On the subpartitions of the ordinary partitions, II

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Abstract

In this note, we provide a new proof for the number of partitions of n having subpartitions of length ℓ with gap d. Moreover, by generalizing the definition of a subpartition, we show what is counted by q-expansion

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{n=0}^{\infty} (-1)^n q^{(an^2+bn)/2}$$

and how fast it grows. Moreover, we prove there is a special sign pattern for the coefficients of q-expansion

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} \left(1 - 2\sum_{n=0}^{\infty} (-1)^n q^{(an^2+bn)/2}\right).$$

Keywords: partition; subpartition; partial theta function.

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1 Introduction

Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m$ be an ordinary partition [1]. In a recent paper [9], the first author defines a subpartition of an ordinary partition as follows. Let us fix a positive integer d. Then, for a given partition, a subpartition with gap d is defined as the longest sequence satisfying $a_1 > a_2 > \cdots > a_s$ and $a_s > a_{s+1}$, where $a_i - a_j \geqslant d$ for all $i < j \leqslant s$. a_{s+1} must be understood as a zero if it comes after the final part. This is a generalization of L. Kolitsch's Rogers-Ramanujan subpartition [10], which is the case $d = 2$. We call the first condition involving d a gap condition and the second condition $a_s > a_{s+1}$ a tail condition. For convenience, we define the subpartition of the empty partition as the empty partition. We define the length of the subpartition with gap d as the number of parts in the subpartition. When the gap d is clear in the context, we will write "the subpartition" instead of "the subpartition with gap d". In [9], the author uses subpartitions to find combinatorial proofs of entries in Ramanujan's lost notebook [11]. Moreover, these subpartitions play a crucial role in obtaining an asymptotic formula for certain q-series involving partial theta functions [8].

Define $p(n)$ to be the number of partitions of n and $p(n, \ell, d)$ to be the number of partitions of n having a subpartition of length ℓ with gap d. In [9], by finding a generating function via a case by case argument, the first author proved that

Theorem 1. *For all nonnegative integers n and ℓ and a positive integer d,*

$$p(n, \ell, d) = p(n - S_{\ell,d}) - p\left(n - S_{\ell+1,d}\right)$$

where, for each nonnegative integer k,

$$S_{k,d} = \begin{cases} 1 + (1+d) + (1+2d) + \cdots + 1 + (k-1)d = \frac{dk^2-(d-2)k}{2}, & \text{if } k \neq 0, \\ 0, & \text{if } k = 0. \end{cases}$$

Example 2. According to Theorem 1, there are 5 partitions of 8 having a subpartition of length 2 with gap 2 as $p(8, 2, 2) = p(8-4) - p(8-9) = p(4) - p(-1) = 5 - 0 = 5$. Here are 5 such partitions and the parts consisting of the subpartition are underlined:

$$\underline{7} + \underline{1}, \quad \underline{6} + \underline{2}, \quad \underline{5} + \underline{3}, \quad \underline{5} + \underline{2} + 1, \quad \text{and} \quad \underline{4} + \underline{2} + 1 + 1.$$

In this note, by employing a combinatorial argument, we give a simpler proof.

Now we further generalize the notion of subpartitions as follows. We introduce a new parameter t and replace the tail condition by $a_s - a_{s+1} \geqslant t$. The case $t = 1$ is the original definition of a subpartition with gap d. Now we define $p(n, \ell, d, t)$ to be the number of partitions of n having a subpartition of length ℓ with gap d and tail condition t. Then, by employing essentially same argument, we can prove the following theorem.

Theorem 3. *For all nonnegative integers n and ℓ, and positive integers d and t,*

$$p(n, \ell, d, t) = p(n - T_{\ell,d,t}) - p(n - T_{\ell+1,d,t}),$$

where, for each nonnegative integer k,

$$T_{k,d,t} = \begin{cases} t + (t+d) + (t+2d) + \cdots + t + (k-1)d = \frac{dk^2+(2t-d)k}{2}, \text{ if } k \neq 0, \\ 0, \text{ if } k = 0. \end{cases}$$

By summing even ℓ's, we see that, for a positive integer a and an integer b with $a+b>0$ and $a \equiv b \pmod 2$, we find that

$$\frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{(an^2+bn)/2} = \sum_{n=0}^{\infty} p_e(n, a, (a+b)/2) q^n, \tag{1}$$

where $(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$ and $p_e(n,d,t)$ is the number of partitions of n having subpartitions of even length with gap d and tail condition t. Here the assumption on a and b is for the positive integrality of $(an^2+bn)/2$ for all positive integers n. From the representation of the partial theta function on the left side of (1), it is not clear at all the positivity of its q-expansion and what it counts. Since n copies of 1 is always counted by $p_e(n,a,(a+b)/2)$, the positivity of q-expansion is now clear from the combinatorial description. The case $a=b=1$ appears Andrews [2] as a generating function for the number of partitions of n in which the first non-occurrence number as a part is odd, which is a conjugation of partition with subpartition of even length with gap d as noted in [8]. When $a=1$ and $b=3$, we have

$$\frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{(n^2+3n)/2} = 1 + q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots .$$

Among 7 partitions of 5, there are 5 partitions having subpartitions of even length with gap 1 with tail condition 2 as follows:

$$4+1, \quad 3+2, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

Moreover, by adopting the argument in [8], we can prove the following theorem.

Theorem 4. *As n tends to infinity, for positive integers d and t,*

$$p_e(n,d,t) \sim \frac{1}{2} p(n).$$

This is a generalization of [8, Theorem 1] and says that asymptotically half of the partitions of n have subpartitions of even length. Much less obviously, there are inequalities between $p_e(n,a,(a+b)/2)$ and $p_o(n,a,(a+b)/2)$, where $p_o(n,a,(a+b)/2) = p(n) - p_e(n,a,(a+b)/2)$, i.e. the number of partitions of n having subpartitions of odd length with gap a and tail condition $(a+b)/2$. These inequalities are unexpected since both $p_e(n,a,(a+b)/2)$ and $p_o(n,a,(a+b)/2)$ are asymptotically $p(n)/2$.

Theorem 5. *For integers a and b satisfying $a > 0$, $a + b > 0$, and $a \equiv b \pmod 2$, we have*

$$p_e(n, a, (a + b)/2) > p_o(n, a, (a + b)/2), \quad \text{if } b > 0,$$
$$p_e(n, a, (a + b)/2) < p_o(n, a, (a + b)/2), \quad \text{if } b < 0,$$

for large enough integers n. Moreover, for $b = 0$ and even integers $a > 2$, we have

$$p_e(n, a, a/2) > p_o(n, a, a/2),$$

for all positive integers n except that the equality holds when $n = 2$ and $a = 4$.

Remark 6. The case $a = 2$ and $b = 0$ was discussed in [8, Theorem 2]. In this case, the sign of $p_e(n, 2, 1) - p_o(n, 2, 1)$ is alternating. This difference is due to that the generating function is essentially modular in this case. The more precise statement in the second part is also due to that $1 - 2\sum_{n=0}^{\infty}(-1)^n q^{(an^2+bn)/2}$ becomes a theta function in these cases.

Remark 7. The conditions on a and b, i.e. $a > 0$, $a + b > 0$, and $a \equiv b \pmod 2$, are needed just for having non-negative integer exponents in the q-expansion.

This paper is organized as follows. In Section 2, we prove the combinatorial results. By adopting the circle method and elementary q-series manipulation, we will prove Theorem 5 in Section 3.

2 Proof of Combinatorial Results

For a given partition λ, we always write it in the form $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$, and for convenience, we define $\lambda_s = 0$ for all integer $s > m$. It is well known [1] that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty}.$$

Now we define $p(n, t, d)$ to be the number of partitions of n having subpartitions of length $\geqslant m$ with gap d. Then, the following lemma immediately implies Theorem 1.

Lemma 8. *For all nonnegative integers m,*

$$p(n, m, d) = p(n - S_{m,d}).$$

Proof. It is enough to show that

$$\sum_{n=0}^{\infty} p(n, m, d)q^n = \frac{q^{S_{m,d}}}{(q; q)_\infty}.$$

By definition, it is clear that the above holds when $t = 0$, and thus we may assume that $t \geqslant 1$. We first observe that $q^{S_{m,d}}$ generates the partition $\pi = (1+(t-1)d, 1+(t-2)d, \ldots, 1)$.

Let λ be a partition generated by $\frac{1}{(q)_\infty}$. We append each part of λ to π beginning with the largest part, and denote the resulting partition as μ, i.e., $\mu_i = \pi_i + \lambda_i$ for all positive integers i. For example, when $\pi = (5,3,1)$ and $\lambda = (4,4,3,2,1)$, we obtain $\mu = (9,7,4,2,1)$. Since the gap between two consecutive parts of π is larger than or equal to d and $\mu_m > \mu_{m+1}$, we see that μ has a subpartition of length at least t, which completes the proof. □

By employing the same argument, we can easily see that $\frac{q^{T_{m,d,t}}}{(q;q)_\infty}$ is a generating function for the number of partitions of n having subpartitions of length $\geqslant m$ with gap d and tail condition t, which implies Theorem 3.

Now we turn to the proof of Theorem 4. To this end, we are going to employ Ingham's Tauberian theorem ([7, Theorem 1] and [4, Theorem 5.3]). To apply the Tauberian theorem, we have to show that $p_e(n,d,t)$ is weakly increasing. To see this, suppose that λ is a partition of n with subpartition of even length. If there is no subpartition in λ, we add a part 1 to the partition λ. Then, the resulting partition is a partition of $n+1$, and since the size of the first two parts remains the same, the length of the subpartition is 0. If λ contains the subpartition, we increase the largest part of λ by 1. Then, the resulting partition is a partition of $n+1$ and this operation does not affect the length of the subpartition. It is clear that this map is an injection, thus we observe that $p_e(n,d,t) \leqslant p_e(n+1,d,t)$. Theorem 4 now immediately follows from [3, Theorem 1] and Ingham's Tauberian theorem.

3 Proof of Theorem 5

For a positive integer a and an integer b with $a+b>0$ and $a \equiv b \pmod 2$, define

$$S_{a,b}(q) := 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{(an^2+bn)/2},$$

$$F_{a,b}(q) = \sum_{N=0}^{\infty}\alpha_{a,b}(N)q^N := \frac{1}{(q;q)_\infty}S_{a,b}(q).$$

Note that $\alpha_{a,b}(N) = p_e(N,a,(a+b)/2) - p_o(N,a,(a+b)/2)$. Therefore, to prove Theorem 5, it suffices to see the sign of $\alpha_{a,b}(N)$. We are going to get an asymptotic formula for $\alpha_{a,b}(N)$ by similar argument of Bringmann and Mahlburg [4]. Main idea of the proof is that we can get an asymptotic formula by focusing on asymptotic behavior of $S_{a,b}(q)$ near $q=1$.

Set $q = e^{2\pi i\tau}$ with $\tau = x+iy$. The following proposition describes an asymptotic behavior of $S_{a,b}(q)$ near $q=1$.

Proposition 9. *Assume $|x| \leqslant y$. As $y \to 0+$,*

$$S_{a,b}(q) = \frac{b}{4}(-2\pi i\tau) + \mathcal{O}(y^2).$$

To prove this result, we need the following Zagier's result on asymptotic expansions for series (the first generalization of Proposition 3 in [12] with a correction on the sign),

Lemma 10. *Suppose that* h *has the asymptotic expansion*

$$h(t)=\sum_{n=0}^{S} b_n t^n + \mathcal{O}\left(t^{S+1}\right)$$

as $t \to 0+$ *and that* h *and all of its derivatives are of rapid decay at infinity, i.e.* $\int_l^\infty |h^{(k)}(x)|\,dx$ *converges for some* $l>0$*. Then, for* $a>0$*, as* $t\to 0+$*,*

$$\sum_{m=0}^{\infty} h((m+a)t)=\frac{1}{t}\int_0^\infty h(x)\,dx-\sum_{n=0}^{S} b_n \frac{B_{n+1}(a)}{n+1}t^n+\mathcal{O}\left(t^{S+1}\right),$$

where $B_n(x)$ *is the* n*-th Bernoulli polynomial.*

Proof of Proposition 9. Let $f_{a,b}(\tau)=(S_{a,b}(q)-1)/2$, i.e.

$$f_{a,b}(\tau)=\sum_{n=1}^{\infty}(-1)^n q^{(an^2+bn)/2}.$$

We can rewrite $f_{a,b}(\tau)$ as follows:

$$f_{a,b}(\tau)=e^{-\frac{b^2}{4a}\pi i\tau} g_{a,b}(\tau),$$

where

$$g_{a,b}(\tau)=\sum_{n=0}^{\infty}\left[e^{4\pi a\left(n+1+\frac{b}{4a}\right)^2 i\tau}-e^{4\pi a\left(n+\frac{1}{2}+\frac{b}{4a}\right)^2 i\tau}\right].$$

We will find asymptotic formulas for the real and imaginary parts of $g_{a,b}(\tau)$. The real part of $g_{a,b}(\tau)$ can be written as

$$\mathrm{Re}\left(g_{a,b}(\tau)\right)=\sum_{n=0}^{\infty}\left[u_{\frac{x}{y}}\left(\left(n+1+\frac{b}{4a}\right)\sqrt{y}\right)-u_{\frac{x}{y}}\left(\left(n+\frac{1}{2}+\frac{b}{4a}\right)\sqrt{y}\right)\right],$$

where

$$u_s(t)=e^{-4\pi a t^2}\cos(4\pi a s t^2)=1-4\pi a t^2+\mathcal{O}(t^4)\quad\text{as}\quad t\to 0+\,.$$

By Lemma 10, for $\dfrac{b}{4a}+\dfrac{1}{2}>0$ (this is the case as $a>0$, $a+b>0$),

$$\begin{aligned}\mathrm{Re}\left(g_{a,b}(\tau)\right)&=\left[\frac{I_u}{\sqrt{y}}-B_1\left(1+\frac{b}{4a}\right)-(-4\pi a)\frac{B_3\left(1+\frac{b}{4a}\right)}{3}y+\mathcal{O}_{\frac{x}{y}}(y^2)\right]\\&\quad-\left[\frac{I_u}{\sqrt{y}}-B_1\left(\frac{1}{2}+\frac{b}{4a}\right)-(-4\pi a)\frac{B_3\left(\frac{1}{2}+\frac{b}{4a}\right)}{3}y+\mathcal{O}_{\frac{x}{y}}(y^2)\right]\\&=-\frac{1}{2}+\frac{b(2a+b)}{8}\pi y+\mathcal{O}_{\frac{x}{y}}(y^2)\end{aligned}$$

with $|I_u| = \left|\int_0^\infty u_s(t)\,dt\right| < \infty$. The imaginary part can be treated similarly.

$$\begin{aligned}
&\operatorname{Im}(g_{a,b}(\tau)) \\
&= \left[\frac{I_v}{\sqrt{y}} - (4\pi a)\frac{B_3\left(1+\frac{b}{4a}\right)}{3}x + \mathcal{O}_{\frac{x}{y}}(y^2)\right] - \left[\frac{I_v}{\sqrt{y}} - (4\pi a)\frac{B_3\left(\frac{1}{2}+\frac{b}{4a}\right)}{3}x + \mathcal{O}_{\frac{x}{y}}(y^2)\right] \\
&= -\frac{b(2a+b)}{8}\pi x + \mathcal{O}_{\frac{x}{y}}(y^2)
\end{aligned}$$

where $v_s(t) = e^{-4\pi a t^2}\sin(4\pi a s t^2)$ and $|I_v| = \left|\int_0^\infty v_s(t)\,dt\right| < \infty$. Together with the assumption $|x| \leqslant y$, we get

$$g_{a,b}(\tau) = -\frac{1}{2} - \frac{b(2a+b)}{8a}(\pi i\tau) + \mathcal{O}(y^2).$$

Therefore, by considering the Taylor expansion of $e^{-\frac{b^2}{4a}\pi i\tau}$,

$$\begin{aligned}
f_{a,b}(\tau) &= \left(1 - \frac{b^2}{4a}\pi i\tau + \mathcal{O}(y^2)\right)\left(-\frac{1}{2} - \frac{b(2a+b)}{8a}(\pi i\tau) + \mathcal{O}(y^2)\right) \\
&= -\frac{1}{2} + \frac{b}{8}(-2\pi i\tau) + \mathcal{O}(y^2),
\end{aligned}$$

as $y \to 0+$ with $|x| \leqslant y$. □

Next, we consider the behavior of $S_{a,b}(q)$ away from $q = 1$.

Proposition 11. *For* $y = \dfrac{1}{2\sqrt{6N}}$ *with* $N > 0$ *and* $y \leqslant |x| \leqslant \frac{1}{2}$, *we have*

$$|S_{a,b}(q)| \ll N^{1/2}.$$

Proof. For $a > 0$ and $a + b > 0$, bounding each term in $S_{a,b}(q)$ gives

$$|S_{a,b}(q)| \leqslant 1 + 2\sum_{n=1}^{\infty} |q|^{n/2} \ll N^{1/2}.$$ □

The following two corollaries describe the behavior of the generating function $F_{a,b}(q)$ near $q = 1$ and away from $q = 1$, respectively.

Corollary 12. *Assume* $y = \dfrac{1}{2\sqrt{6N}}$ *and* $|x| \leqslant y$. *As* $N \to \infty$,

$$F_{a,b}(q) = \left(\frac{b}{4}\right)\frac{e^{\frac{\pi i}{12\tau}}}{\sqrt{2\pi}}(-2\pi i\tau)^{3/2} + \mathcal{O}\left(N^{-5/4}e^{\pi\sqrt{\frac{N}{6}}}\right).$$

Proof. From the asymptotic expansion (3.8) in [4]

$$\frac{1}{(q;q)_\infty} = \sqrt{-i\tau}e^{\frac{\pi i}{12\tau}}\left(1+\frac{2\pi i\tau}{24}+\mathcal{O}(N^{-1})\right),$$

we derive

$$\begin{aligned}\frac{1}{(q;q)_\infty}S_{a,b}(q) &= \sqrt{-i\tau}e^{\frac{\pi i}{12\tau}}\left(1+\frac{2\pi i\tau}{24}+\mathcal{O}(N^{-1})\right)\left(\frac{b}{4}(-2\pi i\tau)+\mathcal{O}(N^{-1}))\right)\\ &= \sqrt{-i\tau}e^{\frac{\pi i}{12\tau}}\left(\frac{b}{4}(-2\pi i\tau)+\mathcal{O}(N^{-1})\right)\end{aligned}$$

by combining with Proposition 9. □

Corollary 13. *If* $y = \dfrac{1}{2\sqrt{6N}}$ *with* $N > 0$ *and* $y \leqslant |x| \leqslant \frac{1}{2}$*,*

$$|F_{a,b}(q)| \ll e^{\pi\sqrt{\frac{N}{6}}-\frac{\sqrt{6N}}{5\pi}}.$$

Proof. Proposition 11 together with the bound from [5, Lemma 3.5]

$$\frac{1}{|(q;q)_\infty|} \ll \sqrt{y}\exp\left[\frac{1}{y}\left(\frac{\pi}{12}-\frac{1}{2\pi}\left(1-\frac{1}{\sqrt{2}}\right)\right)\right]$$

gives the corollary. □

Now, we use the Circle Method with the results on $F_{a,b}(q)$ to see the sign pattern of $\alpha_{a,b}(N)$. By Cauchy's Theorem, we find

$$\alpha_{a,b}(N) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{F_{a,b}(q)}{q^{N+1}}\,dq = \int_{-1/2}^{1/2}F_{a,b}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}\,dx,$$

where $\mathcal{C} = \{|q| = e^{-\frac{\pi}{\sqrt{6N}}}\}$. We separate this integral into two integrals,

$$I' = \int_{|x|\leqslant\frac{1}{2\sqrt{6N}}}F_{a,b}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}\,dx$$

and

$$I'' = \int_{\frac{1}{2\sqrt{6N}}\leqslant|x|\leqslant\frac{1}{2}}F_{a,b}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}\,dx.$$

The first integral I' gives the main contribution to the coefficient $\alpha_{a,b}(N)$.

Proposition 14. *As* $N\to\infty$*,*

$$I' = b\frac{\pi^2}{24\sqrt[4]{24}}N^{-5/4}I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right)+\mathcal{O}\left(N^{-7/4}e^{\pi\sqrt{\frac{2N}{3}}}\right),$$

where $I_s(z)$ *is the modified Bessel function of the first kind.*

Proof. By Corollary 12 with the change of variable $u = 2\sqrt{6N}x$, we deduce

$$\begin{aligned} I' &= \frac{1}{2\sqrt{6N}} \int_{-1}^{1} F_{a,b}\left(e^{\frac{\pi}{\sqrt{6N}}(-1+iu)}\right) e^{\pi\sqrt{\frac{N}{6}}(1-iu)}\, du \\ &= \frac{1}{2\sqrt{6N}} \int_{-1}^{1} \left[\frac{b}{4\sqrt{2\pi}} \left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{3/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}+(1-iu)\right)} + \mathcal{O}\left(N^{-5/4} e^{2\pi\sqrt{\frac{N}{6}}}\right)\right] du \\ &= \frac{b}{4\sqrt{2\pi}} \left(\frac{\pi}{\sqrt{6N}}\right)^{5/2} P_{3/2} + \mathcal{O}\left(N^{-7/4} e^{\pi\sqrt{\frac{2N}{3}}}\right), \end{aligned}$$

where

$$P_s := \frac{1}{2\pi i} \int_{1-i}^{1+i} v^s e^{\pi\sqrt{\frac{N}{6}}\left(v+\frac{1}{v}\right)}\, dv.$$

Lemma 4.2 in [4] shows that

$$P_s = I_{-s-1}\left(\pi\sqrt{\frac{2N}{3}}\right) + \mathcal{O}\left(e^{\frac{3\pi}{2}\sqrt{\frac{N}{6}}}\right),$$

which completes the proof. □

The next proposition shows that the second integral I'' is smaller than the error term in the main asymptotic formula of I'.

Proposition 15. *As* $N \to \infty$,

$$|I''| \ll e^{2\pi\sqrt{\frac{N}{6}} - \frac{\sqrt{6N}}{5\pi}}$$

Proof. By Corollary 13,

$$|I''| \leqslant \int_{\frac{1}{2\sqrt{6N}} \leqslant |x| \leqslant \frac{1}{2}} \left| F_{a,b}\left(e^{-\pi\sqrt{\frac{N}{6}}+2\pi i x}\right) e^{\pi\sqrt{\frac{N}{6}} - 2\pi i N x} \right| dx \ll e^{\pi\sqrt{\frac{N}{6}}} e^{\pi\sqrt{\frac{N}{6}} - \frac{\sqrt{6N}}{5\pi}}. \qquad \square$$

In summary, we have obtained the following asymptotic formula for the coefficient $\alpha_{a,b}(N)$.

Corollary 16. *For an positive integer a and an integer b with $a+b>0$ and $a \equiv b \pmod 2$, as* $N \to \infty$,

$$\alpha_{a,b}(N) = b\frac{\pi^2}{24\sqrt[4]{24}} N^{-5/4} I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + \mathcal{O}\left(N^{-7/4} e^{\pi\sqrt{\frac{2N}{3}}}\right).$$

Now, we are ready to prove Theorem 5. Since

$$I_s(z) \sim \frac{e^z}{\sqrt{2\pi z}},$$

the first part of Theorem 5 follows immediately from Corollary 16.

For the second part of Theorem 5, note that

$$\sum_{n=0}^{\infty} \alpha_{2k,0}(n)q^n = \frac{(q^k;q^{2k})_\infty^2 (q^{2k};q^{2k})_\infty}{(q;q)_\infty},$$

where we set $a = 2k > 2$ and have applied Jacobi's triple product identity [1, Theorem 2.8]. From this, we deduce that

$$\begin{aligned} \frac{(q^k;q^{2k})_\infty^2 (q^{2k};q^{2k})_\infty}{(q;q)_\infty} &= \frac{(q^k;q^{2k})_\infty^2 (q^{2k};q^{2k})_\infty^3}{(q)_\infty (q^{2k};q^{2k})_\infty^2} \\ &= \frac{(q^k;q^k)_\infty^2}{(q;q)_\infty (q^{2k};q^{2k})_\infty} \\ &= \frac{(q^k;q^k)_\infty^k}{(q;q)_\infty (q^k;q^k)_\infty^{k-2} (q^{2k};q^{2k})_\infty}. \end{aligned} \tag{2}$$

Since $\frac{(q^k;q^k)_\infty^k}{(q;q)_\infty}$ is a generating function for the number of k-core partition, it is clear that q-expansion of the above infinite product is nonnegative. By the result of A. Granvile and K. Ono [6], we know that there is a k-core partition of n if $k \geqslant 4$. Thus, when $k \geqslant 4$, the q-expansion of (2) has positive coefficients. When $k = 3$, since the partitions 1 and $1+1$ are 3-core partitions and since $\frac{1}{(q^3;q^3)_\infty} = \sum_{n=0}^{\infty} p(n)q^{3n}$, the coefficients of q-expansion of (2) are always positive. Finally, when $k = 2$, we see that (2) is

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty}.$$

Note that the partitions 1, $1+2$, and $1+2+3$ are 2-core partitions and $\frac{1}{(q^4;q^4)_\infty} = \sum_{n=0}^{\infty} p(n)q^{4n}$. Therefore, the n-th coefficient of q-expansion has positive coefficient provided $n \equiv 0, 1, 3, 6$. Hence, the q-expansion of (2) has positive coefficients except $n = 2$. This completes the proof of the second part of Theorem 5.

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