

On m -closed graphs

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Abstract

A graph is closed when its vertices have a labeling by $[n]$ such that the binomial edge ideal J_G has a quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. In this paper, we generalize this notion and study the so called m -closed graphs. We find equivalent condition to 3-closed property of an arbitrary tree T . Using it, we classify a class of 3-closed trees. The primary decomposition of this class of graphs is also studied.

Keywords: m -closed graph, binomial edge ideal, reduced Gröbner basis, admissible path

1 Introduction and Preliminaries

Suppose G is a simple graph on the vertex set $[n]$ and $R = \mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_n]$ is the polynomial ring over the field \mathbf{k} . The *binomial edge ideal* of G is the ideal

$$J_G = (f_{ij} : \{i, j\} \in E(G) \text{ and } i < j) \subset R,$$

where $f_{ij} = x_i y_j - x_j y_i$. This notion was first introduced in [9] and independently in [13].

Note that any ideal generated by a set of 2-minors of a $2 \times n$ -matrix X of indeterminates may be viewed as the binomial edge ideal of a graph. In [9], the authors compute the reduced Gröbner basis of the binomial edge ideal with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ (we show this order by \prec). In particular, they find the necessary and sufficient conditions in which J_G has a quadratic Gröbner basis. Graphs whose binomial edge ideal has a quadratic Gröbner basis are called closed graphs and the Cohen-Macaulay property of these graphs is studied in [6]. Recently, many authors studied the algebraic properties of some classes of binomial edge ideals. In

particular the regularity and the depth are studied in [1, 6, 7, 10, 12, 15]. But the reduced Gröbner basis obtained in [9] has not been studied in more details.

In this paper, we study the Gröbner basis of J_G where G is a simple graph. We call G an m -closed graph when its vertices can be labeled by $[n]$ such that the elements of the reduced Gröbner basis of J_G have degree at most m , and m is the least integer with this property for G . Note that by this definition, a closed graph is a 2-closed graph.

In Section 2 we study some basic properties of m -closed graphs. In particular, we show that a cycle C_n ($n > 3$) is m -closed where $m = \begin{cases} \frac{n}{2} + 1 & n \text{ is even;} \\ \frac{n+1}{2} + 1 & n \text{ is odd.} \end{cases}$ (see Theorem 8). Using it we conclude that in each m -closed graph, any cycle with at least $2m - 1$ vertices has a chord.

The notion of weakly closed graphs has been introduced in [11] as a generalization of closed graphs. The final result of section 2 shows that each weakly closed graph is m -closed for some $m \leq 4$ (see Theorem 12). In Section 3 we study 3-closed property of trees and we show that a tree T with n vertices is 3-closed if and only if it is not a path and there exists a labeling of its vertices such that $d(i, i + 1) \leq 2$ for each $i < n$ (see Theorem 13). The class of 3-closed trees and the number of elements of the reduced Gröbner basis of J_T for a 3-closed labeling is also studied by means of the bipartite graph G^* attached to a simple graph G corresponding to the generators of J_G (see Definition 16, Theorem 17 and Corollary 18).

In Section 4, we study a class of trees constructed from caterpillar trees. We characterize the minimal primary decomposition of this class of trees (see Theorem 20). Also, we show that they are 3-closed. For some other trees constructed by caterpillar trees we show 3-closed property (see Theorem 21). To prove Theorem 21, we need an algorithm to give a 3-closed labeling to the vertices of a caterpillar tree such that 1 is assigned to an arbitrary vertex. This is provided in algorithm 22 presented in the Appendix section.

In the following, we review some definitions and results from [9] which we need in the next sections.

Definition 1. Let G be a simple graph on $[n]$, and let i and j be two vertices of G with $i < j$. A path $i = i_0, i_1, \dots, i_r = j$ from i to j is called *admissible*, if

- (i) $i_k \neq i_\ell$ for $k \neq \ell$;
- (ii) for each $k = 1, \dots, r - 1$ one has either $i_k < i$ or $i_k > j$;
- (iii) for any proper subset $\{j_1, \dots, j_s\}$ of $\{i_1, \dots, i_{r-1}\}$, the sequence $\{i, j_1, \dots, j_s, j\}$ is not a path.

Given an admissible path $\pi : i = i_0, i_1, \dots, i_r = j$ from i to j , where $i < j$, we associate the monomial

$$u_\pi = \left(\prod_{i_k > j} x_{i_k} \right) \left(\prod_{i_\ell < i} y_{i_\ell} \right).$$

By [3, Chapter 2, Proposition 6], the reduced Gröbner basis of J_G with respect to \prec is unique. We have:

Theorem 2. [9, Theorem 2.1] *Let G be a simple graph on $[n]$. Then the set of binomials*

$$\mathcal{G} = \bigcup_{i < j} \{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is the reduced Gröbner basis of J_G .

By [9, Theorem 3.2], we can write J_G as a finite intersection of prime ideals. In fact, corresponding to each subset $S \subset [n]$ we have the prime ideal

$$P_S(G) = \left(\bigcup_{i \in S} \{x_i, y_i\} \right) + J_{\tilde{G}_1} + \cdots + J_{\tilde{G}_{c(S)}},$$

where $G_1, \dots, G_{c(S)}$ are the connected components of the induced subgraph on the vertices $[n] \setminus S$, and \tilde{G}_ℓ is the complete graph on the vertices of G_ℓ for all ℓ . Then

$$J_G = \bigcap_{S \subset [n]} P_S(G). \tag{1}$$

Moreover, $\dim R/J_G = \max\{(n - |S|) + c(S) : S \subset [n]\}$ and hence $\dim R/J_G \geq n + c(G)$, where $c(G)$ is the number of the connected components of G . Equation (1) also shows that J_G is a radical ideal. If G is a connected graph then $P_\emptyset(G) = J_{K_n}$ is a minimal prime ideal of J_G . Note that if S is an arbitrary subset of $[n]$ the prime ideal $P_S(G)$ is not necessary a minimal prime ideal of J_G . The next lemma detects the minimal prime ideals of J_G when G is a connected graph. Note that for $S \subset [n]$, by $c(S)$ we mean $c(G_{[n] \setminus S})$.

Lemma 3. [9, Corollary 3.9] *Let G be a connected graph on the vertex set $[n]$ and $S \subset [n]$. Then $P_S(G)$ is a minimal prime ideal of J_G if and only if $S = \emptyset$, or $S \neq \emptyset$ and for each $i \in S$ one has $c(S \setminus \{i\}) < c(S)$.*

2 m -closed graphs

In this section we study the reduced Gröbner basis of J_G . As Theorem 2 shows the reduced Gröbner basis depends on the labeling of the vertices of G . We recall that a labeling of G is a bijection $V(G) \simeq [n] = \{1, \dots, n\}$, and given a labeling, we typically assume $V(G) = [n]$.

The graph G is called closed with respect to the given labeling if J_G has a quadratic Gröbner basis with respect to \prec . By [9, Theorem 1.1] we have:

Theorem 4. *Let G be a simple graph on the vertex set $[n]$. G is closed if and only if the following condition is satisfied:*

For every two edges $\{i, j\}$ and $\{k, \ell\}$ in $E(G)$ with $i < j$ and $k < \ell$, one has $\{j, \ell\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = \ell$.

Let G be a graph, we recall that the *clique complex* of G , denoted $\Delta(G)$, is the simplicial complex on $[n]$ whose faces are the cliques of G . The graph G is closed if and only if there exists a labeling of G such that all facets of $\Delta(G)$ are intervals $[a, b] \subset [n]$ (see [6, Theorem 2.2]). Closed graphs are studied in more details in [2, 4].

Following the definition of closed graph we introduced m -closed graphs.

Definition 5. Let m be a positive integer. We say that a graph G with vertex set $V(G) = \{v_1, \dots, v_n\}$ is m -closed, if its vertices can be labeled by $[n]$ such that for this labeling all the elements of \mathcal{G} are of degree $\leq m$, and m is the least integer with this property for G .

Moreover, a labeling of the vertices of G is called an m -closed labeling if the reduced Gröbner basis of J_G is in degree m and less than m with respect to this labeling.

By the above definition a closed graph is a 2-closed graph. the cycle C_4 (cycle with 4 vertices) is 3-closed and C_5 is 4-closed.

By Theorem 2, a graph G is m -closed if and only if, there exists a labeling for its vertices such that each admissible path in G has at most m vertices and in each labeling of the vertices, there exists an admissible path of length ℓ where $\ell \geq m - 1$.

We recall that a bridge is an edge whose removal from a graph increases the number of components. If e is a bridge of a connected graph G , and H_1 and H_2 are the connected components of $G \setminus e$, we write $G \setminus e = H_1 \sqcup H_2$.

In the following we find some information about m -closed graphs.

Proposition 6. (i) Let G be a graph and ℓ be the length of the longest induced path of G . Then G is m -closed for some $m \leq \ell + 1$.

(ii) Let G be a graph and H be an ℓ -closed induced subgraph of G . Then G is m -closed for some $m \geq \ell$.

(iii) Let e be a bridge of a connected graph G and $G \setminus e = H_1 \sqcup H_2$. If H_1 is m -closed and H_2 is ℓ -closed ($\ell \geq m$), then G is ℓ -closed provided that there exists an m -closed labeling of H_1 in which 1 is the label of the end point of e in H_1 and there exists an ℓ -closed labeling of H_2 in which 1 is the label of the endpoint of e in H_2 .

Proof. Part (i) and (ii) follow from the definition of an admissible path and m -closed property.

For part (iii), assume that H_1 is an m -closed graph on $[n_1]$, H_2 is an ℓ -closed graph on $[n_2]$ and 1 is the label of the end points of e in each H_i ($i = 1, 2$). We give a labeling to G by assigning to each vertex i of H_1 the new label $n_1 - i + 1$ and to each vertex i of H_2 the new label $n_1 + i$. So, by this labeling $e = \{n_1, n_1 + 1\}$. It is easy to see that the graph $G = H_1 \cup \{n_1, n_1 + 1\} \cup H_2$ is an ℓ -closed graph on $[n_1 + n_2]$. \square

A natural question to ask is that if the reduced Gröbner basis of J_G has an element of degree m , can we conclude that it also has an element of degree ℓ for each $1 < \ell < m$. This is not true in general, as the following example shows:

Example 7. Let G be the path on $[5]$ with $E(G) = \{\{1, 4\}, \{3, 4\}, \{3, 5\}, \{2, 5\}\}$. Then \mathcal{G} has an element of degree 5 while it doesn't have any element of degree 4.

For a simple graph G on $[n]$, and $m \geq 3$, if the reduced Gröbner basis of J_G has an element of degree m , then it has an element of degree 3. In fact, G is not closed and by Theorem 4, there exist two edges $\{i, j\}$ and $\{i, \ell\}$ in $E(G)$ with $i < j$, $i < \ell$ and $\{j, \ell\} \notin E(G)$, or there exist two edges $\{i, j\}$ and $\{k, j\}$ in $E(G)$ with $i < j$, $k < j$ and $\{i, k\} \notin E(G)$. So j, i, ℓ or i, j, k is an admissible path of length 2. So, \mathcal{G} has an element of degree 3.

Therefore, if G is an m -closed graph, in each labeling of its vertices, there exists an admissible path of length 2. But as we have seen in the above example, we can not extend Theorem 4 to check if a labeling is a 3-closed labeling or not.

We recall that if I is an ideal of R , the leading term ideal of I with respect to \prec is the monomial ideal of R which is generated by $(LT_{\prec}(f) \mid 0 \neq f \in I)$ where $LT_{\prec}(f)$ is the leading term of f with respect to \prec . We write $LT_{\prec}(I)$ for the leading term ideal of I .

If G is a graph, it is clear that for any arbitrary labeling of the vertices of G , $|\mathcal{G}| = \mu(LT_{\prec}(J_G)) \geq \mu(J_G)$ ($\mu(I)$ is the minimal number of homogeneous generators of I). Moreover, G is a closed graph if and only if there exists a labeling in which $\mu(LT_{\prec}(J_G)) = \mu(J_G)$. So If G is a non-closed graph on $[n]$ and $\mu(LT_{\prec}(J_G)) = \mu(J_G) + 1$, then G is 3-closed.

It is well known by [9, Proposition 1.2], that a closed graph is chordal. In the following we find a generalization of this necessary condition for m -closed property. For this we need the following theorem about cycles:

Theorem 8. Let C_n be the cycle on $n \geq 4$ vertices. Then C_n is m -closed where

$$m = \begin{cases} \frac{n}{2} + 1 & n \text{ is even;} \\ \frac{n+1}{2} + 1 & n \text{ is odd.} \end{cases}$$

Proof. Let C_n be the cycle on n vertices and m be as defined in the theorem. To show the result, we first prove that in any labeling of the vertices of C_n , one can find an admissible path with at least m vertices.

In an arbitrary labeling of the vertices of C_n , one of the following situation happens:

Case 1: For all $i \in \{1, \dots, n-1\}$, $d(i, i+1) = 1$. This case happens if and only if we give successive integers to the vertices. i. e., $(E(C_n) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\})$. So $P : 1, n, n-1, \dots, 3$ is an admissible path with $n-1$ vertices and $n-1 \geq m$.

Case 2: There exists $i \in \{1, \dots, n-1\}$, $d(i, i+1) = \ell \geq 2$. So we have two admissible paths

$$P_1 : i, j_1, \dots, j_{\ell-1}, i+1 \text{ and } P_2 : i, j'_1, \dots, j'_{n-\ell-1}, i+1$$

where $\{i, i+1\} \sqcup \{j_1, \dots, j_{\ell-1}\} \sqcup \{j'_1, \dots, j'_{n-\ell-1}\} = [n]$, P_1 has $\ell+1$ vertices and P_2 has $n-\ell+1$ vertices.

In the case that n is even, if $\ell+1 < \frac{n}{2} + 1$ and $n-\ell+1 < \frac{n}{2} + 1$, then $n+2 < n+2$ which is a contradiction. So, one of the paths P_1 and P_2 has at least m vertices.

Now assume that n is odd. Since $d(i, i+1) = \ell$, we have $\ell \leq n - \ell$. Moreover, $\ell = n - \ell$ if and only if $n = 2\ell$ which is a contradiction. So, $\ell < n - \ell$.

If $n - \ell + 1 < \frac{n+1}{2} + 1$, then by $1 + \ell < n - \ell + 1 < \frac{n+1}{2} + 1$ we have $n + 2 < n + 2$ which is a contradiction. So P_2 has at least m vertices.

So in each labeling of the vertices of C_n , we have an admissible path with at least m vertices.

Now, if we find a labeling of the vertices of C_n such that each admissible path has at most m vertices, the conclusion follows.

Suppose that:

$$V(C_n) = \{v_1, v_2, \dots, v_n\}, \quad E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$$

If n is even, we do as follows:

1. $S = \{v_1, v_2, \dots, v_n\}$,
2. label v_1 as 1,
3. $i = 1$,
4. **While** $i < n$ **do**
 - (a) Pick $v_j \in S$ such that $d(i, v_j) = m - 1$ and label v_j as $i + 1$,
 - (b) If $i + 2 < n$, label v_{j+1} as $i + 2$,
 - (c) $i := i + 2$.

By this labeling of the vertices, for each i , $d(i, i+1) = m - 1$ if i is odd and $d(i, i+1) = 1$ if i is even. So we have some admissible path with m vertices.

If n is odd, we do as follows:

For each $1 \leq i < m$, label v_i as $2i - 1$ and for each $m \leq i \leq n$, label v_i as $2(i - m + 1)$. By this labeling, for each i , $d(i, i+1) = m - 2$ and for each i there is a unique admissible path with m vertices between i and $i + 1$.

Now assume that $P : j_1, \dots, j_t$ ($t > m$) is an admissible path in C_n . So, $j_t > j_1 + 1$.

If n is odd, by the fact that $d(i, i+1) = m - 2$ for each i , we conclude $j_1 + 1 \in V(P)$ which is a contradiction.

Assume that n is even. If j_1 is odd, as above we conclude that $j_1 + 1 \in V(P)$ which is the desired contradiction. If j_1 is even and $j_1 + 1 \notin V(P)$, then $P' = j_1 + 1, j_1, \dots, j_{t-1}$ is a path with t vertices. Since $d(j_1 + 1, j_1 + 2) = m - 1$, $j_1 + 2 \in V(P')$. So $j_t > j_1 + 2$ and $j_1 + 2 \in V(P)$ again a contradiction. \square

The next corollaries are the generalization of the fact that a closed graph is chordal. These results are immediate consequences of Proposition 6 and Theorem 8.

Corollary 9. *If G is an m -closed graph, then each cycle of G with $2m - 1$ or more vertices has a chord.*

Corollary 10. *Let G be an m -closed graph and*

$$\ell = \max\{t \mid \exists \text{ an induced cycle with } t \text{ vertices in } G\}.$$

$$\text{If } \ell \geq 4, \text{ then } m \geq \begin{cases} \frac{\ell}{2} + 1 & \ell \text{ is even;} \\ \frac{\ell+1}{2} + 1 & \ell \text{ is odd.} \end{cases}$$

A generalization of the notion of closed graph is *weakly closed graph* which has been introduced in [11]. Let G be a graph. G is said to be weakly closed if there exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$, i is adjacentable with j (for the definition of adjacentable see [11, Definition 1.2]). The following theorem is a characterization of weakly closed graphs.

Theorem 11. [11, Theorem 1.9] *Let G be a graph. Then the following conditions are equivalent:*

1. G is weakly closed.
2. There exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$ or $\{k, j\} \in E(G)$.

In the following we relate the m -closed graphs to weakly closed graphs.

Theorem 12. *Let G be a weakly closed graph. Then G is m -closed for some $m \leq 4$.*

Proof. Suppose that G is a weakly closed graph on $[n]$. Then by Theorem 11, for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$ or $\{k, j\} \in E(G)$.

We prove that each admissible path of G has at most 4 vertices. Assume to the contrary that there exists an admissible path $P : i = i_1, i_2, \dots, i_{m-1}, i_m = j$ with $m \geq 5$ vertices. Note that $i < j$. If $i_2 > j$, then $i < j < i_2$ and $\{i, i_2\} \in E(G)$. So $\{i, j\} \in E(G)$ or $\{i_2, j\} \in E(G)$ which is a contradiction. If $i_{m-1} < i$, then $i_{m-1} < i < j$ and $\{i_{m-1}, j\} \in E(G)$. So $\{i_{m-1}, i\} \in E(G)$ or $\{i, j\} \in E(G)$. Again, it is a contradiction. Therefore $i_2 < j$ and $i_{m-1} > i$. Since P is an admissible path, we have $i_2 < i$ and $i_{m-1} > j$.

Let

$$t = \min\{r \mid 2 < r \leq m - 1, i_r > j\}.$$

So $i_{t-1} < i < j < i_t$ and $\{i_{t-1}, i_t\} \in E(G)$. If $t = 3$, then $\{i_2, j\} \in E(G)$ or $\{j, i_3\} \in E(G)$ which is impossible because $m \geq 5$ and P is an admissible path. If $t > 3$, then $\{i_{t-1}, i\} \in E(G)$ or $\{i, i_t\} \in E(G)$. This case also is impossible since P is an admissible path.

So, in any case we get a contradiction. Thus $m \leq 4$ and the result follows. \square

Note that the converse of Theorem 12 is not true since C_5 is 4-closed and not weakly closed.

3 3-closed trees

In the following we characterize 3-closed trees. Let G be a simple graph on the vertex set $[n]$ and \mathcal{G} has no element of degree more than 3, then $d(i, i + 1) \leq 2$ for each i . But the converse is not true in general. For example, let C be the cycle on the vertex set $[n]$ and with the edge set $\{\{1, 3\}, \{3, 4\}, \{2, 4\}, \{2, 5\}, \{1, 5\}\}$. Then for each i , $d(i, i + 1) \leq 2$ but C is 4-closed.

We recall that by [9, Corollary 1.3], a tree is a closed graph if and only if it is a path. Next result shows that a 3-closed labeling for a tree T is a labeling in which $d(i, i + 1) \leq 2$ for each i .

Theorem 13. *Let T be a tree with n vertices and assume that T is not a path. Then T is 3-closed if and only if there exists a labeling for $V(T)$ such that $d(i, i + 1) \leq 2$ for each i .*

Proof. Assume to the contrary that there exists a tree T on the vertex set $[n]$ such that $d(i, i + 1) \leq 2$ for each i , and T has an admissible path of length at least 3. Let

$$m - 1 = \max\{\ell(P) \mid P \text{ is an admissible path}\}$$

and

$$i_1 = \max\{t \mid \text{there exists an admissible path of length } m - 1 \text{ starting from } t\}.$$

Then $m > 3$ and we can consider an admissible path like $P : i_1, i_2, \dots, i_m$. Since T is a tree, $d(i_1, i_m) \geq 3$. So, $i_1 + 1 \neq i_m$ which shows that $i_1 < i_1 + 1 \leq i_m - 1 < i_m$. Therefore $i_1 + 1 \notin \{i_2, i_3, \dots, i_{m-1}\}$. Moreover, by $d(i_1, i_1 + 1) \leq 2$, one of the following situations happens:

Case a: $\{i_1, i_1 + 1\} \in E(T)$. In this case, $i_1 + 1, i_1, \dots, i_m$ is an admissible path of length m which is a contradiction by our choice of m .

Case b: $\{i_1 + 1, i_2\} \in E(T)$. In this case, $i_1 + 1, i_2, i_3, \dots, i_m$ is an admissible path of length $m - 1$ which is a contradiction by our choice of i_1 .

Case c: There exists $j \in [n] \setminus \{i_2, \dots, i_m\}$ such that $i_1 + 1, j, i_1$ is a path. In this case, consider the path $P' : i_1 + 1, j, i_1, i_2, \dots, i_m$. Since $\ell(P') = m + 1$, by our choice of m , P' is not an admissible path. So, $i_1 < i_1 + 1 < j < i_m$. It is easy to see that $P'' : j, i_1, i_2, \dots, i_m$ is an admissible path of length m which is again a contradiction by our choice of m . \square

Remark 14. By Theorem 13, a labeling of a tree T is a 3-closed labeling if and only if $d(i, i + 1) \leq 2$ for each $1 \leq i < n$. This is not true for an arbitrary 3-closed graph. For example, Let G be a graph with

$$V(G) = \{v_1, \dots, v_5\} \quad \text{and} \quad E(G) = \{\{v_1, v_2\}, \{v_2, v_4\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}\}.$$

Then G is a bipartite 3-closed graph. If we assign i to each vertex v_i , then $d(i, i + 1) \leq 2$ for each $1 \leq i < 5$ but this is not a 3-closed labeling of G .

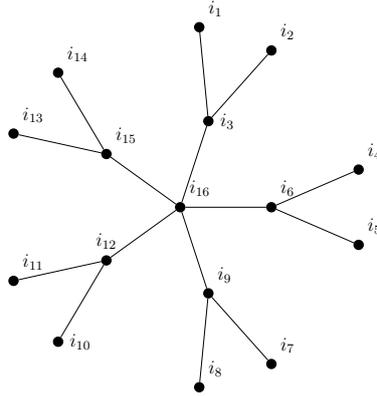


Figure 1:

Next we give an example of a tree which is not 3-closed.

Example 15. Consider the above tree on 16 vertices (Figure 1). We prove that T is not 3-closed. By contradiction assume that there exists a labeling of $V(T)$ such that

$$d(k, k + 1) \leq 2 \quad \text{for all } k \in \{1, \dots, 15\}. \quad (2)$$

Without loss of generality, we can assume that $\{1, 16\} \cap \{i_7, i_8, \dots, i_{15}\} = \emptyset$. So,

$$\{i_j - 1, i_j + 1\} \subset \{1, 2, \dots, 16\} \quad \text{for all } j \in \{7, 8, \dots, 15\}. \quad (3)$$

If $i_7 < i_8$ and they are not two successive integers, then by (2) $\{i_7 - 1, i_7 + 1, i_8 + 1\} \subseteq \{i_9, i_{16}\}$ which is a contradiction. So, we can assume that $i_8 = i_7 + 1$. By a similar argument, we should also have, $i_{11} = i_{10} + 1$ and $i_{14} = i_{13} + 1$.

Again, by (2) and (3) we can easily see that

$$i_{16} = i_7 - 1 \text{ or } i_{16} = i_7 + 2,$$

and

$$i_{16} = i_{10} - 1 \text{ or } i_{16} = i_{10} + 2,$$

and

$$i_{16} = i_{13} - 1 \text{ or } i_{16} = i_{13} + 2.$$

So, $i_7 = i_{10}$ or $i_7 = i_{13}$ or $i_{10} = i_{13}$ which is a contradiction.

Definition 16. Let G be a graph on the vertex set $[n]$, we associate to G a bipartite graph G^* where

$$V(G^*) = \{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_n\}, \quad E(G^*) = \{x_i y_j \mid \{i, j\} \in E(G) \text{ and } i < j\}.$$

Note that if G is a closed graph, for a closed labeling of G , $LT_{\prec}(J_G) = I(G^*)$ where $I(G^*)$ is the edge ideal of the graph G^* .

Conversely, if H is a bipartite graph on the vertex set $\{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_n\}$ such that for each $\{x_i, y_j\} \in E(H)$ we have $i < j$, then we can associate to H a simple graph H_* on the vertex set $[n]$ in a natural way ($(H_*)^* = H$).

Note that if T is a tree, then T^* is also a tree. In the following, we give a characterization of 3-closed trees by means of Definition 16.

Theorem 17. *Let \mathcal{T}_n be the set of all bipartite graphs H on the vertex set $\{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_n\}$ with the following properties:*

1. $\{x_i, y_j\} \in E(H) \implies i < j$.
2. for all $i \in \{1, \dots, n-1\}$ one of the following conditions holds:
 - $\{x_i, y_{i+1}\} \in E(H)$.
 - There exists $j > i+1$, $\{x_i, y_j\}, \{x_{i+1}, y_j\} \in E(H)$.
 - There exists $j < i$, $\{x_j, y_i\}, \{x_j, y_{i+1}\} \in E(H)$.
3. $|E(H)| = n-1$

Then a tree with n vertices is 3-closed if and only if T is not a path and there exists $H \in \mathcal{T}_n$ such that $T \cong H_*$.

Proof. If T is a 3-closed graph on $[n]$, then, by Theorem 13, $d(i, i+1) \leq 2, \forall 1 \leq i < n$. So T^* satisfies condition 2. Since $|E(T)| = |E(T^*)| = n-1$, the conclusion follows from the fact that $T = (T^*)_*$.

Conversely, if H satisfies condition 1 then H_* is defined and is a graph on $[n]$. By condition 2, in H_* , $d(i, i+1) \leq 2$ for each i and moreover H_* is connected. Now since $|E(H_*)| = n-1 = |V(H_*)| - 1$, H_* is a tree. So, by Theorem 13, H_* is a 3-closed tree. \square

In the next corollary, we find the number of elements of the reduced Gröbner basis of a 3-closed tree.

Corollary 18. *Let T be a tree on the vertex set $[n]$ and $d(i, i+1) \leq 2$ for all $i \in \{1, \dots, n-1\}$. Then $|\mathcal{G}| = n-1 + \beta_{13}(I(T^*))$.*

Proof. Let G be a simple graph on $[n]$ and $K_3(G)$ = the number of triangles of G . Then by [15, Theorem 2.2], $\beta_{13}(J_G) = 2K_3(G)$. So, for an arbitrary tree T , $\beta_{13}(J_T) = 0$.

Now, if $d(i, i+1) \leq 2$, then by Theorem 13, $LT_{\prec}(J_T)$ is generated in degrees 2 and 3. So, $\beta_{23}(LT_{\prec}(J_T)) = 0$ and

$$\beta_{13}(LT_{\prec}(J_T)) = \beta_{13}(\langle x_i y_j \mid i < j, \{i, j\} \in E(T) \rangle) = \beta_{13}(I(T^*)).$$

Moreover, by [14] the graded Betti numbers of J_T is obtained from the graded Betti numbers of $LT_{\prec}(J_T)$ by a sequence of consecutive cancelations. So

$$\beta_{03}(LT_{\prec}(J_T)) = \beta_{13}(LT_{\prec}(J_T)) = \beta_{13}(I(T^*))$$

and the conclusion follows. \square

We remark that if G is an arbitrary 3-closed graph, for a 3-closed labeling, the same argument as the proof of Corollary 18 shows that $|\mathcal{G}| = |E(G)| + \beta_{13}(I(G^*)) - 2K_3(G)$.

4 Binomial Edge Ideals of Caterpillar Trees

In this section, we study the binomial edge ideals of caterpillar trees and some trees constructed from this kind of trees. First we recall its definition.

Definition 19. A caterpillar tree is a tree T with the property that it contains a path P such that any vertex of T is either a vertex of P or it is adjacent to a vertex of P .

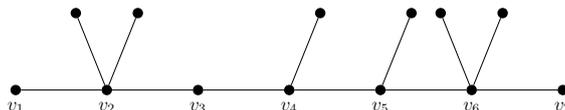


Figure 2:

Note that the path P in the definition of a caterpillar tree is a longest induced path of T and we call it the central path of T . Figure 2 is an example of a caterpillar tree with the central path $P : v_1, v_2, \dots, v_7$.

Caterpillar trees were first studied by Harary and Schwenk [8]. These graphs have some applications in chemistry and physics [5].

Let T be a caterpillar tree and ℓ be the length of its longest induced path. By [6, Theorem 1.1] $\text{depth}(R/J_T) = |V(T)| + 1$ and by [1, Theorem 4.1] $\text{reg}(R/J_T) = \ell$. In the following we describe the minimal primary decomposition of J_T . We recall that since J_T is a radical ideal, to know the minimal primary decomposition of J_T , it is enough to characterize its minimal prime ideals.

Theorem 20. Let T be a caterpillar tree, $P : v_1, \dots, v_l$ be the central path of T and $S \subset V(T)$. Then $P_S(T)$ is a minimal prime ideal of J_T if and only if $S = \emptyset$ or $S = \{v_{i_1}, \dots, v_{i_k}\} \subseteq \{v_1, \dots, v_l\}$ where $1 < i_1 < \dots < i_k < l$ satisfy the following conditions:

- If $\deg(v_{i_j}) = 2$, then $d(v_{i_j}, v_{i_{j+1}}) \geq 2$ and $d(v_{i_j}, v_{i_{j-1}}) \geq 2$
- If $\deg(v_{i_j}) = 3$, then $d(v_{i_j}, v_{i_{j+1}}) \geq 2$ or $d(v_{i_j}, v_{i_{j-1}}) \geq 2$.

Proof. We prove that each prime ideal corresponding to a set S , where S is satisfying in the mentioned conditions, is a minimal prime ideal by induction on the number of vertices in the set S .

For $k = 1$ the statement is obvious. Now assume theorem is true for each S with $|S| = m$ and $S' = \{v_{i_1}, \dots, v_{i_{m+1}}\}$ has the mentioned conditions. If $S = \{v_{i_1}, \dots, v_{i_m}\}$, by induction hypothesis, $P_S(T)$ is a minimal prime ideal of J_T . Let $d = \deg(v_{i_{m+1}})$ and $d' = \deg(v_{i_m})$.

Depending on $d(v_{i_m}, v_{i_{m+1}})$, we distinguish the following cases:

Case 1: $d(v_{i_m}, v_{i_{m+1}}) \geq 2$. In this case it is easy to see that $c(S') = c(S) + d - 1$ and for all $j \in \{1, \dots, m\}$, $c(S' \setminus \{v_{i_j}\}) = c(S \setminus \{v_{i_j}\}) + d - 1$.

Case 2: $d(v_{i_m}, v_{i_{m+1}}) = 1$. In this case, $d \geq 3$ and $d' \geq 3$. A straightforward observation shows that $c(S') = c(S) + d - 2$ and for all $j \in \{1, \dots, m - 1\}$, $c(S' \setminus \{v_{i_j}\}) = c(S \setminus \{v_{i_j}\}) + d - 2$. Moreover for deleting the vertex v_{i_m} , one of the following situations happens:

- (a) $m = 1$ or $d(v_{i_{m-1}}, v_{i_m}) = 2$. One can see $c(S' \setminus \{v_{i_m}\}) = c(S') - (d' - 2)$.
- (b) $d' \geq 4$. In this case, $c(S') \geq c(S' \setminus \{v_{i_m}\}) + (d' - 3)$.

It is obvious that in all of the above situations, $c(S' \setminus \{v_{i_j}\}) < c(S')$ for all $j \in \{1, \dots, m + 1\}$. So, Lemma 3 implies $P_{S'}(T)$ is a minimal prime ideal of J_T .

Now assume that $S \subset V(T)$ is not as described in the theorem. So, one of the following situation happens:

- 1) There exists a vertex v of degree 1 in S . In this case, $c(S \setminus \{v\}) \geq c(S)$. So, by Lemma 3, $P_S(T)$ is not a minimal prime ideal of J_T .
- 2) For some j , $\deg(v_{i_j}) = 2$, and ($d(v_{i_j}, v_{i_{j+1}}) = 1$ or $d(v_{i_j}, v_{i_{j-1}}) = 1$). Without loss of generality assume that $d(v_{i_{j-1}}, v_{i_j}) = 1$. Since $v_{i_{j-1}}$ and v_{i_j} are connected through just one edge, removing the vertex v_{i_j} doesn't change the number of connected components of $T_{V(T) \setminus S}$, meaning that $c(S \setminus \{v_{i_j}\}) = c(S)$. Again, by Lemma 3, $P_S(T)$ is not a minimal prime ideal of J_T .
- 3) For some j , $\deg(v_{i_j}) = 3$, $d(v_{i_j}, v_{i_{j+1}}) = 1$ and $d(v_{i_j}, v_{i_{j-1}}) = 1$. In this situation also straightforward observation shows that $c(S \setminus \{v_{i_j}\}) = c(S)$. So, $P_S(T)$ is not a minimal prime ideal of J_T .

So the conclusion follows. □

For example, if T is the caterpillar tree described in Figure 2, then by Theorem 20, it is easy to find all minimal prime ideals of J_T and see that $\dim(R/J_T) = 19$.

Finally, we prove that caterpillar trees and some trees constructed by caterpillar trees are 3-closed.

Theorem 21. (a) *Let T be a caterpillar tree. Then T is 3-closed.*

(b) *Let $T = T_1 \cup B \cup T_2$ where T_1 and T_2 are two caterpillar trees and B is a bridge between T_1 and T_2 , and the endpoints of B are chosen from the vertices of the central paths of T_1 and T_2 respectively. Then T is 3-closed.*

More generally,

(c) *Let T be a tree and $T = T_1 \cup B \cup T_2$ where T_1 , T_2 and B are caterpillar trees, and the endpoints of the central path of B are chosen from the vertices of T_1 and T_2 respectively. Then T is 3-closed.*

Proof. (a) Let $n = |T|$, it is enough to find a labeling of $V(T)$ such that $d(i, i + 1) \leq 2$ for each $1 \leq i < n$.

Let $P : v_1, \dots, v_\ell$ be the central path of T , and for each $1 \leq j \leq \ell$, $N'_T(v_j) = N_T(v_j) \setminus V(P)$ is determined the leaf neighbors of the vertex v_j .

We do as follows:

label v_1 as 1; $t = 2$; $j = 2$;

While $j \leq \ell$ **do**

 label v_j as t ; $t = t + 1$;

$S := N'_T(v_j)$;

While $S \neq \emptyset$ **do**;

$v :=$ pick $v \in S$ such that v is the rightmost leaf of v_j ;

 label v as t ;

$t = t + 1$; $S = S \setminus \{v\}$;

end;

$j = j + 1$;

end

It is easy to see that by this labeling of $V(T)$, $d(i, i + 1) \leq 2$ for all $1 \leq i < n$.

(b) By proposition 6, it is enough to show that for each caterpillar tree T and each vertex v of its central path, there exists a 3-closed labeling in which 1 is assigned to v . This fact follows from Algorithm 22.

(c) Without loss of generality, we can assume that the endpoints of the central path of B are chosen from the vertices of the central paths of T_1 and T_2 respectively. Because if this is not the case and for example $\{v\} = V(T_1) \cap V(B)$ where v is not a vertex of the central path of T_1 , then there exists a vertex w of the central path of T_1 such that $e = \{v, w\} \in E(T_1)$. So we can replace T_1 with $T_1 \setminus e$ and B with $B \cup e$. We can also assume that $E(T_1)$, $E(T_2)$ and $E(B)$ are pairwise disjoint sets.

Let $v \in V(B) \cap V(T_1)$ and $w \in V(B) \cap V(T_2)$. By Algorithm 22, there exists a 3-closed labeling of $V(T_1)$ that assigns $n_1 = |V(T_1)|$ to v . By part (a) of the proof there exists a 3-closed labeling of $V(B)$ with integers $n_1, \dots, n_2 = n_1 + |V(B)| - 1$ that assigns n_1 to v and n_2 to w . Again by Algorithm 22 there exists a 3-closed labeling of $V(T_2)$ with integers $n_2, \dots, n_3 = n_2 + |V(T_2)| - 1$ that assigns n_2 to w . All together we get a 3-closed labeling of T and the conclusion follows. \square

By [11, Proposition 3.2], a tree T is weakly closed if and only if T is a caterpillar tree. So, by Theorem 21, If T is a weakly closed graph, then T is m -closed for some $m \leq 3$.

5 Appendix

In the following we introduce an algorithm to label the vertices of a caterpillar tree T with integers $1, \dots, n$ such that $d(i, i + 1) \leq 2$ for all $1 \leq i < n$. Suppose that the central path of T is $P : v_1, \dots, v_\ell$ and for each $1 \leq j \leq \ell$, $N'_T(v_j) = N_T(v_j) \setminus V(P)$ is determined the leaf neighbors of the vertex v_j .

The algorithm works as follows. First a candidate for 1 is found by choosing an arbitrary vertex of the central path which is called v_{i_0} . We then go through the vertices in the central path. If v_{i_0+1} has some leaf neighbors, we label them $2, \dots, t$ from right to left, and then we label v_{i_0+2} as $t+1$. Otherwise we label v_{i_0+2} as 2. Then we set $j = i_0 + 2$ and this process is repeated for the next vertices of the v_j until we reach the endpoint of P . In the return path from v_l to v_1 and then from v_1 to v_{i_0} the similar process is repeated until every vertex is labeled.

Algorithm 22. Labeling algorithm of caterpillars trees:

Input: A caterpillar tree T with the central path $P : v_1, \dots, v_\ell$.

Output: A 3-closed labeling of T

$v_{i_0} :=$ one of the vertices on the central path;

$j := i_0$; label v_{i_0} as 1; $t := 2$;

While $j < \ell - 1$ **do**

$S := N'_T(v_{j+1})$;

While $S \neq \emptyset$ **do**;

$v :=$ pick $v \in S$ such that v is the rightmost leaf of v_{j+1} ; label v as t ;

$t = t + 1$; $S = S \setminus \{v\}$;

end;

label v_{j+2} as t ;

$j = j + 2$; $t = t + 1$;

end

If $j == \ell - 1$

label v_ℓ as t ;

$j = \ell$; $t = t + 1$;

Otherwise

label $v_{\ell-1}$ as t ;

$j = \ell - 1$; $t = t + 1$;

end

While $j > 2$ **do**

$S := N'_T(v_{j-1})$;

While $S \neq \emptyset$ **do**;

$v :=$ pick $v \in S$ such that v is the rightmost leaf of v_{j-1} ; label v as t ;

$t = t + 1$; $S = S \setminus \{v\}$;

end;

label v_{j-2} as t ;

$j = j - 2$; $t = t + 1$;

end

If $j == 2$ and $i_0 > 1$

label v_1 as t ;

$j = 1$; $t = t + 1$;

Otherwise

If $i_0 > 2$

label v_2 as t ;

```

     $j = 2; t = t + 1;$ 
  end;
end ;
While  $j < i_0 - 2$  do
   $S := N'_T(v_{j+1});$ 
  While  $S \neq \emptyset$  do;
     $v :=$  pick  $v \in S$  such that  $v$  is the rightmost leaf of  $v_{j+1}$ ; label  $v$  as  $t$ ;
     $t = t + 1; S = S \setminus \{v\};$ 
  end;
  label  $v_{j+2}$  as  $t$ ;
   $j = j + 2; t = t + 1;$ 
end
If  $j == i_0 - 2$ 
   $S := N'_T(v_{i_0-1});$ 
  While  $S \neq \emptyset$  do
     $v :=$  pick  $v \in S$  such that  $v$  is the rightmost leaf of  $v_{i_0-1}$ ; label  $v$  as  $t$ ;
     $t = t + 1; S = S \setminus \{v\};$ 
  end;
end;
end;
```

Remark 23. If one wants to give a 3-closed labeling to a caterpillar tree T in such a way that 1 is assigned to $v \in N'_T(v_{i_0})$ for some $1 < i_0 < \ell$, it is enough to label v as 1, v_{i_0} as 2, set $N'_T(v_{i_0}) = N'_T(v_{i_0}) \setminus \{v\}$ and start with $t := 3$ instead of $t := 2$.

Moreover, if one wants to give a 3-closed labeling to a caterpillar tree T in such a way that $n = |V(T)|$ is assigned to an arbitrary vertex v , it is enough to apply Algorithm 22, by labeling v as 1 and at the end changing the label i of each vertex to $n - i + 1$.

Example 24. Here, we give an example of a labeled caterpillar tree using Algorithm 22. Note that 12 is the label of v_1 , 11 is the label of v_2 , 1 is the label of v_3 and so on .

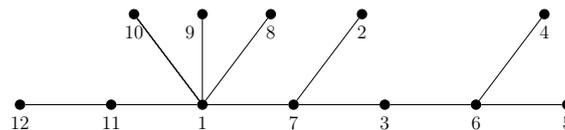


Figure 3:

Finally, we give an example of a 3-closed tree described in Theorem 21(part b). Note that the labeling is given by Algorithm 22, and Proposition 6.

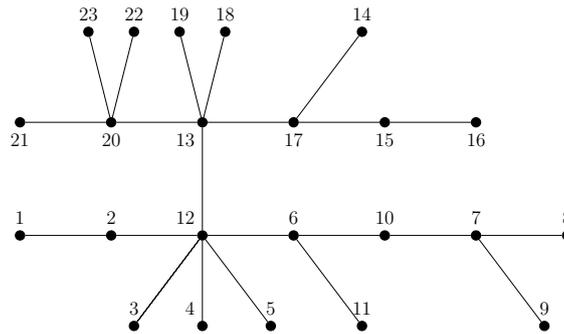


Figure 4:

Acknowledgements

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