. These sets are in connection with binary codes. There are 2^a with $a = \frac{1}{3}(n + 1)(n^2 + 2n + 3)$ different such sets in $\mathrm{PG}(n, 4)$ and they can be all algebraically described [28]. They have also been studied in [7] where the following is proved:

Theorem 23. [7] *If X is a set of class $[1, 3, 5]_1$ in $\mathrm{PG}(n, 4), n \geqslant 3$ containing a hyperplane $\overline{\Pi}$, then it is one of the following:*

- *the hyperplane $\overline{\Pi}$;*
- *the union of three hyperplanes $\overline{\Pi}, \overline{\Pi}', \overline{\Pi}''$ from a pencil;*
- *the union of $\overline{\Pi}$ with the projection into an $(n - i)$-dimensional projective subspace $\mathrm{PG}(n-i, 4)$ $(4 \leqslant i \leqslant n+1)$ of a nonsingular quadric Q of an $(n-i+1)$-dimensional projective subspace $\mathrm{PG}(n - i + 1, 4)$ from a point v not on Q (different from the nucleus if $n - i + 1$ is even).*

From this result we can obtain, with a proof similar to that of Theorem 22, the following characterization theorem also for sets of type $(0, 2, 4)_1$ in $\mathrm{AG}(n, 4)$.

Theorem 24. *If X is a set of type $(0, 2, 4)_1$ in $\mathrm{AG}(n, 4), n \geqslant 3$ then it is one of the following:*

- *the union of 2 parallel hyperplanes;*
- *a cylinder with vertex an $(n-3)$-dimensional projective subspace at infinity $\mathcal{V}_\infty$ and base a hyperoval M in a plane Π, whose line at infinity L_∞ is skew with $\mathcal{V}_\infty$;*
- *a cylinder with vertex an $(n-3)$-dimensional projective subspace at infinity $\mathcal{V}_\infty$ and base the complement of a hyperoval in a plane Π whose line at infinity L_∞ is skew with $\mathcal{V}_\infty$;*
- *a cylinder with vertex an i-dimensional projective subspace at infinity $\mathcal{V}_\infty$ $(-1 \leqslant i \leqslant n - 4)$ and base $M = K \setminus \Sigma'_\infty$, where K is the projection of a nonsingular quadrics Q of a $\mathrm{PG}(n - i, 4)$ from a point v (different from the nucleus $\mathcal{N}$ of Q if $n - i$ is*

even) into an $(n-i-1)$-dimensional projective subspace $\mathrm{PG}(n-i-1,4)$*, contained in* $\mathrm{PG}(n-i,4)$*, not on* v *and skew with* $\mathcal{V}_\infty$ *and* $\Sigma'_\infty = \mathrm{PG}(n-i-2,4) = K \cap \Sigma_\infty$ *is the hyperplane of* $\mathrm{PG}(n-i-1,4)$ *contained in* K*.*

3.4 Examples satisfying Property (*)

In the previous theorems, propositions and observations we have described all the sets of class $[0,m,q]_1$ in $\mathrm{AG}(n,q)$. We will now decide which of these examples satisfy Property $(*)$ and which do not satisfy Property $(*)$. Recall that from [19, Proposition 6.6], the open cases in the affine spaces are for $(q,m)=(16,4)$ and $(q,m)=(4,2)$. In both cases the possible candidate examples for satisfying Property $(*)$ must have secant lines meeting in affine Baer sublines (see [19, Proposition 6.6 (ii)]).

Proposition 25. *Let* X *be the example of set of type* $(0,4,16)_1$ *in* $\mathrm{AG}(n,16)$*,* $n \geqslant 2$ *given by a Cyl*$(\mathcal{V}_\infty, M)$*, where* M *is a* 4*-maximal arc in a plane* Π *whose line at infinity* L_∞ *is skew with the* $(n-3)$*-dimensional projective subspace* $\mathcal{V}_\infty$ *at infinity. It does not satisfy Property* $(*)$*.*

Proof. From [1] and [13] we know that in $\mathrm{PG}(2,16)$ there are exactly two non-isomorphic 4-maximal arcs. They are both of Denniston type and can be described in the following way. Let ξ be a primitive element of $\mathrm{GF}(16)$ such that $\xi^4+\xi+1=0$ and put $A=\{0,1,\xi,\xi^4\}$ and $A'=\{0,1,\xi^5,\xi^{10}\}$. Then the two non-isomorphic 4-maximal arcs of $\mathrm{PG}(2,16)$ give the following two non-isomorphic 4-maximal arcs in $\mathrm{AG}(2,16)$:

$$M = \cup_{\lambda\in A}\,\{(x,y) : x^2+\alpha xy+y^2+\lambda=0\} \text{ and}$$
$$M' = \cup_{\lambda\in A'}\,\{(x,y) : x^2+\xi xy+y^2+\lambda=0\}.$$

From [19, Proposition 6.6 (ii)] if a set of points of $\mathrm{AG}(n,16)$ satisfies Property $(*)$, then the secant lines must meet it in affine Baer sublines. We will show that neither M nor M' have all secant lines meeting them in affine Baer sublines. Indeed consider the lines $L: y=0$ and $L': y=1$ we have $L\cap M=\{(x,0): x\in\{0,1,\xi^2,\xi^8\}\}$ that is not an affine Baer subline and $L'\cap M'=\{(x,1): x\in\{0,\xi,\xi^{12},\xi^{13}\}\}$ that is not an affine Baer subline. Hence X does not satisfy Property $(*)$. □

Proposition 26. *Let* X *be the example of set of type* $(0,2,4)_1$ *in* $\mathrm{AG}(n,4)$ *coming from the projection of a nonsingular quadric* Q *of a* $\mathrm{PG}(n-i,4)$*. It does not satisfy Property* $(*)$*.*

Proof. We divide two cases:

Case 1. $n-i$ is even: If Q is a nonsingular quadric of $\mathrm{PG}(n-i,4)$ with $n-i$ an even number, then Q is a parabolic quadric. Let $\mathcal{N}$ be the nucleus of Q. Consider the $\mathrm{Cone}(\mathcal{V}_\infty, K)$ with vertex an i-dimensional $(-1\leqslant i\leqslant n-4)$ projective subspace $\mathcal{V}_\infty$ of $\mathrm{PG}(n,4)$ and base K, the projection of the nonsingular quadric $\mathcal{Q}$ of $\mathrm{PG}(n-i,4)$ from a point $v\neq\mathcal{N}$ into a $\mathrm{PG}(n-i-1,4)$ not on v. The set K contains a hyperplane Σ'_∞ of $\mathrm{PG}(n-i-1,4)$ and a special point $\mathcal{N}'\in\Sigma_\infty$. The corresponding set $X=$Cyl$(\mathcal{V}_\infty, M)$, where $M = K \setminus \Sigma'_\infty$ is of type $(0,2,4)_1$ in $\mathrm{AG}(n,4)$ and for the existence of the point $\mathcal{N}'$ it follows easily that X does not satisfy Property $(*)$.

Case 2. $n-i$ is odd: If Q is a nonsingular quadric of $\mathrm{PG}(n-i,4)$ with $n-i$ and odd number, then Q can be either an elliptic quadric Q^- or an hyperbolic quadric Q^+. The linear group stabilizing these quadrics are $G^- = PGO^-(n-i+1,4)$ and $G^+ = PGO^+(n-i+1,4)$, respectively. If v is a point of $\mathrm{PG}(n-i,4)$ not on $Q^\pm$, the group $G^\pm$ has three orbits on the lines on v, the secant lines to $Q^\pm$, the tangent lines to $Q^\pm$ and the external lines to $Q^\pm$. In the example X=Cyl$(\mathcal{V}_\infty, M)$ where $M = K\setminus\Sigma'_\infty$, K being the projection of $Q^\pm$ from v into a $(n-i)$-dimensional projective subspace $\mathrm{PG}(n-i,4)$ of $\mathrm{PG}(n,4)$ we have that the projection of the tangent lines to $Q^\pm$ on v give the hyperplane Σ'_∞ of $\mathrm{PG}(n-i-1,4)$ contained in the set K, the projection of the secant lines to $Q^\pm$ give the set $M = K \setminus \Sigma'_\infty$ and the projection of the external lines to $Q^\pm$ give the complement of M in $\mathrm{AG}(n-i-1,4) = \mathrm{PG}(n-i-1,4) \setminus \Sigma'_\infty$. Let (L,L') be a pair of lines on v with L an external line to $Q^\pm$ and L' a secant line to $Q^\pm$ and let (R,R') be another pair of lines on v with R an external line to $Q^\pm$ and R' a secant line to $Q^\pm$. By Witt's theorem, without loss of generality, the two pairs of lines span the same type of plane, that is a plane meeting $Q^\pm$ in a nonsingular conic Γ. The stabiliser of the point v not on Γ (and different from the nucleus of Γ) in $PGO(3,q)$ has order q. It acts transitively on the secant lines on v, and the stabilizer of a secant line on v and the point v is an elation with centre v. Thus each external line on v lies in a different orbit on pairs. So we have $q/2 = 2$ orbits on such pairs (L,L'). Hence it is not transitive on such pairs. This gives that the stabiliser of X =Cyl$(\mathcal{V}_\infty, M)$ in $A\Gamma L(n+1,q)$ is not transitive on pair of points (x,y) with $x \in X$ and $y \notin X$. So X does not satisfy Property $(*)$. $\square$

Theorem 27. *A set X of points of $\mathrm{AG}(n,q)$, $n \geqslant 2$ satisfies Property $(*)$ if and only either X or X^c is one of the following:*

- *An affine subspace $\mathrm{AG}(i,q)$ for $i \in \{0,\ldots,n\}$ of $\mathrm{AG}(n,q)$;*
- *A cylinder Cyl$(\mathcal{V}_\infty, M)$ of $\mathrm{AG}(n,4)$, where M is either a hyperoval or the complement of a hyperoval of a plane Π whose line at infinity is skew with the $(n-3)$-dimensional projective subspace $\mathcal{V}_\infty$;*
- *A pair of parallel hyperplanes of $\mathrm{AG}(n,4)$;*
- *Four parallel hyperplanes of $\mathrm{AG}(n,16)$ with the secant lines meeting the set in affine Baer sublines.*

Proof. From [19, Proposition 6.6] if X (or X^c) is not an affine subspace, then $q \in \{4,16\}$, X is a set of class $[0,\sqrt{q},q]_1$ and the secant lines to X meet X in affine Baer sublines. From the previous propositions X can be one of the examples in the statement of this proposition. If X is the union of two parallel hyperplanes in $\mathrm{AG}(n,4)$ or the union of four parallel hyperplanes in $\mathrm{AG}(n,16)$ with the secant lines meeting in affine Baer sublines, then since $A\Gamma L(n,q)$ is transitive on parallel classes we may fix our favorite parallel class $\mathcal{P} = \{\Pi_a : x_1 = a\}_{a\in\mathrm{GF}(q)}$ and since the stabiliser of a parallel class inside $A\Gamma L(n,q)$ is

2-transitive on the hyperplanes of the parallel class we may assume that $X = \Pi_0 \cup \Pi_1$ for $q = 4$ and that $X = \Pi_0 \cup \Pi_1 \cup \Pi_a \cup \Pi_b$ for $q = 16$. In this last case since every secant line meets X in an affine Baer subline, we have that $\{0, 1, a, b\} = \mathrm{GF}(4)$ hence $b = a^2$ and if ξ is a primitive element of GF(16), then $a = \xi^5, b = \xi^{10}$. Note that a primitive element ξ of GF(16) has order 15, so ξ^5 has order 3, and therefore ξ^5 is a primitive element of GF(4), and hence it is a zero of $x^2 + x + 1$. We will show that both these examples satisfy Property $(*)$ by showing that in both cases the following hold (see Proposition 7):
(1) G_X is transitive on the points of X;
(2) $G_{X,O}$ is transitive on the points of X^c, O being the origin.
For $q = 4$, the stabiliser in $A\Gamma L(n, 4)$ of X, G_X contains the following maps:

$$x' = Ax + c \text{ with } A_1 = (1, 0, \ldots, 0) \text{ being the first line of } A \text{ and } c^T = (1, c_2, \ldots, c_n),$$

that interchange Π_0 and Π_1 and $G_{\{X,\Pi_0\}}$, the stabiliser of Π_0 inside G_X, is transitive on the points of Π_0 since it contains all translations by vectors in Π_0. Hence (1) of Proposition 7 holds for $q = 4$.
For $q = 16$, the stabiliser in $A\Gamma L(n, 16)$ of X, G_X contains the following maps:

$$x' = Ax^\sigma + c \text{ with } A_1 = (a_{11}, 0, \ldots, 0) \text{ and } c^T = (c_1, c_2, \ldots, c_n),$$

where σ is one of the four Frobenius maps $z \mapsto z^{2^i}, i \in \{0, 1, 2, 3\}$, $a_{11} \in \{1, \xi^5, \xi^{10}\}, c_1 \in \{0, 1, \xi^5, \xi^{10}\}$. From this it follows easily that also in this case G_X is transitive on the hyperplanes contained in X. Moreover again G_{X,Π_0}, the stabiliser in G_X of Π_0 is transitive on the points of Π_0. Hence (1) of Proposition 7 holds also for $q = 16$.
Let $G_{X,O}$ be the stabiliser in G_X of the origin O in both examples. We will prove that (2) of Proposition 7 holds. For $q = 4$, the group $G_{X,O}$ contains the following maps:

$$x' = Ax^\sigma \text{ with } A_1 = (1, 0, \ldots, 0),$$

where σ is one of the two Frobenius maps $z \mapsto z^{2^i}$ with $i \in \{0, 1\}$. When σ is the Frobenius map $z \mapsto z^2$, then the above collineations interchanges the two hyperplanes contained in X^c and if σ is the identity, then the collineations stabilize the two hyperplanes contained in X^c and act transitively on the points of each. So (2) of Proposition 7 holds.
For $q = 16$, the group $G_{X,O}$ contains the following maps:

$$x' = Ax^\sigma \text{ with } A_1 = (a_{11}, 0, \ldots, 0),$$

where σ is one of the four Frobenius maps $z \mapsto z^{2^i}, i \in \{0, 1, 2, 3\}$ and $a_{11} \in \{1, \xi^5, \xi^{10}\}$. Again from this it is easy to see that the above collineations act transitively on the hyperplanes of X^c, the stabiliser inside $G_{X,O}$ of any of the hyperlanes of $\mathcal{P}$ contained in X^c is transitive on the points of the chosen hyperplane. Hence also for $q = 16$ we have proved that (2) Proposition 7 holds. Finally let X be the cylinder with base an hyperoval in $\mathrm{AG}(n, 4)$. we may assume that X is the following:

$$X = \{(a, b, x_3, \ldots, x_n) \in \mathrm{AG}(n, 4) : a^2 + \xi b^2 + ab + 1 = 0 \text{ or } a = b = 0\}$$

since all hyperovals in $\mathrm{AG}(2,4)$ are projectively equivalent, then all cylinders with base hyperovals of $\mathrm{AG}(2,4)$ are equivalent under a collineation of $A\Gamma L(n,4)$. The stabiliser G_X of X in $A\Gamma L(n,4)$ contains the following maps

$$x' = Ax^{\sigma} + c^T$$

with $A_1 = (a_{11}, a_{12}, 0, \ldots, 0), A_2 = (a_{21}, a_{22}, 0, \ldots, 0)$ being the first two lines of A and σ being one of the two Frobenius maps $z \mapsto z^{2^i}$, with $i \in \{0,1\}$ and the following maps

$$(a', b')^T = A'((a,b)^T)^{\sigma} + (c_1, c_2)^T,$$

where $A'_1 = (a_{11}, a_{12})$, $A'_2 = (a_{21}, a_{22})$, are in the stabiliser inside $A\Gamma L(2,4)$ of the hyperoval $M := \{(a,b) \in \mathrm{AG}(2,4) : a^2 + \xi b^2 + ab + 1 = 0\} \cup \{(0,0)\}$. Since the stabiliser in $A\Gamma L(2,4)$ is transitive on the points of M, then it follows that G_X, the stabiliser of the cylinder X in $A\Gamma L(n,4)$ is transitive on points of X. So also for this example (1) of Proposition 7 holds. Let $G_{X,O}$ be the stabiliser in G_X of the origin O of $\mathrm{AG}(n,4)$. This group contains the following maps:

$$x' = Ax^{\sigma}$$

with $A_1 = (a_{11}, a_{12}, 0, \ldots, 0), A_2 = (a_{21}, a_{22}, 0, \ldots, 0)$ and σ being one of the two Frobenius maps $z \mapsto z^{2^i}$ with $i \in \{0,1\}$ and the following maps

$$(a', b')^T = A'((a,b)^T)^{\sigma}$$

where $A'_1 = (a_{11}, a_{12}) A'_2 = (a_{21}, a_{22})$ are in the stabiliser inside $A\Gamma L(2,4)$ of the hyperoval $M := \{(a,b) \in \mathrm{AG}(2,4) : a^2 + \xi b^2 + ab + 1 = 0\} \cup \{(0,0)\}$ and of the origin $(0,0)$. Since this last group is transitive on the points of M^c in $\mathrm{AG}(2,4)$ (see [19]) it follows that $G_{X,O}$ is transitive on the points of X^c. Hence again (2) Proposition 7 holds and hence also this example satisfies Property $(*)$. Similarly can be shown that also if X is the cylinder on the complement of the hyperoval of $\mathrm{AG}(2,4)$ Property $(*)$ is satisfied. $\square$

Observation 28. *Regarding the open cases of Liebler and Praeger [19] we have shown that there are examples in the affine spaces* $\mathrm{AG}(n,q)$ *of sets of type* $(0,m,q)_1$ *for both* $(m,q) = (4,16)$ *and* $(m,q) = (2,4)$ *satisfying Property* $(*)$. *Moreover no other examples, different from the ones in the previous theorem can exist.*

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