Some spectral properties of uniform hypergraphs

Jiang Zhou*
College of Science
College of Computer Science and Technology
Harbin Engineering University
Harbin, PR China
zhoujiang04113112@163.com

Lizhu Sun
School of Science
Harbin Institute of Technology
Harbin, PR China
sunlizhu678876@126.com

Wenzhe Wang
College of Science
Harbin Engineering University
Harbin, PR China
690564734@qq.com

Changjiang Bu
College of Science
Harbin Engineering University
Harbin, PR China
buchangjiang@hrbeu.edu.cn

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Abstract

For a k-uniform hypergraph H, we obtain some trace formulas for the Laplacian tensor of H, which imply that $\sum_{i=1}^{n} d_i^s$ $(s = 1, \ldots, k)$ is determined by the Laplacian spectrum of H, where $d_1, \ldots, d_n$ is the degree sequence of H. Using trace formulas for the Laplacian tensor, we obtain expressions for some coefficients of the Laplacian polynomial of a regular hypergraph. We give some spectral characterizations of odd-bipartite hypergraphs, and give a partial answer to a question posed by Shao et al (2014). We also give some spectral properties of power hypergraphs, and show that a conjecture posed by Hu et al (2013) holds under certain conditons.

Keywords: Hypergraph eigenvalue; Adjacency tensor; Laplacian tensor; Signless Laplacian tensor; Power hypergraph

1 Introduction

Recently, the research on spectral theory of hypergraphs has attracted extensive attention [1,5-8,11,13,14,16-18]. We first introduce some necessary concepts and notations. For a

*Corresponding author.

positive integer n, let $[n] = \{1, \ldots, n\}$. An order k dimension n tensor $\mathcal{A} = (a_{i_1 \cdots i_k}) \in \mathbb{C}^{n\times\cdots\times n}$ is a multidimensional array with n^k entries, where $i_j \in [n]$, $j = 1, \ldots, k$. We sometimes write $a_{i_1\cdots i_k}$ as $a_{i_1\alpha}$, where $\alpha = i_2 \cdots i_k$. When $k = 1$, $\mathcal{A}$ is a column vector of dimension n. When $k = 2$, $\mathcal{A}$ is an $n \times n$ matrix. The *unit tensor* of order $k \geqslant 2$ and dimension n is a diagonal tensor $\mathcal{I}_n = (\delta_{i_1 i_2 \cdots i_k})$ such that $\delta_{i_1 i_2\cdots i_k} = 1$ if $i_1 = i_2 = \cdots = i_k$, and $\delta_{i_1 i_2\cdots i_k} = 0$ otherwise. In [15], Shao defined the following product of tensors, which is a generalization of the matrix multiplication.

Definition 1. [15] Let $\mathcal{A}$ and $\mathcal{B}$ be order $m \geqslant 2$ and order $k \geqslant 1$, dimension n tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m - 1)(k - 1) + 1$ and dimension n with entries

$$c_{i\alpha_1\ldots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m\in[n]} a_{ii_2\ldots i_m} b_{i_2\alpha_1} \cdots b_{i_m\alpha_{m-1}} \ (i \in [n],\ \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}).$$

Let $\mathcal{A}$ be an order $k \geqslant 2$ dimension n tensor, and let $x = (x_1, \ldots, x_n)^\top$. From Definition 1, the product $\mathcal{A}x$ is a vector in $\mathbb{C}^n$ whose i-th component is (see Example 1.1 in [15])

$$(\mathcal{A}x)_i = \sum_{i_2,\ldots,i_k\in[n]} a_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k}.$$

The concept of tensor eigenvalues was posed in [9, 12]. If there exists a nonzero vector $x \in \mathbb{C}^n$ such that $\mathcal{A}x = \lambda x^{[k-1]}$, then λ is called an *eigenvalue* of $\mathcal{A}$, x is an *eigenvector* of λ, where $x^{[k-1]} = (x_1^{k-1}, \ldots, x_n^{k-1})^\top$. The *determinant* of $\mathcal{A}$, denoted by $\det(\mathcal{A})$, is the resultant of the system of polynomials $f_i(x_1, \ldots, x_n) = (\mathcal{A}x)_i$ $(i = 1, \ldots, n)$. The *characteristic polynomial* of $\mathcal{A}$ is defined as $\phi_{\mathcal{A}}(\lambda) = \det(\lambda\mathcal{I}_n - \mathcal{A})$, where $\mathcal{I}_n$ is the unit tensor of order k and dimension n. It is known that eigenvalues of $\mathcal{A}$ are exactly roots of $\phi_{\mathcal{A}}(\lambda)$ [12]. The multiset of roots of $\phi_{\mathcal{A}}(\lambda)$ (counting multiplicities) is the *spectrum* of $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$. The maximal modulus of eigenvalues of $\mathcal{A}$ is called the *spectral radius* of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$. More details on eigenvalues and characteristic polynomials of tensors can be found in [4, 12].

A hypergraph H is called *k-uniform* if each edge of H contains exactly k distinct vertices. Let $V(H)$ and $E(H)$ denote the vertex set and the edge set of H, respectively. In [13], Qi defined the Laplacian and the signless Laplacian tensor of a uniform hypergraph as follows.

Definition 2. [7, 13] The *adjacency tensor* of a k-uniform hypergraph H, denoted by $\mathcal{A}_H$, is an order k dimension $|V(H)|$ tensor with entries

$$a_{i_1 i_2\cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } i_1 i_2 \cdots i_k \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}_H$ be an order k dimension $|V(H)|$ diagonal tensor whose diagonal entries are vertex degrees of H. The tensors $\mathcal{L}_H = \mathcal{D}_H - \mathcal{A}_H$ and $\mathcal{Q}_H = \mathcal{D}_H + \mathcal{A}_H$ are the *Laplacian* *tensor* and the *signless Laplacian tensor* of H, respectively. Eigenvalues of $\mathcal{A}_H$, $\mathcal{L}_H$ and $\mathcal{Q}_H$ are called eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of H, respectively. Characteristic polynomials of $\mathcal{L}_H$ and $\mathcal{Q}_H$ are called Laplacian polynomial and signless Laplacian polynomial of H, respectively.

This paper is organized as follows. In Section 2, we give some trace formulas for the Laplacian tensor of a uniform hypergraph, and obtain expressions for some coefficients of the Laplacian polynomial of a regular hypergraph. In Section 3, we give some spectral characterizations of odd-bipartite hypergraphs. In Section 4, we give some spectral properties of power hypergraphs.

2 Laplacian spectra and degree sequence of hypergraphs

Traces of tensors are very useful in the study of spectral theory of tensors. The *d-th order trace* of an order $k \geqslant 2$ dimension n tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$ is defined as [1, 4, 10]

$$Tr_d(\mathcal{T}) = (k-1)^{n-1} \sum_{d_1+\cdots+d_n=d} \prod_{i=1}^{n} \frac{1}{(d_i(k-1))!} \left(\sum_{y \in [n]^{k-1}} t_{iy} \frac{\partial}{\partial a_{iy}} \right)^{d_i} \mathrm{tr}(A^{d(k-1)}),$$

where $A = (a_{ij})$ is an $n \times n$ auxiliary matrix, and $\frac{\partial}{\partial a_{iy}} = \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_k}}$ if $y = i_2 \cdots i_k$. The codegree d coefficient of the characteristic polynomial of $\mathcal{T}$ can be expressed in terms of $Tr_1(\mathcal{T}), \ldots, Tr_d(\mathcal{T})$ (see [4, Theorem 6.3]). It is also known that $Tr_t(\mathcal{T}) = \sum_{\lambda \in \sigma(\mathcal{T})} \lambda^t$ for any $t \in [n(k-1)^{n-1}]$ (see [4, Theorem 6.10]). Hence $Tr_d(\mathcal{T})$ is an important invariant in the spectral theory of tensors.

Shao et al [16] give a graph theoretical formula for $Tr_d(\mathcal{T})$. In order to describe this formula, we introduce some notations in [16]. For an integer $d > 0$, we define

$$\mathcal{F}_d = \{(i_1\alpha_1, \ldots, i_d\alpha_d) | 1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant n;\ \alpha_1, \ldots, \alpha_d \in [n]^{k-1}\}.$$

For $F = (i_1\alpha_1, \ldots, i_d\alpha_d) \in \mathcal{F}_d$ and an order $k \geqslant 2$ dimension n tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$, we write $\pi_F(\mathcal{T}) = \prod_{j=1}^{d} t_{i_j\alpha_j}$. Let $p_i(F)$ be the total number of times that the index i appears in F. If $p_i(F)$ is a multiple of k for any $i \in [n]$, then F is called *k-valent*.

Definition 3. [16] Let $F = (i_1\alpha_1, \ldots, i_d\alpha_d) \in \mathcal{F}_d$, where $i_j\alpha_j \in [n]^k$, $j = 1, \ldots, d$. Then
(1) Let $E(F) = \bigcup_{j=1}^{d} E_j(F)$, where $E_j(F)$ is the arc multi-set

$$E_j(F) = \{(i_j, v_1), \ldots, (i_j, v_{k-1})\}$$

if $\alpha_j = v_1 \cdots v_{k-1}$.
(2) Let $b(F)$ be the product of the factorials of the multiplicities of all the arcs of $E(F)$.
(3) Let $c(F)$ be the product of the factorials of the outdegrees of all the vertices in the arc multi-set $E(F)$.
(4) Let $W(F)$ be the set of all closed walks W with the arc multi-set $E(F)$.

Shao et al give a graph theoretical formula for $Tr_d(\mathcal{T})$ as follows (see equation (3.5) in [16]).

Lemma 4. [16] *Let $\mathcal{T} = (T_{i_1\cdots i_k})$ be an order $k \geqslant 2$ dimension n tensor. Then*

$$Tr_d(\mathcal{T}) = (k-1)^{n-1} \sum_{F \in \mathcal{F}'_d} \frac{b(F)}{c(F)} \pi_F(\mathcal{T}) |W(F)|,$$

where $\mathcal{F}'_d = \{F | F \in \mathcal{F}_d,\ F \text{ is k-valent}\}$.

For a k-uniform hypergraph H, Cooper and Dutle [1] proved that $Tr_d(\mathcal{A}_H) = 0$ for $d \in [k-1]$. We give some trace formulas for the Laplacian (signless Laplacian) tensor of uniform hypergraphs as follows.

Theorem 5. *Let H be a k-uniform hypergraph with degree sequence $d_1, \dots, d_n$. Then*

$$\begin{aligned} Tr_t(\mathcal{L}_H) &= Tr_t(\mathcal{Q}_H) = (k-1)^{n-1} \sum_{i=1}^n d_i^t,\ t = 1, \dots, k-1, \\ Tr_k(\mathcal{L}_H) &= (-1)^k k^{k-1} (k-1)^{n-k} |E(H)| + (k-1)^{n-1} \sum_{i=1}^n d_i^k, \\ Tr_k(\mathcal{Q}_H) &= k^{k-1} (k-1)^{n-k} |E(H)| + (k-1)^{n-1} \sum_{i=1}^n d_i^k. \end{aligned}$$

Proof. By Lemma 4, we have

$$Tr_t(\mathcal{L}_H) = (k-1)^{n-1} \sum_{F \in \mathcal{F}'_t} \frac{b(F)}{c(F)} \pi_F(\mathcal{L}_H) |W(F)|, \tag{1}$$

where $\mathcal{F}'_t = \{F | F \in \mathcal{F}_t,\ F \text{ is k-valent}\}$. For $F = (i_1\alpha_1, \dots, i_t\alpha_t) \in \mathcal{F}_t$, if $\pi_F(\mathcal{L}_H) = \prod_{j=1}^t (\mathcal{L}_H)_{i_j\alpha_j} \neq 0$, then $i_j\alpha_j = i_j i_j \cdots i_j \in [n]^k$ or $i_j\alpha_j \in E(H)$ for any $1 \leqslant j \leqslant t$.

Let $F \in \mathcal{F}_t$ satisfies $\pi_F(\mathcal{L}_H) \neq 0$. If $t < k$, then F is k-valent if and only if $F = (i_1 i_1 \cdots i_1, \dots, i_t i_t \cdots i_t)$. In this case, $|W(F)| \neq 0$ if and only if $i_1 = \cdots = i_t$. Let $F_i = (ii\cdots i, \dots, ii\cdots i) \in \mathcal{F}'_t$ $(t < k)$. From Eq. (1) and Definition 3, we have

$$\begin{aligned} Tr_t(\mathcal{L}_H) &= (k-1)^{n-1} \sum_{i=1}^n \frac{b(F_i)}{c(F_i)} \pi_{F_i}(\mathcal{L}_H) |W(F_i)| \\ &= (k-1)^{n-1} \sum_{i=1}^n \frac{(t(k-1))!}{(t(k-1))!} d_i^t = (k-1)^{n-1} \sum_{i=1}^n d_i^t. \end{aligned}$$

Similar with the above procedure, we can also get $Tr_t(\mathcal{Q}_H) = (k-1)^{n-1} \sum_{i=1}^n d_i^t$, $t = 1, \dots, k-1$.

Let $F \in \mathcal{F}_k$ satisfies $\pi_F(\mathcal{L}_H) \neq 0$. Then F is k-valent and $|W(F)| \neq 0$ if and only if $F = (ii\cdots i, \dots, ii\cdots i)$ or $F = (i_1\alpha_1, \dots, i_k\alpha_k)$, where $i_1\alpha_1, \dots, i_k\alpha_k$ correspond to the same edge $i_1 i_2 \cdots i_k \in E(H)$. Let $F_i = (ii\cdots i, \dots, ii\cdots i) \in \mathcal{F}'_k$. From Eq. (1) and Definition 3, we have

$$\begin{aligned} Tr_k(\mathcal{L}_H) &= (-1)^k Tr_k(\mathcal{A}_H) + (k-1)^{n-1} \sum_{i=1}^{n} \frac{b(F_i)}{c(F_i)} \pi_{F_i}(\mathcal{L}_H) |W(F_i)| \\ &= (-1)^k Tr_k(\mathcal{A}_H) + (k-1)^{n-1} \sum_{i=1}^{n} \frac{(k(k-1))!}{(k(k-1))!} d_i^k \\ &= (-1)^k Tr_k(\mathcal{A}_H) + (k-1)^{n-1} \sum_{i=1}^{n} d_i^k. \end{aligned}$$

From the proof of [1, Theorem 3.15], we have $Tr_k(\mathcal{A}_H) = k^{k-1}(k-1)^{n-k}|E(H)|$. Hence

$$Tr_k(\mathcal{L}_H) = (-1)^k k^{k-1}(k-1)^{n-k}|E(H)| + (k-1)^{n-1} \sum_{i=1}^{n} d_i^k.$$

Similar with the above procedure, we can also get

$$Tr_k(\mathcal{Q}_H) = k^{k-1}(k-1)^{n-k}|E(H)| + (k-1)^{n-1} \sum_{i=1}^{n} d_i^k.$$

□

Remark. Note that traces of a tensor are determined by its spectrum [3, Theorem 6.3]. For a k-uniform hypergraph H, by Theorem 5, we know that $\sum_{i=1}^{n} d_i^s$ $(s = 1, \dots, k)$ is determined by the Laplacian (signless Laplacian) spectrum of H, where $d_1, \dots, d_n$ is the degree sequence of H.

Let $p_t(\mathcal{M})$ denote the codegree t coefficient of the characteristic polynomial of a tensor $\mathcal{M}$.

Lemma 6. *Let $\mathcal{M}$ be an order $k \geqslant 2$ dimension n tensor. Then*

$$t! p_t(\mathcal{M}) = \det \begin{pmatrix} -Tr_t & Tr_1 & Tr_2 & \cdots & Tr_{t-1} \\ -Tr_{t-1} & t-1 & Tr_1 & \cdots & Tr_{t-2} \\ -Tr_{t-2} & 0 & t-2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Tr_1 \\ -Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $Tr_t = Tr_t(\mathcal{M})$, $t \in [n(k-1)^{n-1}]$.

Proof. From [4, Theorem 6.10], we have

$$\begin{pmatrix} t & Tr_1 & Tr_2 & \cdots & Tr_{t-1} \\ 0 & t-1 & Tr_1 & \cdots & Tr_{t-2} \\ 0 & 0 & t-2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Tr_1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} p_t(\mathcal{M}) \\ p_{t-1}(\mathcal{M}) \\ \vdots \\ p_2(\mathcal{M}) \\ p_1(\mathcal{M}) \end{pmatrix} = \begin{pmatrix} -Tr_t \\ -Tr_{t-1} \\ \vdots \\ -Tr_2 \\ -Tr_1 \end{pmatrix}.$$

We can obtain the expression of $t!p_t(\mathcal{M})$ from Cramer's rule. $\square$

A uniform hypergraph H is called *d-regular* if each vertex of H has degree d. The following are some coefficients of the Laplacian (signless Laplacian) polynomial of regular hypergraphs.

Theorem 7. *Let H be a d-regular k-uniform hypergraph with n vertices. Then*

$$\begin{aligned} p_t(\mathcal{L}_H) &= p_t(\mathcal{Q}_H) = (-1)^t d^t \binom{n(k-1)^{n-1}}{t},\ t = 1, \dots, k-1, \\ p_k(\mathcal{L}_H) &= (-1)^{k+1} k^{k-3} (k-1)^{n-k} nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}, \\ p_k(\mathcal{Q}_H) &= -k^{k-3}(k-1)^{n-k} nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}. \end{aligned}$$

Proof. By Lemma 6, we have

$$t!p_t(\mathcal{L}_H) = \det \begin{pmatrix} -Tr_t & Tr_1 & Tr_2 & \cdots & Tr_{t-1} \\ -Tr_{t-1} & t-1 & Tr_1 & \cdots & Tr_{t-2} \\ -Tr_{t-2} & 0 & t-2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Tr_1 \\ -Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{2}$$

where $Tr_t = Tr_t(\mathcal{L}_H)$. Since H is d-regular, by Theorem 5, we have $Tr_i = dTr_{i-1} = n(k-1)^{n-1}d^i$, $i = 2, \dots, k-1$. If $t < k$, then by Eq. (2), we have

$$\begin{aligned} t!p_t(\mathcal{L}_H) &= \det \begin{pmatrix} 0 & Tr_1 & Tr_2 & \cdots & Tr_{t-1} \\ 0 & t-1 & Tr_1 & \cdots & Tr_{t-2} \\ \vdots & 0 & t-2 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & Tr_1 \\ d-Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & Tr_1-(t-1)d & 0 & \cdots & 0 \\ 0 & t-1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & Tr_1-2d & 0 \\ 0 & \vdots & \ddots & 2 & Tr_1 \\ d-Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix} \\ &= (-1)^t \prod_{i=0}^{t-1} (Tr_1 - id). \end{aligned}$$

Since $Tr_1 = n(k-1)^{n-1}d$, we have

$$\begin{aligned} p_t(\mathcal{L}_H) &= (-1)^t \frac{\prod_{i=0}^{t-1}(n(k-1)^{n-1}d - id)}{t!} = (-1)^t d^t \frac{\prod_{i=0}^{t-1}(n(k-1)^{n-1} - i)}{t!} \\ &= (-1)^t d^t \binom{n(k-1)^{n-1}}{t}. \end{aligned}$$

Similar with the above procedure, we can also get

$$p_t(\mathcal{Q}_H) = (-1)^t d^t \binom{n(k-1)^{n-1}}{t},\ t = 1, \ldots, k-1.$$

Since H is d-regular, by Theorem 5, we have $Tr_k = (-1)^k k^{k-2}(k-1)^{n-k}nd + dTr_{k-1}$ and $Tr_i = dTr_{i-1} = n(k-1)^{n-1}d^i$, $i = 2, \ldots, k-1$. From Eq. (2), we have

$$\begin{aligned}
k!p_k(\mathcal{L}_H) &= \det \begin{pmatrix} (-1)^{k+1}k^{k-2}(k-1)^{n-k}nd & Tr_1 & Tr_2 & \cdots & Tr_{k-1} \\ 0 & k-1 & Tr_1 & \cdots & Tr_{k-2} \\ \vdots & 0 & k-2 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & Tr_1 \\ d - Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix} \\
&= \det \begin{pmatrix} (-1)^{k+1}k^{k-2}(k-1)^{n-k}nd & Tr_1 - (k-1)d & 0 & \cdots & 0 \\ 0 & k-1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & Tr_1 - 2d & 0 \\ 0 & \vdots & \ddots & 2 & Tr_1 \\ d - Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix} \\
&= (-1)^{k+1}k^{k-2}(k-1)^{n-k}(k-1)!nd + (-1)^k d^k \prod_{i=0}^{k-1}(n(k-1)^{n-1} - i).
\end{aligned}$$

$$p_k(\mathcal{L}_H) = (-1)^{k+1}k^{k-3}(k-1)^{n-k}nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}.$$

Similar with the above procedure, we can also get

$$p_k(\mathcal{Q}_H) = -k^{k-3}(k-1)^{n-k}nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}.$$

□

3 Eigenvalues and odd-bipartite hypergraphs

A k-uniform hypergraph H is called *odd-bipartite*, if there exists a proper subset V_1 of $V(H)$ such that each edge of H contains exactly odd number of vertices in V_1 [6, 17]. Spectral characterizations of odd-bipartite hypergraphs will be investigated in this section. We first give some auxiliary lemmas. The following lemma can be obtained from equation (2.1) in [15].

Lemma 8. *Let $\mathcal{A} = (a_{i_1 \cdots i_k})$ be an order $k \geqslant 2$ dimension n tensor, and let $P = (p_{ij}), Q = (q_{ij})$ be $n \times n$ matrices. Then*

$$(P\mathcal{A}Q)_{i_1 \cdots i_k} = \sum_{j_1, \ldots, j_k \in [n]} a_{j_1 \cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}.$$

Lemma 9. [6] *Let H be a connected k-uniform hypergraph. A nonzero vector x is an eigenvector of $\mathcal{Q}_H$ corresponds to the zero eigenvalue if and only if there exist nonzero $\gamma \in \mathbb{C}$ and integers α_i such that $x_i = \gamma \exp(\frac{2\alpha_i \pi}{k}\sqrt{-1})$ for each $i \in V(H)$, and*

$$\sum_{j \in e} \alpha_j = \sigma_e k + \frac{k}{2}$$

for some integer σ_e associated with each $e \in E(H)$.

Weakly irreducible tensors are defined in [3]. It is known that a k-uniform hypergraph H is connected if and only if $\mathcal{A}_H$ is weakly irreducible [11].

Lemma 10. [17, 19] *Let $\mathcal{A}$ be an order k dimension n nonnegative weakly irreducible tensor. If $\rho(\mathcal{A})\exp(\theta\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$, then there exists a diagonal matrix U with unit diagonal entries such that*

$$\mathcal{A} = \exp(-\theta\sqrt{-1})U^{-(k-1)}\mathcal{A}U.$$

For a tensor $\mathcal{T}$, let $H\sigma(\mathcal{T}) = \{\lambda | \lambda \in \sigma(\mathcal{T}),\ \lambda \text{ has a real eigenvector}\}$. For a connected k-uniform hypergraph G, Shao et al [17] proved that

$$H\sigma(\mathcal{L}_G) = H\sigma(\mathcal{Q}_G) \Longrightarrow \sigma(\mathcal{L}_G) = \sigma(\mathcal{Q}_G).$$

Shao et al wish to know whether the reverse implication is true. We show that the reverse is true when k is not divisible by 4.

Theorem 11. *Let G be a connected k-uniform hypergraph, and k is not divisible by 4. Then the following are equivalent:*
(1) k is even and H is odd-bipartite.
(2) $H\sigma(\mathcal{L}_G) = H\sigma(\mathcal{Q}_G)$.
(3) $\sigma(\mathcal{L}_G) = \sigma(\mathcal{Q}_G)$.
(4) 0 is a signless Laplacian eigenvalue of G.

Proof. From [17, Theorem 2.2], we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Since 0 is always an eigenvalue of $\mathcal{L}_G$ (see [13]), we have (3) $\Rightarrow$ (4). Next we prove that (4) $\Rightarrow$ (1).

If 0 is an eigenvalue of $\mathcal{Q}_G$, then by Lemma 9, there exists a vertex labeling $f : V(G) \to [k]$ such that

$$\sum_{i \in e} f(i) \equiv \frac{k}{2} \pmod{k}$$

for each $e \in E(G)$. Hence k is even. Since k is not divisible by 4, we know that $\frac{k}{2}$ is odd. So $\sum_{i \in e} f(i)$ is odd for each $e \in E(G)$. Let $V_1 = \{u | u \in V(G),\ f(u) \text{ is odd}\}$. For any $e \in E(G)$, since $\sum_{i \in e} f(i)$ is odd, e contains exactly odd number of vertices in V_1. Hence G is odd-bipartite. □

When $k = 2$, Theorem 11 becomes a classic result in spectral graph theory, i.e., a connected graph G is bipartite if and only if 0 is a signless Laplacian eigenvalue of G. It is also well known that a connected graph G is bipartite if and only if $-\rho(\mathcal{A}_G)$ is an eigenvalue of G. We generalize this result as follows.

Theorem 12. *Let H be a connected k-uniform hypergraph, and k is not divisible by 4. Then the following are equivalent:*
(1) k is even and H is odd-bipartite.
(2) $-\rho(\mathcal{A}_H)$ is an eigenvalue of H.

Proof. From [17, Theorem 2.3], we have (1) $\Rightarrow$ (2). If (2) holds, then by Lemma 10, there exists a diagonal matrix U with unit diagonal entries such that $\mathcal{A}_H = -U^{-(k-1)}\mathcal{A}_H U$. By Lemma 8, we have

$$a_{i_1 i_2 \cdots i_k} = -a_{i_1 i_2 \cdots i_k} u_{i_1}^{-(k-1)} u_{i_2} \cdots u_{i_k},$$

where u_{i_j} is the diagonal entry of U corresponds to vertex i_j $(j = 1, \ldots, k)$. For any edge $e = i_1 i_2 \cdots i_k \in E(H)$, we get

$$u_{i_1}^{-(k-1)} u_{i_2} \cdots u_{i_k} = -1,\ u_{i_1} u_{i_2} \cdots u_{i_k} = -u_{i_1}^k.$$

Similarly, we have $u_{i_1} u_{i_2} \cdots u_{i_k} = -u_{i_1}^k = \cdots = -u_{i_k}^k$. Since $u_{i_1}, \ldots, u_{i_k}$ are unit complex number, there exist θ and integers $\alpha_{i_1}, \ldots, \alpha_{i_k}$ such that $u_{i_j} = \exp(\frac{2\pi\alpha_{i_j}+\theta}{k}\sqrt{-1})$, $j = 1, \ldots, k$. Then

$$\begin{aligned} u_{i_1} u_{i_2} \cdots u_{i_k} = \exp(\frac{k\theta + 2\pi\sum_{j=1}^k \alpha_{i_j}}{k}\sqrt{-1}) &= -u_{i_1}^k = -\exp(\theta\sqrt{-1}), \\ \exp(\frac{2\pi\sum_{j=1}^k \alpha_{i_j}}{k}\sqrt{-1}) &= -1. \end{aligned}$$

Hence $\sum_{j=1}^k \alpha_{i_j} \equiv \frac{k}{2} \pmod{k}$, k is even. Since k is not divisible by 4, $\sum_{j=1}^k \alpha_{i_j}$ is odd for any edge $e = i_1 i_2 \cdots i_k \in E(H)$. Let $V_1 = \{u | u \in V(H), \alpha_u \text{ is odd}\}$. For any $e = i_1 i_2 \cdots i_k \in E(H)$, since $\sum_{j=1}^k \alpha_{i_j}$ is odd, e contains exactly odd number of vertices in V_1. Hence H is odd-bipartite. □

Let H be a connected k-uniform hypergraph. If 0 is an eigenvalue of $\mathcal{Q}_H$, then by the proof of Theorem 11, we know that there exists a vertex labeling $f : V(H) \to [k]$ such that $\sum_{i \in e} f(i) \equiv \frac{k}{2} \pmod{k}$ for each $e \in E(H)$. We pose the following conjecture.

Conjecture. Let H be a connected k-uniform hypergraph. Then the following are equivalent:
(1) k is even and H is odd-bipartite.
(2) 0 is a signless Laplacian eigenvalue of H.
(3) $-\rho(\mathcal{A}_H)$ is an eigenvalue of H.
(4) There exists a vertex labeling $f : V(H) \to [k]$ such that $\sum_{i \in e} f(i) \equiv \frac{k}{2} \pmod{k}$ for each $e \in E(H)$.

4 Eigenvalues of power hypergraphs

A vertex with degree one is called a *core vertex* [7]. For a k-uniform hypergraph H, if $e \in E(H)$ contains core vertices, then we use $H-e$ to denote a k-uniform sub-hypergraph of H obtained by deleting the edge e and all core vertices in e.

Theorem 13. *Let H be a k-uniform hypergraph, and let $e \in E(H)$ be an edge contains at least two core vertices. If λ is an eigenvalue of $H-e$, then λ is an eigenvalue of H.*

Proof. Suppose that x is an eigenvector of the eigenvalue λ of $H-e$. Let y be a column vector of dimension $|V(H)|$ such that $y_u = x_u$ if $u \in V(H-e)$, and $y_u = 0$ if $u \in V(H)$ is a core vertex in e. Since $\mathcal{A}_{H-e}x = \lambda x^{[k-1]}$, we have $\mathcal{A}_H y = \lambda y^{[k-1]}$. So λ is an eigenvalue of H. □

In [7], Hu et al defined power hypergraphs as follows.

Definition 14. [7] Let G be an ordinary graph (i.e. 2-uniform hypergraph). For any $k \geqslant 3$, the kth power of G, denoted by G^k, is a k-uniform hypergraph with edge set $E(G^k) = \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}\} | e \in E(G)\}$, and vertex set $V(G^k) = V(G) \cup \{i_{e,j} | e \in E(G), j \in [k-2]\}$.

Some examples of power hypergraphs are given in [7, Fig.1]. From Definition 14, we know that each edge of a power hypergraph G^k contains two adjacent vertices in $V(G)$ and $k-2$ core vertices not in $V(G)$.

If H is a connected k-uniform hypergraph, then $\mathcal{A}_H$ and $\mathcal{Q}_H$ are both weakly irreducible [13]. So we obtain the following lemma from [13, Theorem 2.2].

Lemma 15. *Let H be a connected k-uniform hypergraph. If λ is an eigenvalue of $\mathcal{A}_H$ ($\mathcal{Q}_H$) with a positive eigenvector, then $\lambda = \rho(\mathcal{A}_H)$ ($\lambda = \rho(\mathcal{Q}_H)$).*

Theorem 16. *If $\lambda \neq 0$ is an eigenvalue of a graph G, then $\lambda^{\frac{2}{k}}$ is an eigenvalue of G^k. Moreover, $\rho(\mathcal{A}_{G^k}) = \rho(\mathcal{A}_G)^{\frac{2}{k}}$.*

Proof. Suppose that x is an eigenvector of the eigenvalue $\lambda \neq 0$ of graph G. Then $\sum_{j \in N_G(i)} x_j = \lambda x_i$ for any $i \in V(G)$, where $N_G(i)$ is the set of all neighbors of i in G. Let y be a column vector of dimension $|V(G^k)|$ such that $y_u = (x_u)^{\frac{2}{k}}$ if $u \in V(G)$, and $y_u = (\lambda^{-1} x_i x_j)^{\frac{1}{k}}$ if $u \in V(G^k) \setminus V(G)$ is a core vertex in the edge contains two adjacent vertices $i, j \in V(G)$. For any $i \in V(G)$, by $\sum_{j \in N_G(i)} x_j = \lambda x_i$, we have

$$(\mathcal{A}_{G^k} y)_i = \sum_{j \in N_G(i)} (\lambda^{-1} x_i x_j)^{\frac{k-2}{k}} (x_j)^{\frac{2}{k}} = \lambda^{\frac{2}{k}} (x_i)^{\frac{2(k-1)}{k}} = \lambda^{\frac{2}{k}} (y_i)^{k-1}.$$

For any $u \in V(G^k) \setminus V(G)$, we have

$$(\mathcal{A}_{G^k} y)_u = (\lambda^{-1} x_i x_j)^{\frac{k-3}{k}} (x_i)^{\frac{2}{k}} (x_j)^{\frac{2}{k}} = \lambda^{\frac{2}{k}} (\lambda^{-1} x_i x_j)^{\frac{k-1}{k}} = \lambda^{\frac{2}{k}} (y_u)^{k-1}.$$

Hence $\lambda^{\frac{2}{k}}$ is an eigenvalue of G^k with an eigenvector y.

If G is connected and $\lambda = \rho(\mathcal{A}_G)$, then we can choose x as a positive eigenvector of $\rho(\mathcal{A}_G)$. In this case, y is a positive eigenvector of the eigenvalue $\rho(\mathcal{A}_G)^{\frac{2}{k}}$ of G^k. Lemma 15 implies that $\rho(\mathcal{A}_{G^k}) = \rho(\mathcal{A}_G)^{\frac{2}{k}}$ when G is connected.

If G has $r \geqslant 2$ components $G_1, \dots, G_r$, then

$$\rho(\mathcal{A}_{G^k}) = \max\{\rho(\mathcal{A}_{G_1^k}), \dots, \rho(\mathcal{A}_{G_r^k})\} = \max\{\rho(\mathcal{A}_{G_1})^{\frac{2}{k}}, \dots, \rho(\mathcal{A}_{G_r})^{\frac{2}{k}}\} = \rho(\mathcal{A}_G)^{\frac{2}{k}}.$$

□

We can obtain the following result from Theorem 16.

Corollary 17. *For any nontrivial graph G, we have $\lim_{k\to\infty} \rho(\mathcal{A}_{G^k}) = 1$. Moreover, $\{\rho(\mathcal{A}_{G^k})\}$ is a strictly decreasing sequence if $\rho(\mathcal{A}_G) > 1$.*

The following corollary follows from Theorem 13 and 16.

Corollary 18. *If $\lambda \neq 0$ is an eigenvalue of any subgraph of a graph G, then $\lambda^{\frac{2}{k}}$ is an eigenvalue of G^k for $k \geqslant 4$.*

Let P_n and S_n be the path and the star of order n, respectively. The following result was proved by Li et al [8]. Here we give a different proof.

Corollary 19. *Let T be a tree with n vertices. Then*

$$\rho(\mathcal{A}_{P_n^k}) \leqslant \rho(\mathcal{A}_{T^k}) \leqslant \rho(\mathcal{A}_{S_n^k}),$$

where the left equality holds if and only if $T = P_n$, and the right equality holds if and only if $T = S_n$.

Proof. Among all trees with n vertices, P_n is the unique tree with the smallest adjacency spectral radius, and S_n is the unique tree with the largest adjacency spectral radius [2]. By Theorem 16, we have

$$\rho(\mathcal{A}_{P_n^k}) \leqslant \rho(\mathcal{A}_{T^k}) \leqslant \rho(\mathcal{A}_{S_n^k}),$$

where the left equality holds if and only if $T = P_n$, and the right equality holds if and only if $T = S_n$. □

Theorem 20. *If $\alpha \neq 0$ is an eigenvalue of a d-regular graph G, then the roots of $(x - d)(x-1)^{\frac{k-2}{2}} - \alpha = 0$ are signless Laplacian eigenvalues of G^k. Moreover, $\rho(\mathcal{Q}_{G^k})$ is the largest real root of $(x-d)(x-1)^{\frac{k-2}{2}} - d = 0$.*

Proof. Suppose that x is an eigenvector of the eigenvalue $\alpha \neq 0$ of graph G. Then $\sum_{j\in N_G(i)} x_j = \alpha x_i$ for any $i \in V(G)$, where $N_G(i)$ is the set of all neighbors of i in G. Let $\lambda \in \mathbb{C}$ be any number such that $(\lambda - d)(\lambda - 1)^{\frac{k-2}{2}} = \alpha$, then $\lambda \neq 1$. Let y be a column vector of dimension $|V(G^k)|$ such that $y_u = (x_u)^{\frac{2}{k}}$ if $u \in V(G)$, and $y_u = (\lambda - 1)^{-\frac{1}{2}}(x_i x_j)^{\frac{1}{k}}$

if $u \in V(G^k)\setminus V(G)$ is a core vertex in the edge contains two adjacent vertices $i, j \in V(G)$. For any $i \in V(G)$, by $\sum_{j\in N_G(i)} x_j = \alpha x_i$ and $(\lambda - d)(\lambda - 1)^{\frac{k-2}{2}} = \alpha$, we have

$$(\mathcal{Q}_{G^k} y)_i = d(x_i)^{\frac{2(k-1)}{k}} + \sum_{j\in N_G(i)} (\lambda - 1)^{-\frac{k-2}{2}} (x_i x_j)^{\frac{k-2}{k}} (x_j)^{\frac{2}{k}} = \lambda (x_i)^{\frac{2(k-1)}{k}} = \lambda (y_i)^{k-1}.$$

For any $u \in V(G^k) \setminus V(G)$, we have

$$\begin{aligned}(\mathcal{Q}_{G^k} y)_u &= (\lambda - 1)^{-\frac{k-1}{2}} (x_i x_j)^{\frac{k-1}{k}} + (\lambda - 1)^{-\frac{k-3}{2}} (x_i x_j)^{\frac{k-3}{k}} (x_i)^{\frac{2}{k}} (x_j)^{\frac{2}{k}} \\ &= \lambda (\lambda - 1)^{-\frac{k-1}{2}} (x_i x_j)^{\frac{k-1}{k}} = \lambda (y_u)^{k-1}.\end{aligned}$$

Hence λ is a signless Laplacian eigenvalue of G^k with an eigenvector y.

If G is connected and $\alpha = d = \rho(\mathcal{A}_G)$, then we can choose x as a positive eigenvector of $\rho(\mathcal{A}_G)$. In this case, y is a positive eigenvector of the signless Laplacian eigenvalue λ of G^k. Lemma 15 implies that $\rho(\mathcal{Q}_{G^k})$ is the largest real root of $(x-d)(x-1)^{\frac{k-2}{2}} - d = 0$ when G is connected.

If G has $r \geqslant 2$ components $G_1, \ldots, G_r$, then

$$\rho(\mathcal{Q}_{G^k}) = \max\{\rho(\mathcal{Q}_{G_1^k}), \ldots, \rho(\mathcal{Q}_{G_r^k})\}.$$

Since $G_1, \ldots, G_r$ are connected d-regular graphs, we know that $\rho(\mathcal{Q}_{G^k}) = \rho(\mathcal{Q}_{G_1^k}) = \cdots = \rho(\mathcal{Q}_{G_r^k})$ is equal to the largest real root of $(x-d)(x-1)^{\frac{k-2}{2}} - d = 0$. $\square$

The following corollary follows from Theorem 20.

Corollary 21. *For any d-regular graph G, we have $\lim_{k\to\infty} \rho(\mathcal{Q}_{G^k}) = d$. Moreover, $\rho(\mathcal{Q}_{G^k})$ is a strictly decreasing sequence if $d > 1$.*

Remark. In [7, Conjecture 4.1], Hu et al conjectured that $\rho(\mathcal{Q}_{G^k})$ is a strictly decreasing sequence for any graph G and even k. By Corollary 21, this conjecture holds when G is regular of degree $d > 1$.

The proof of the following theorem is similar with that of Theorem 20. So we omit it.

Theorem 22. *If $\alpha \neq 0$ is an eigenvalue of a d-regular graph G, then the roots of $(d - x)(1-x)^{\frac{k-2}{2}} - \alpha = 0$ are Laplacian eigenvalues of G^k.*

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