A 64-dimensional counterexample to Borsuk's conjecture

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Abstract

Bondarenko's 65-dimensional counterexample to Borsuk's conjecture contains a 64-dimensional counterexample. It is a two-distance set of 352 points.

1 Introduction

In 1933 Karol Borsuk [2] asked whether each bounded set in the n-dimensional Euclidean space (containing at least two points) can be divided into $n+1$ parts of smaller diameter. (The diameter of a set X is the least upper bound of the distances of pairs of points in X.) The hypothesis that the answer to that question is positive became famous under the name *Borsuk's conjecture*.

The first counterexamples were given by Jeff Kahn and Gil Kalai [7] who showed that Borsuk's conjecture is false for $n = 1325$ and gave an exponential lower bound $c^{\sqrt{n}}$ with $c = 1.2$ for the number of parts needed for large n. Subsequently, several authors found counterexamples in lower dimensions.

In 2013 Andriy V. Bondarenko [1] constructed a 65-dimensional two-distance set S of 416 vectors that cannot be divided into fewer than 84 parts of smaller diameter. That was not just the first known two-distance counterexample to Borsuk's conjecture but also a considerable reduction of the lowest known dimension the conjecture fails in in general.

This article presents a 64-dimensional subset of S of size 352 that cannot be divided into fewer than 71 parts of smaller diameter, thus producing a two-distance counterexample to Borsuk's conjecture in dimension 64.

2 Euclidean representation of strongly regular graphs

We very briefly repeat the basic facts. More details can be found in [1] and [3].

A finite graph Γ without loops or multiple edges is called a $\text{srg}(v, k, \lambda, \mu)$, where srg abbreviates 'strongly regular graph', when it has v vertices, is regular of valency k, where

$0 < k < v-1$, and any two distinct vertices x, y have λ common neighbours when x and y are adjacent (notation: $x \sim y$), and μ common neighbours otherwise (notation: $x \not\sim y$).

The adjacency matrix A of Γ is the matrix of order v defined by $A_{xy} = 1$ if $x \sim y$ and $A_{xy} = 0$ otherwise. Let I be the identity matrix of order v, and let J be the matrix of order v with all entries equal to 1. Then A is a symmetric matrix with zero diagonal such that $AJ = JA = kJ$ and $A^2 = kI + \lambda A + \mu(J - I - A)$. It follows that the eigenvalues of A are k, r, s, with $k \geqslant r \geqslant 0 > s$, where r, s are the two solutions of $x^2 + (\mu - \lambda)x + \mu - k = 0$, so that $(A - rI)(A - sI) = \mu J$. The multiplicities of k, r, s are $1, f, g$ (respectively), where $1 + f + g = v$ and $k + fr + gs = 0$.

The matrix $M = A - sI - \frac{k-s}{v}J$ has rank f, so that the map $x \mapsto \overline{x}$ that sends each vertex x to row x of M is a representation of Γ in $\mathbb{R}^f$, and the inner product $(\overline{x}, \overline{y})$ depends only on whether $x = y$, $x \sim y$ or $x \not\sim y$.

3 The $G_2(4)$ graph

There exists a graph Γ that is a srg(416,100,36,20) with automorphism group $G_2(4)$:2 acting rank 3, with point stabilizer J_2:2, see, e.g., Hubaut [4], pp. 370, 372. Here $v = 416$, $k = 100$, $r = 20$, $s = -4$ and $f = 65$, $g = 350$, so that $M = A + 4I - \frac{1}{4}J$ and we have $M^2 = 24M = 24A + 96I - 6J$. This means that

$$(\overline{x}, \overline{y}) = \begin{cases} 90 & \text{if } x = y \\ 18 & \text{if } x \sim y \\ -6 & \text{if } x \not\sim y, \end{cases}$$

and $\|\overline{x} - \overline{y}\|^2 = 144$ when $x \sim y$, and $\|\overline{x} - \overline{y}\|^2 = 192$ when $x \not\sim y$.

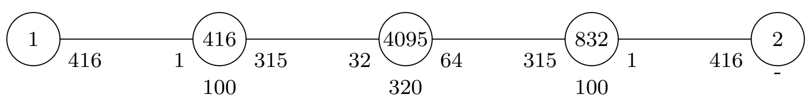
This graph Γ has maximal clique size 5 (because each point neighbourhood is a srg(100,36,14,12), that has point neighbourhoods srg(36,14,4,6), which has bipartite point neighbourhoods).

Bondarenko's example S is the image of Γ in $\mathbb{R}^{65}$. Any subset of smaller diameter corresponds to a clique and therefore has size at most 5. Since $|S| = 416$, at least 84 subsets of smaller diameter are needed to cover the set.

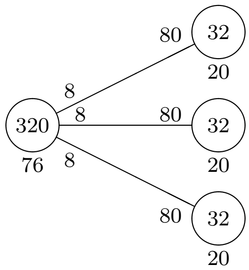
4 Structure of the $G_2(4)$ graph

The graph Γ occurs as point neighbourhood in the Suzuki graph Σ, which is a srg(1782, 416,100,96) (cf. [4]). For two nonadjacent vertices a, b of Σ, we can identify the set of 416 neighbours of a with the vertex set X of Γ, and then the 96 common neighbours of a and b form a 96-subset B of X.

The graph Σ has a triple cover $3 \cdot \Sigma$ constructed by Leonard Soicher [8]. It is distance-transitive with intersection array $\{416, 315, 64, 1; 1, 32, 315, 416\}$ on 5346 vertices.



We see that the 96-subset B is the union of three mutually nonadjacent subsets B_1, B_2 and B_3 of size 32. Put $C = X \setminus B$ so that $|C| = 320$. Since $3 \cdot \Sigma$ is tight (cf. [6]), the partition $\{B_1, B_2, B_3, C\}$ of X is regular (a.k.a. equitable) with diagram



(that is, each vertex in B_1 has 20 neighbours in B_1, none in B_2, B_3, and 80 in C, etc.).

Now we define $T = \{\overline{x} \mid x \in B_1 \cup C\} \subseteq \mathbb{R}^{65}$. Let u be the vector

$$u = \sum_{y \in B_2} \overline{y} - \sum_{y \in B_3} \overline{y}.$$

Then u is a vector in our $\mathbb{R}^{65}$, and for all $x \in T$ we have $(u, x) = 0$. On the other hand, $(u, u) = 64 \cdot 576 \neq 0$. It follows that T lies in the hyperplane $u^{\perp}$, a copy of $\mathbb{R}^{64}$. Because any subset of smaller diameter contains at most 5 vectors, we proved

Theorem 1 *There is a 2-distance set T of size 352 in $\mathbb{R}^{64}$ such that any partition of T into parts of smaller diameter has at least 71 parts.*

Remark For more explicit constructions and a corresponding computer program, see [5].

References

[1] Andriy V. Bondarenko, *On Borsuk's conjecture for two-distance sets*, Discrete and Computational Geometry **51** (2014) 509–515.

[2] Karol Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933) 177–190.

[3] Andries E. Brouwer & Willem H. Haemers, *Spectra of graphs*, Springer, 2012.

[4] Xavier L. Hubaut, *Strongly regular graphs*, Discr. Math. **13** (1975) 357–381.

[5] Thomas Jenrich, *A 64-dimensional two-distance counterexample to Borsuk's conjecture*, 2013-08-25, arXiv:1308.0206v5 .

[6] Aleksandar Jurišić, Jack Koolen & Paul Terwilliger, *Tight distance-regular graphs*, J. Alg. Combin. **12** (2000) 163-197.

[7] Jeff Kahn & Gil Kalai, *A counterexample to Borsuk's conjecture*, Bull. Amer. Math. Soc. (New Series) **29** (1993) 60–62.

[8] Leonard H. Soicher, *Three new distance-regular graphs*, Europ. J. Combin. **14** (1993) 501–505.