

Bruhat Order on Partial Fixed Point Free Involutions

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Abstract

The order complex of inclusion poset PF_n of Borel orbit closures in skew-symmetric matrices is investigated. It is shown that PF_n is an EL-shellable poset, and furthermore, its order complex triangulates a ball. The rank-generating function of PF_n is computed and the resulting polynomial is contrasted with the Hasse-Weil zeta function of the variety of skew-symmetric matrices over finite fields.

Keywords: Bruhat-Chevalley order, partial fixed-point-free involutions, EL-shellability, rank generating function.

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1 Introduction

This paper is a continuation of our earlier investigations [5], [7] on the Bruhat order on certain sets of involutions, and our notation follows these references closely:

- $\mathbb{C}$: field of complex numbers,
- S_n : symmetric group of $n \times n$ permutation matrices,
- R_n : rook monoid of $n \times n$ partial permutation matrices,
- I_n : involutions in S_n ,
- F_n : fixed-point-free involutions in S_n ,
- PI_n : partial involutions in R_n ,
- Mat_n : all $n \times n$ matrices over $\mathbb{C}$,
- Sym_n : all $n \times n$ symmetric matrices over $\mathbb{C}$,
- GL_n : invertible $n \times n$ matrices over $\mathbb{C}$,
- B_n : Borel group of invertible upper triangular matrices from GL_n .

In addition to the above list of notation, we consider Skew_n , the space of all $n \times n$ skew-symmetric matrices over $\mathbb{C}$, and PF_n , the set of all fixed-point-free partial involutions. The purpose of this article is to investigate some combinatorial properties of PF_n . In some sense, this is the final step of our program for showing that the sets of partial permutations R_n , PI_n , and PF_n all share the same algebraic combinatorial properties.

Let X be a variety on which a Borel group B acts algebraically. Let W denote the set of B -orbits in X , and define the B -ordering $\leq$ on W by

$$\mathcal{O}_1 \leq \mathcal{O}_2 \iff \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}, \quad \mathcal{O}_1, \mathcal{O}_2 \in W. \quad (1)$$

Study of this basic combinatorial set-up is important for group theory. Indeed, suppose G is a linear algebraic group with a Borel subgroup B . Then the double cosets of B in G are equivalent to the orbits of $B \times B$ acting on $X = G$ via $(g, h) \cdot x = gxh^{-1}$. Furthermore, $B \times B$ -orbits in X are parameterized by the ‘Weyl group’ of G (the *Bruhat-Chevalley decomposition*). We have a well-known special case, when $G = \text{GL}_n$. Then, $B_n \times B_n$ -orbits are parameterized by S_n , and the induced partial ordering is the *Bruhat-Chevalley ordering* on S_n .

In [18], by generalizing Bruhat-Chevalley decomposition to linear algebraic monoids, Renner constructs a rich family of orbit posets. In particular, among other things, he shows that the orbits of the Borel group action

$$(g, h) \cdot A = gAh^{-1}, \quad g, h \in B_n, \quad A \in \text{Mat}_n. \quad (2)$$

are parameterized by R_n . Basic combinatorial properties of $B_n \times B_n$ -ordering on R_n are investigated in [6].

In [19], Richardson and Springer investigate the Borel orbits in the setting of symmetric spaces. In particular, they show that the set of involutions I_n of S_n parameterizes the Borel

orbits in the symmetric space $\mathrm{SL}_n/\mathrm{SO}_n$, and furthermore, the corresponding B_n -ordering on I_n agrees with the restriction of the Bruhat-Chevalley ordering from S_n (see [20]). Here SL_n is the special linear group and SO_n is its special orthogonal subgroup. Also in [19], they show that B_n -orbits in $\mathrm{SL}_n/\mathrm{Sp}_n$ are parameterized by $F_n \subset I_n$.

The monoid of matrices Mat_n can be viewed as a partial compactification of GL_n , and similarly, the set of all symmetric matrices (respectively, set of all skew-symmetric matrices) can be viewed as a partial compactification of $\mathrm{SL}_n/\mathrm{SO}_n$ (respectively, of $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$). Similar to the construction of R_n , by using suitable modifications of the method of Gauss-Jordan elimination, it is shown in [22] for $X = \mathrm{Sym}_n$, and in [8] for $X = \mathrm{Skew}_n$ that the B_n -orbits of the action

$$g \cdot A = (g^{-1})^\top A g^{-1}, \quad g \in B_n, \quad A \in X \quad (3)$$

are parameterized by PI_n and PF_n , respectively. Further combinatorial properties of the B_n -ordering on PI_n and on PF_n are investigated by the second author in the papers [1] (joint with E. Bagno) and [8].

There is an interesting relation between PF_n and the set of invertible involutions: Let $x \in PF_n$ be a partial fixed-point-free involution with determinant 0. We denote by $\tilde{x}$ the completion of x to an involution in I_n by adding the missing diagonal entries. For example,

$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \tilde{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define $\phi : PF_n \rightarrow I_n$ by setting

$$\phi(x) = \begin{cases} \tilde{x} & \text{if } x \in PF_n \setminus F_n, \\ x & \text{otherwise.} \end{cases} \quad (4)$$

It is not difficult to check that ϕ is a bijection between PF_n and I_n such that $\phi(x) = x$ for all $x \in F_n$. Now that we have two sets in bijection with corresponding B_n -orderings, it is natural to ask for their comparison. This is one of the goals of our paper.

Recall that the order complex $\Delta(P)$ of a poset P is the abstract simplicial complex consisting of all chains in P . Important topological information on a simplicial complex is hidden in the orderings of its facets (which corresponds to the maximal chains in P). If the facets are ordered in a way that the intersection of a facet with all the preceding facets is a simplicial subcomplex of codimension 1, then the complex is called *shellable*. In this case, it is known that the simplicial complex has the homology type of a sphere, or of a ball. For posets, a purely combinatorial criteria for checking the shellability condition is found by Björner in [2], and it is called the “lexicographic shellability” of P .

A finite graded poset P with the unique maximum $\hat{1}$ and the unique minimum $\hat{0}$ is called *EL-shellable*, if there exists a map $f = f_r : C(P) \rightarrow \Gamma$ from the set of covering relations $C(P)$ of P into a totally ordered set Γ satisfying

1. in every interval $[x, y] \subseteq P$ of length $k > 0$ there exists a unique saturated chain

$$\mathbf{c} : x_0 = x < x_1 < \cdots < x_{k-1} < x_k = y$$

such that the entries of the sequence

$$f(\mathbf{c}) = (f(x_0, x_1), f(x_1, x_2), \dots, f(x_{k-1}, x_k)) \quad (5)$$

are weakly increasing.

2. The sequence (5) is lexicographically smallest among all sequences of the form

$$(f(x_0, x'_1), f(x'_1, x'_2), \dots, f(x'_{k-1}, x_k)),$$

where $x_0 < x'_1 < \cdots < x'_{k-1} < x_k$.

In literature there are different versions of lexicographic shellability and EL-shellability is known to imply the others (see [23]). A brief history of the shellability questions in Borel orbit posets is as follows: In [11], Edelman proves that BC-order on S_n is EL-shellable. Shortly after, Proctor in [16] shows that all classical Weyl groups are EL-shellable. Around the same time, in [3], Björner and Wachs show that Bruhat-Chevalley ordering on all Coxeter groups, as well as on all sets of minimal-length coset representatives (quotients) in Coxeter groups are “dual CL-shellable” (a weaker alternative to EL-shellability). A decade after the introduction of CL-shellability, in [10], M. Dyer shows that Bruhat-Chevalley ordering on all Coxeter groups and all quotients are EL-shellable. As an application of EL-shellability, using Dyer’s methods, in [24], L. Williams shows that the poset of cells of a cell decomposition for totally non-negative part of a flag variety is EL-shellable. In the papers [13] and [12] A. Hultman, although avoids showing lex. shellability, obtains the same topological consequences for the Bruhat-Chevalley ordering on “twisted involutions” in Coxeter groups.

There are various directions that the results of [3] are extended. For semigroups, in [17], Putcha shows that “ J -classes in Renner monoids” are CL-shellable. In [4], the first author shows that for the special Renner monoid R_n , not only the J -classes are lex. shellable, but also the whole rook monoid R_n is EL-shellable. In [7], the first and the third authors show that PI_n is EL-shellable. In [5], we show that F_n is also EL-shellable, and furthermore, its order complex is a ball of appropriate dimension. In [14], Incitti shows that I_n is EL-shellable, and in [15] he shows that the B -order on involutions in all classical Weyl groups are EL-shellable.

Contributing to the above literature, we show in this paper that PF_n is an EL-shellable poset. Moreover, we show that the order complex of PF_n triangulates a ball of dimension $n(n-1)/2$. On the other hand, it is known that the order complex of I_n triangulates a sphere of dimension $\lfloor n^2/4 \rfloor$ (see [14], page 255).

The structure of our paper is as follows. In the next section we introduce basic notation for poset theory. In particular, we recollect some known, basic facts about Bruhat-Chevalley ordering on rooks and partial involutions. In Subsection 2.3, we compare the length functions of PF_n and PI_n .

Unfortunately, PF_n is not a connected subposet of PI_n , hence we are not able to directly utilize our earlier results from [7]. Therefore, we devote all of Section 3 for the review of the covering relations of I_n , F_n , and of PI_n in order for describing the covering relations of PF_n next.

In Section 4 we present our proof of EL-shellability of PF_n . As an application of this result, in Section 5, we determine the homotopy type of the order complex of the proper part of PF_n , namely PF_n with its smallest and the largest elements excluded.

In the final section of our paper, we investigate the length-generating functions of certain subposets of PF_n . In particular, we relate our length generating function computations to the number of rational points of the variety of skew-symmetric matrices of fixed rank defined over a finite field.

2 Preliminaries

Notation: Let m be a positive integer. We denote the set $\{1, \dots, m\}$ by $[m]$. The rank of a matrix $x \in \text{Mat}_n$ is denoted by $rk(x)$.

2.1 Poset terminology

All of our posets are assumed to be finite, graded, and furthermore, they are assumed to possess a minimal and a maximal element, denoted by $\hat{0}$ and $\hat{1}$, respectively. We reserve the letter P as the name of a generic such poset and denote by $\ell : P \rightarrow \mathbb{N}$ (or, by ℓ_P , if needed) the length function on P . The set of all covering relations in P is denoted by $C(P)$. If $(x, y) \in C(P)$, then we write $y \rightarrow x$ to mean that y covers x .

Recall that the Möbius function of P is defined recursively by the formula

$$\begin{aligned}\mu([x, x]) &= 1, \\ \mu([x, y]) &= - \sum_{x \leq z < y} \mu([x, z])\end{aligned}$$

for all $x \leq y$ in P . As customary, we denote by $\Delta(P)$ the order complex of P . It is well known that $\mu(\hat{0}, \hat{1})$ is equal to the “reduced Euler characteristic” $\tilde{\chi}(\Delta(P))$ of the topological realization of $\Delta(P)$. See Proposition 3.8.6 in [21].

Let Γ denote a finite totally ordered set and let g be a Γ -valued function defined on $C(P)$. Then g is called an *R-labeling* of P , if for every interval $[x, y]$ in P , there exists a unique chain $x = x_1 \leftarrow x_2 \leftarrow \dots \leftarrow x_{n-1} \leftarrow x_n = y$ such that

$$g(x_1, x_2) \leq g(x_2, x_3) \leq \dots \leq g(x_{n-1}, x_n). \quad (6)$$

Thus, P is EL-shellable, if it has an *R-labeling* $g : C(P) \rightarrow \Gamma$ such that for each interval $[x, y]$ in P the sequence (6) is lexicographically smallest among all sequences of the form

$$(g(x, x'_2), g(x'_2, x'_3), \dots, g(x'_{k-1}, y)),$$

where $x \leftarrow x_2 \leftarrow' \cdots \leftarrow x'_{k-1} \leftarrow y$.

For $S \subseteq [n]$, by P_S we denote the subset $P_S = \{x \in P : \ell(x) \in S\}$, and denote by μ_S the Möbius function of the poset $\hat{P}_S$ that is obtained from P_S by adjoining a smallest and a largest element, if they are missing. For an R -labeling $g : C(P) \rightarrow \Gamma$ of P , it is well known that the quantity $(-1)^{|S|-1} \mu_S(\hat{0}_{\hat{P}_S}, \hat{1}_{\hat{P}_S})$ is equal to the number of maximal chains $x_0 = \hat{0} \leftarrow x_1 \leftarrow \cdots \leftarrow x_n = \hat{1}$ in P for which the sequence $(g(x_0, x_1), \dots, g(x_{n-1}, x_n))$ has descent set S , that is to say, for which $\{i \in [n] : g(x_{i-1}, x_i) \geq g(x_{i+1}, x_i)\} = S$. See Theorem 3.14.2 in [21].

2.2 B -order on partial involutions

The notation F_n , I_n , PI_n , R_n , S_n , Skew_n , and Sym_n are as in the introduction.

Recall that R_n parameterizes the $B_n \times B_n$ -orbits in Mat_n . For the purposes of this paper, it is more natural for us to look at the inclusion poset of $B_n^\top \times B_n$ -orbit closures in R_n , whose dual poset we denote by $(R_n, \leq_{\text{Rook}})$. Here $B_n^\top$ is the Borel subgroup of all lower triangular matrices from GL_n .

In [8], the second author shows that the Borel orbits in Skew_n are parameterized by those elements $x \in \text{Skew}_n$ such that

1. the entries of x are either 0, 1 or -1,
2. any non-zero entry of x that is above the main diagonal is a +1,
3. in every row and column of x there exists at most one non-zero entry.

Note that when -1's in x are replaced by +1's, the resulting matrix $\tilde{x}$ is a partial involution with no diagonal entry. In other words, $\tilde{x}$ is a fixed-point-free partial involution. It is obvious that this correspondence is a bijection, hence PF_n parameterizes the Borel orbits in Skew_n .

Containment relations among the closures of Borel orbits in Skew_n define a partial ordering on PF_n . We denote its dual by $\leq_{\text{Skew}}$. Similarly, on PI_n we have the dual of the partial ordering induced from the containment relations among the Borel orbit closures in Sym_n . We denote this dual partial ordering by $\leq_{\text{Sym}}$.

2.3 Combinatorial approach to the posets R_n, PI_n, PF_n

There is a combinatorial method for deciding when two elements x and y from $(R_n, \leq_{\text{Rook}})$ (respectively, from $(PI_n, \leq_{\text{Sym}})$, or from $(PF_n, \leq_{\text{Skew}})$) are comparable with respect to $\leq_{\text{Rook}}$ (respectively, with respect to $\leq_{\text{Sym}}$, or $\leq_{\text{Skew}}$). We denote by $Rk(x)$ the matrix whose i, j -th entry is the rank of the upper left $i \times j$ submatrix of x . Hence, $Rk(x)$ is an $n \times n$ matrix with non-negative integer entries. We call $Rk(x)$, the *rank-control matrix* of x .

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two matrices of the same size with real number entries. We write $A \leq B$ if $a_{i,j} \leq b_{i,j}$ for all i and j . Then

$$x \leq_{\text{Rook}} y \iff Rk(y) \leq Rk(x). \quad (7)$$

The same criterion holds for the posets $\leq_{Sym}$ and $\leq_{Skew}$.

We recall some fundamental facts about the covering relations of $\leq_{Sym}$ and $\leq_{Skew}$. Our references are [1] and [8]. Let $Rk(x) = (r_{i,j})_{i,j=1}^m$ denote the rank-control matrix of an $m \times m$ matrix x . As a notation we set $r_{0,i} = 0$ for $i = 0, \dots, m$ and define

$$\rho_{\leq}(x) = \#\{(i, j) : 1 \leq i \leq j \leq n \text{ and } r_{i,j} = r_{i-1,j-1}\}, \quad (8)$$

$$\rho_{<}(x) = \#\{(i, j) : 1 \leq i < j \leq n \text{ and } r_{i,j} = r_{i-1,j-1}\}. \quad (9)$$

Then the length function ℓ_{PF_n} of the poset PF_n is equal to the restriction of $\rho_{<}$ to PF_n . Furthermore, x covers y if and only if $Rk(x) \leq Rk(y)$ and $\ell_{PF_n}(x) - \ell_{PF_n}(y) = 1$.

The length function of PF_n differs from the length function of PI_n in two ways: The ranks of two matrices $y < x$ in PF_n differ by a multiple of 2, and the smallest element in PI_n is the identity matrix, which is not in PF_n . The minimal element in PF_n is given by the matrix with the largest rank-control matrix. This means that in the case when n is even $\ell_{PF_n}(x) = \ell_{PI_n}(x) - \frac{n-rk(x)}{2} - \frac{n}{2}$. We subtract $\frac{n-rk(x)}{2}$ so that the length function increases only by 1 if the rank drops by 2 and we subtract $\frac{n}{2}$ because the minimal element has to have length zero. Similarly, when n is odd we have to subtract $\frac{n-1-rk(x)}{2}$ and $\frac{n+1}{2}$. Summarizing, we see that for all n the length function $\ell_{PF_n}(x)$ of PF_n is given by

$$\begin{aligned} \ell_{PF_n}(x) &= \ell_{PI_n}(x) - \frac{n - rk(x)}{2} - \frac{n}{2} \\ &= \ell_{PI_n}(x) - \frac{2n - rk(x)}{2} \\ &= \rho_{<}(x) - \frac{2n - rk(x)}{2}. \end{aligned} \quad (10)$$

Example 1. When $n = 6$, the smallest element is

$$\omega_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\text{and when } n = 5, \text{ the smallest element is } \omega_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3 An EL-labeling of PI_n

We recall some results on the covering relations of I_n , F_n , and of PI_n [14, 5, 7].

3.1 EL-labeling of I_n

For a permutation $\sigma \in S_n$, a *rise* of σ is a pair of indices $1 \leq i_1, i_2 \leq n$ such that

$$i_1 < i_2 \text{ and } \sigma(i_1) < \sigma(i_2).$$

A rise (i_1, i_2) is called *free*, if there is no $k \in [n]$ such that

$$i_1 < k < i_2 \text{ and } \sigma(i_1) < \sigma(k) < \sigma(i_2).$$

For $\sigma \in S_n$, define its *fixed point set*, its *exceedance set* and its *defect set* to be

$$\begin{aligned} I_f(\sigma) &= \text{Fix}(\sigma) = \{i \in [n] : \sigma(i) = i\}, \\ I_e(\sigma) &= \text{Exc}(\sigma) = \{i \in [n] : \sigma(i) > i\}, \\ I_d(\sigma) &= \text{Def}(\sigma) = \{i \in [n] : \sigma(i) < i\}, \end{aligned}$$

respectively.

Given a rise (i_1, i_2) of σ , its *type* is defined to be the pair (a, b) , if $i_1 \in I_a(\sigma)$ and $i_2 \in I_b(\sigma)$, for some $a, b \in \{f, e, d\}$. We call a rise of type (a, b) an *ab-rise*. On the other hand, two kinds of *ee*-rises have to be distinguished from each other; an *ee*-rise is called *crossing*, if $i_1 < \sigma(i_1) < i_2 < \sigma(i_2)$, and it is called *non-crossing*, if $i_1 < i_2 < \sigma(i_1) < \sigma(i_2)$.

The rise (i_1, i_2) of an involution $\sigma \in I_n$ is called *suitable* if it is free and if its type is one of the following: (f, f) , (f, e) , (e, f) , (e, e) , (e, d) .

A *covering transformation*, denoted $ct_{(i_1, i_2)}(\sigma)$, of a suitable rise (i_1, i_2) of σ is the involution obtained from σ by moving the 1's from the black dots to the white dots as depicted in Figure 1.

It is shown in [14] that if τ and σ are two involutions in I_n , then

$$\tau \text{ covers } \sigma \text{ in } \leq_{Sym} \iff \tau = ct_{(i_1, i_2)}(\sigma), \text{ for some suitable rise } (i_1, i_2) \text{ of } \sigma.$$

Let Γ denote the totally ordered set $[n] \times [n]$ with respect to lexicographic ordering. In the same paper, Incitti shows that the labeling defined by

$$f_\Gamma((\sigma, ct_{(i_1, i_2)}(\sigma))) := (i_1, i_2) \in \Gamma \tag{11}$$

is an EL-labeling, hence, $(I_n, \leq_{Sym})$ is an EL-shellable poset.

3.2 EL-labeling of F_{2n}

Recall that F_{2n} is a connected graded subposet of I_{2n} . Therefore, its covering relations are among the covering relations of I_{2n} . On the other hand, within F_{2n} we use two types of covering transformations, only: a non-crossing *ee*-rise and an *ed*-rise. These moves correspond to the items numbered 4 and 6 in Table 1 of [14]. It is shown in [5] that these covering labels is an EL-labeling for F_{2n} .

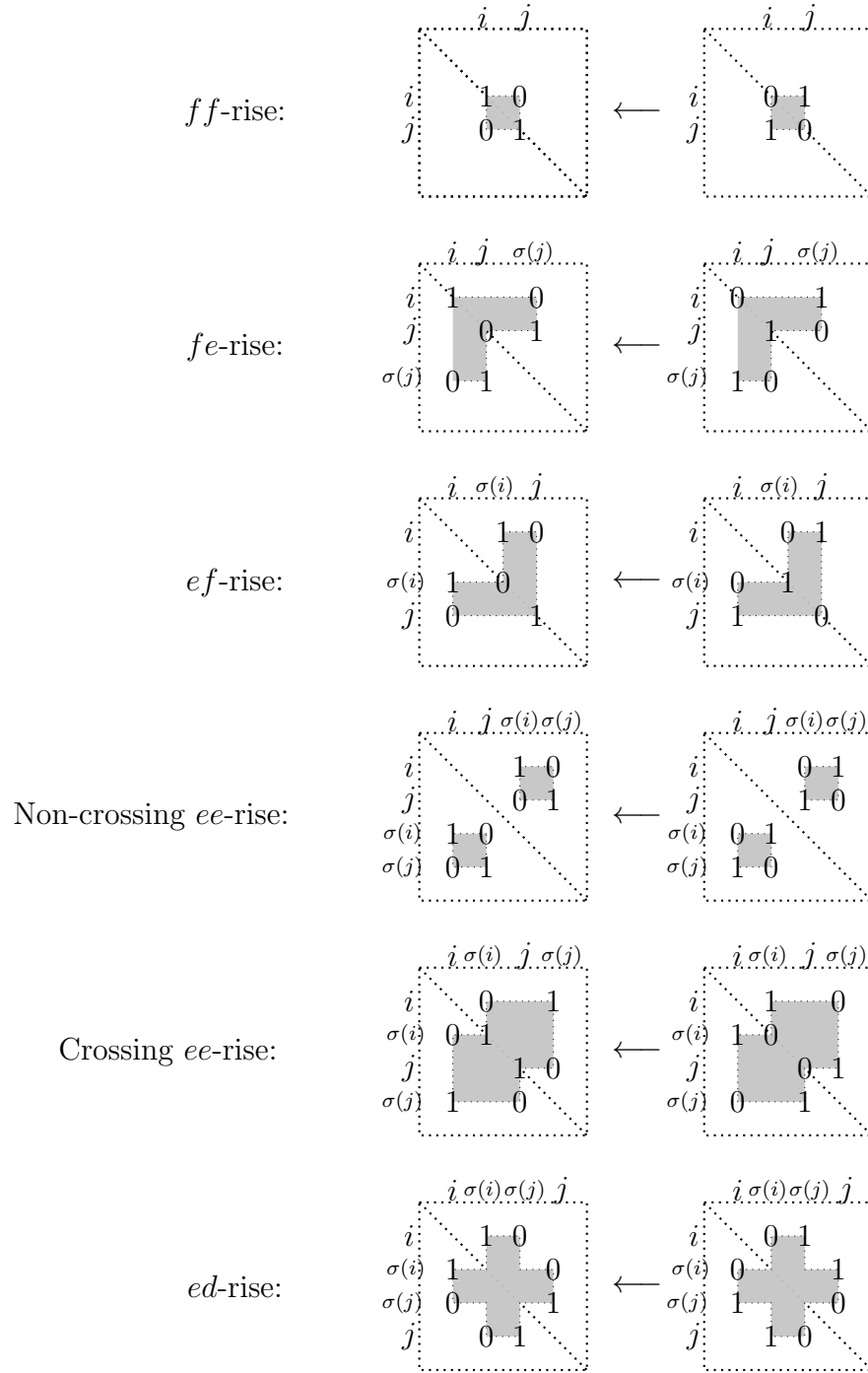


Figure 1: Covering transformations $\sigma \leftarrow \tau = ct_{(i,j)}(\sigma)$ of I_n .

3.3 EL-labeling of PI_n

When two partial involutions x and y have the same zero rows and zero columns, the covering relation $x \rightarrow y$ is the same as in the case of invertible involutions.

Example 2.

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that $x \rightarrow y$ if and only if the invertible involution $\tilde{x}$, that is obtained from x by removing the rows and columns of x with no non-zero entries, covers the invertible involution $\tilde{y}$ that is obtained from y by removing its rows and columns with zeros only.

Moving down a non-zero entry along the diagonal gives a covering relation:

Example 3.

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Another type of covering relation is obtained by the moving of off-diagonal pairs (i, j) and (j, i) , where $i > j$ to down/right, or to right/down available positions.

Example 4. There are two cases:

1. $y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is covered by $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,
2. $y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is covered by $x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

When a down/right move is performed on y (as in part 2. of Example 4), there may not be any available positions to place the non-zero entries of x . In this case, the pushed entries are placed on the diagonal. If there are no available diagonal entries for both of the 1's, then one of them is pushed out of the matrix.

Example 5. Once again, there are two moves of similar nature:

$$1. \ y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$2. \ y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the light of the above examples, we label a covering relation $x \rightarrow y$ in PI_n as follows.

- Definition 6.**
1. As in Example 2, if the covering relation $x \rightarrow y$ is derived from the covering relation $\tilde{x} \rightarrow \tilde{y}$ of invertible involutions that are obtained from x and y , respectively, then we use the labeling $\tilde{x} \rightarrow \tilde{y}$ as defined in [14].
 2. If the covering relation results from a move as in Example 3, namely from a diagonal push where the element that is pushed from is at the position (i, i) , then we label it by (i, i) .
 3. Suppose $x \rightarrow y$ is as in Example 4, or 5. Observe that, in all of these covering relations, one of the 1's is pushed down and the other is pushed right. Let i denote the column index of the first 1 that is pushed to the right, and let j denote the index of the resulting column. Then we label the covering by (i, j) .

To illustrate the third labeling let us present a few more examples.

Example 7.

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding labeling here is $(3, 5)$.

Example 8.

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding labeling here is $(1, 3)$.

Example 9.

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding labeling here is $(2, 3)$.

Definition 10. If x covers y with label (i, j) , then we refer to it as an (i, j) -covering and say that y is obtained from x by an (i, j) -move. More briefly, we call a covering relation a c -cover, if it is derived from an involution; a d -cover, if it is obtained by a shift of a diagonal element; an r -cover, if it is derived from a right/down, or from a down/right move. The corresponding moves of 1's are referred to as c -, d - and r -moves.

Lemma 11 (Lemma 16, [7]). *Let x and y be two partial involutions. Then x covers y if and only if one of the following is true:*

1. x is obtained from y by a c -move as in Example 2.
2. Without removing a suitable rise, x is obtained from y by one of the following moves:
 - (a) a d -move, as in Example 3,
 - (b) an r -move, as in Example 4, or as in Example 5.

It is shown in [7] that the covering labelings defined in Definition 10 is an EL-labeling for PI_n .

4 An EL-labeling of PF_n

Covering relations of F_n are covering relations in I_n , as well. Unfortunately, this is not the case for PF_n relative to PI_n . In other words, as a subposet of PI_n , PF_n is not connected. For example, when $n = 2$, there are only two partial fixed-point-free involutions: $x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, hence x covers y as a partial fixed-point-free involution.

However, viewed as a partial involution x does not cover y since $y < \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} < x$.

Lemma 12. *Suppose $x \rightarrow y$ in PF_n . Then either x covers y as an element of PI_n , or there exists $z \in PI_n$ such that $x \rightarrow z$ by an d -cover as an element of PI_n , and $z \rightarrow y$ by an r -cover in PI_n , where at each step the rank drops by 1. Furthermore, in the first case, there are two possibilities:*

1. $x \rightarrow y$ is an r -cover in PI_n , or

2. $x \rightarrow y$ is a c -cover corresponding to a non-crossing ee , or to an ed -rise in PI_n .

Proof. Obviously, if x covers y in PI_n and if both x and y are members of PF_n , then x covers y in PF_n , also. Thus, the last assertion follows from Lemma 11.

We proceed with the assumption that $x, y \in PF_n$ but x does not cover y in PI_n . Towards a contradiction, assume that there does not exist $z \in PI_n$ as in the conclusion of the lemma. This means that the open interval $(y, x) = \{z \in PI_n : y < z < x\}$ lies in $PI_n \setminus PF_n$. In other words, any $z \in (y, x)$ has to have a non-zero diagonal entry. This eliminates the possibility of $z \rightarrow y$ being a c -cover (see Figure 1). Clearly, $z \rightarrow y$ cannot be a d -cover, neither.

We continue with the assumption that z is obtained from y by an r -move, which places two symmetric entries on the diagonal. In this case, another r -move is possible in y involving the same 1's. (To construct an example to this situation, start with y as in Example 9.) Let z_1 denote this new element from PF_n . By comparing their rank-control matrices, we see that $Rk(x) < Rk(z_1)$, hence $y < z_1 < x$. This contradicts with our assumption that the interval (y, x) lies in $PI_n \setminus PF_n$. Therefore, z covers y by an r -move, by deleting a 1 from y and placing another to diagonal. Then by a d -move removing this diagonal 1 we obtain x . Thus we obtain a contradiction to our initial assumption. $\square$

Remark 13. Let x and y be two elements from PF_n such that x covers y by an r -move. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ denote x and y in one-line notation. Then exactly one of the following statements is true:

1. x is obtained from y by replacing exactly two entries of $y = (y_1, \dots, y_n)$ by 0's.
2. There exists $i \in [n]$ such that x is obtained from y by replacing y_i by the number x_i , setting y_i -th entry of y to 0 and replacing the x_i -th entry of y (which is a 0) by i .

In the light of Lemma 12 we give the following definition.

Definition 14. 1. If the covering relation is derived from a c -move, then we use the labeling as defined in [7] and transform this label (i, j) into $(n - i, n - j)$.

2. If the covering relation $x \rightarrow y$ results from an r -move, then we define the label to be $(i + n, j)$, where $x > y$ results from y by moving the 1 in column i to row j . If the 1 is pushed out of the matrix, then we set $j = n + 1$.

In the case of invertible fixed-point-free involutions we show in [5] that the lexicographically largest chain is the only decreasing chain. Since the label is transformed from (i, j) to $(n - i, n - j)$ now the lexicographically smallest chain is increasing. The reason the label of r -moves is shifted by n in the first coordinate is to ensure that every r -cover has a bigger label than any c -cover. In Figure 2, we illustrate the Definition 14.

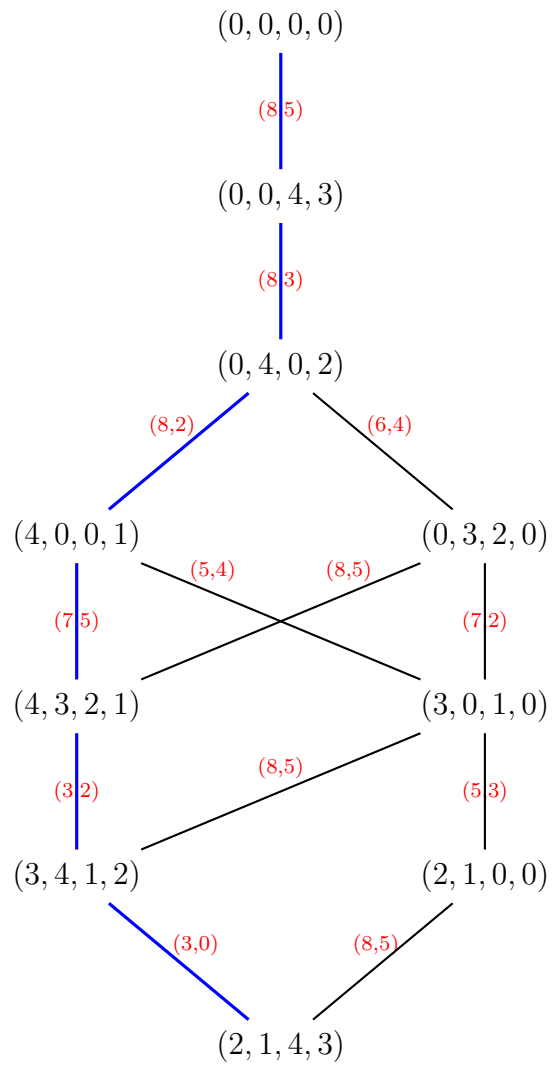


Figure 2: The EL-labeling of PF_4 .

Proposition 15. *Let $y < x$ be two partial fixed-point-free involutions from PF_n , and let*

$$\mathbf{c} : x = x_1 < x_2 < \cdots < x_{s+1} = y$$

denote the saturated chain whose sequence of labels $f(\mathbf{c})$, as defined in Definition 14, is lexicographically smallest among all such sequences. Then $f(\mathbf{c})$ is a weakly increasing sequence.

Proof. Towards a contradiction assume that $f(\mathbf{c})$ is not weakly increasing. Then there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $\mathbf{c}$ such that $f((x_{t-1}, x_t)) > f((x_t, x_{t+1}))$. We have 4 cases to consider:

Case 1: $\text{type}((x_{t-1}, x_t)) = c$, and $\text{type}((x_t, x_{t+1})) = c$,

Case 2: $\text{type}((x_{t-1}, x_t)) = r$, and $\text{type}((x_t, x_{t+1})) = r$,

Case 3: $\text{type}((x_{t-1}, x_t)) = c$, and $\text{type}((x_t, x_{t+1})) = r$,

Case 4: $\text{type}((x_{t-1}, x_t)) = r$, and $\text{type}((x_t, x_{t+1})) = c$.

In each of these cases, we either produce an immediate contradiction by showing that the two moves are interchangeable (hence $\mathbf{c}$ is not the smallest chain), or we construct an element $z \in [x, y] \cap PF_n$ which covers x_{t-1} , and such that $f((x_{t-1}, z)) < f((x_{t-1}, x_t))$. Since we assume that $f(\mathbf{c})$ is the lexicographically smallest Jordan-Hölder sequence, the existence of such an element z is a contradiction, also.

To this end, suppose that the label of the first move ($x_t \rightarrow x_{t-1}$) is (i, j) , and the second move ($x_{t+1} \rightarrow x_t$) is labeled by (k, l) .

Case 1: Follows from the proof for invertible fixed-point-free involutions, see [5].

Case 2: If $i = k$, then $l > j$. In this case, we interchange the two moves to obtain our desired contradiction. Therefore we continue with assuming $k < i$. If $k - n = j$ then $j < i - n$ and $(m + n, l)$ is possible in x_{t-1} with $m < j < i$, where $(m, i - n)$ is the position of the 1 in x_{t-1} . If $k - n \neq j$ then either the two moves are interchangeable, or (k, l) removes a suitable rise in x_{t-1} which corresponds to a move with a smaller label than (i, j) .

Case 3: This case is impossible since every c -move has a smaller label than any r -move.

Case 4: If the r -cover labeled (i, j) is the covering relation with the lexicographically smallest label then there is no suitable rise in x_{t-1} . The c -move has to involve one of the moved 1's since otherwise there is a suitable rise in x_{t-1} . For this, one of the moved 1's has to have a 1 to the upper left or the lower right in x_t that was not to the upper left or lower right of it in x_{t-1} . Since the 1's are moved right and down respectively, it is impossible that there is a 1 to the lower right in x_t that is not to the lower right in x_{t-1} . If the c -cover corresponds to the suitable rise $(m, i - n)$ (with label $(n - m, i)$), then (i, j) is not the r -move with the smallest label in x_{t-1} since in this case $(m + n, j)$ is possible in x_{t-1} with $(n + m, j) < (i, j)$. If the c -cover corresponds to the rise (m, j) , then the r -move $(m + n, i - n)$ is possible in x_{t-1} which again has a smaller label than (i, j) .

□

Proposition 16. *We retain the notation from (the proof of) Proposition 15. Then $f(\mathbf{c})$ is the unique increasing chain in $[y, x]$.*

Proof. We use induction on the length $s + 1$ of the interval $[y, x]$ to prove that no other chain is lexicographically increasing. Clearly, if x covers y , there is nothing to prove, so, we assume that for any interval of length $k \leq s$ there exists a unique increasing maximal chain.

Assume that there exists another increasing chain

$$\mathbf{c}' : y = x_0 < x'_1 < \cdots < x'_s < x_{s+1} = x.$$

Since the length of the chain

$$x'_1 < \cdots < x'_s < x_{s+1} = x$$

is s , by the induction hypothesis, it is the lexicographically smallest chain between x'_1 and x . We are going to find contradictions to each of the following possibilities:

- Case 1: $\text{type}(x_0, x_1) = c$, and $\text{type}(x_0, x'_1) = c$,
- Case 2: $\text{type}(x_0, x_1) = r$, and $\text{type}(x_0, x'_1) = r$,
- Case 3: $\text{type}(x_0, x_1) = c$, and $\text{type}(x_0, x'_1) = r$,
- Case 4: $\text{type}(x_0, x_1) = r$, and $\text{type}(x_0, x'_1) = c$.

In each of these cases we will construct a partial fixed-point-free involution $z \in [y, x]$ such that z covers x'_1 and $f((x'_1, z)) < f((x'_1, x'_2))$, contradicting the induction hypothesis. To this end, let $f((x_0, x_1)) = (i, j)$, $f((x_0, x'_1)) = (k, l)$ and assume that $(k, l) < (i, j)$.

Case 1: Done in the proof for the invertible case, see [5].

Case 2: It is impossible for $i = k$ since there is only one r -move for each 1. Therefore assume that $i < k$. Let the moved 1's be on the symmetric positions $(i - n, m)$ and $(m, i - n)$ in x_0 . If $k = m + n$ then $(l + n, j)$ is possible in x'_1 with $(l + n, j) < (k, l)$. If $k \neq m + n$ then either the two moves are interchangeable or the suitable rise $(n - i, n - k)$ is possible in x'_1 .

Case 3: Since no r -move can remove a suitable rise, there exists a legal c -move in x'_1 . But this c -move has a smaller label than (k, l) which is our desired contradiction.

Case 4: This case is not possible because every c -move has a smaller label than any r -move.

□

Combining previous two propositions, we have our first main result:

Theorem 17. *The poset PF_n is an EL-shellable poset.*

5 The order complex of PF_n

In [5], it is shown that the order complex $\Delta(F_n)$ of fixed-point-free involutions triangulates a ball of dimension $n^2 - n - 2$. In this section we obtain a similar result for PF_n .

Lemma 18. For all $n \geq 2$,

$$\dim \Delta(PF_n) = \ell(PF_n) = n + (n-1) + \cdots + 1 - n = \binom{n}{2}.$$

Proof. Straightforward by using (10). □

We continue by analyzing the intervals of length two.

Lemma 19. Each length two interval $[y, x] \subseteq PF_n$ has at most four elements.

Proof. Just as in the proof of Theorem 17, if $y < z < x$, then there are 4 cases to consider:

- Case 1: $\text{type}((y, z)) = c$, and $\text{type}((z, x)) = c$,
- Case 2: $\text{type}((y, z)) = r$, and $\text{type}((z, x)) = r$,
- Case 3: $\text{type}((y, z)) = c$, and $\text{type}((z, x)) = r$,
- Case 4: $\text{type}((y, z)) = r$, and $\text{type}((z, x)) = c$.

In the first case, $[y, x]$ is isomorphic to an interval in F_m for some $m \leq n$, and therefore, it has at most 4 elements (since F_m is a connected subposet of I_m , which is Eulerian).

In the second case, we look at the one-line notations of y and x . If z is obtained from y by setting two non-zero entries of y to 0's, and if, at the same time, x is obtained from z by setting two non-zero entries of z to 0's, then y and x differ at exactly 4 entries. Therefore, $[y, x]$ contains at most one other element other than z , which is obtained from y by setting two entries of y to 0's. If z is obtained by increasing the i -th entry y_i of y to z_i , and if, at the same time, x is obtained from z by increasing the i -th entry z_i of z to x_i , then $[y, x]$ has exactly 3 elements. If z is obtained from y by increasing the i -th entry y_i of y to z_i , and if x is obtained from z with no overlap with the replaced/increased entries of y , then $[y, x]$ has exactly 4 elements. Finally, if z is obtained by increasing the i -th entry y_i of y to z_i , and x is obtained from z by replacing the z_i -th entry of z by 0, then y and x differ at exactly at 4 positions. Therefore, the interval $[y, x]$ have at most 4 elements.

Since the arguments of Case 3 and Case 4 are identical, we handle Case 3 only. Suppose that there exist more than 4 elements in $[y, x]$. Since one of the elements $y < z < x$ is obtained from y by a c -move, the covering type of any other $y < z_1 < x$ is not of type c . Otherwise, to obtain x from z we need to apply another c -move to z . But then the matrix ranks of y and x would be the same. Therefore, we conclude that if $z_1 \neq z$ and $y < z_1 < x$, then z_1 is obtained from y by an r -move, and x is obtained from z_1 by a c -move. Now it is clear that it is impossible to have another element $y < z_2 < x$ such that z_2 covers y by an r -move and $z_2 \notin \{z, z_1\}$. Therefore $[y, x]$ have exactly 4 elements and the proof is complete. □

We know from [9] that a pure, shellable simplicial complex Δ of which every $\dim \Delta - 1$ face is contained in at most two facets is homeomorphic to either a ball, or a sphere. By Lemma 19, we see that $\Delta(PF_n)$ satisfies this property.

Theorem 20. Let $\widetilde{PF}_n$ denote the proper part of PF_n , namely the subposet obtained from PF_n by removing its smallest and the largest elements. For $n \geq 3$, the order complex $\Delta(PF_n)$ triangulates a ball of dimension $\dim \Delta(PF_n) - 2 = \binom{n}{2} - 2$.

Proof. By the discussion above, it is enough to show that the reduced Euler characteristic of $\Delta(\widetilde{PF}_n)$ is 0.

By Hall's Theorem (see Chapter 3, [21]), we know that the reduced Euler characteristic of an order complex of a poset P is equal to the value of the Möbius function $\mu_{\widehat{P}}$ on the interval $[\widehat{0}, \widehat{1}]$, where $\widehat{P}$ is P with a $\widehat{0}$ (a smallest element) and a $\widehat{1}$ (a largest element) adjoined. Therefore, it is enough to show that $\mu_{PF_n}([\widehat{0}, \widehat{1}]) = 0$, where $\widehat{0} = (0, \dots, 0)$ and $\widehat{1} = (0, \dots, 0, n, n-1)$.

Let PF_n^* denote the dual of PF_n . By abuse of notation we use $\widehat{0}$ for the smallest element of PF_n^* although it is $\widehat{1}$ of PF_n . Similarly, we denote the largest element of PF_n^* by $\widehat{1}$. Now, since $\mu_{PF_n}([\widehat{0}, \widehat{1}]) = \mu_{PF_n^*}([\widehat{0}, \widehat{1}])$, we are going to show that the later value is zero.

It is easy to see that the cardinality of the set $\{x \in PF_n^* : \ell_{PF_n^*}([\widehat{0}, x]) \leq 3\}$ is 1, for $n \geq 3$. Indeed, if $\ell_{PF_n^*}(x) = 3$, then in one-line notation $x = (0, \dots, 0, n, 0, n-2)$, and $[\widehat{0}, x] = \{\widehat{0} < 0 < z_0 < x\}$, where $z_0 = (0, \dots, 0, n, n-1)$.

For simplicity, let us denote $\mu_{PF_n^*}$ by μ , and denote the length function $\ell_{PF_n^*}$ by ℓ . We prove by induction that $\mu([\widehat{0}, z]) = 0$ for all z with $\ell(z) > 1$. Our base case is when $\ell(z) = 2$. In this case, $[\widehat{0}, z]$ is a chain of length 2 by the discussion in the previous paragraph, and hence, the corresponding value is 0. Now assume that $\mu([\widehat{0}, z]) = 0$ for all z with $2 \leq \ell(z) \leq s$, and let $z' \in PF_n^*$ be an element with $\ell(z') = s+1$. Since

$$\mu([\widehat{0}, z']) = - \sum_{\widehat{0} \leq z < z'} \mu([\widehat{0}, z]) = -(\mu([\widehat{0}, \widehat{0}]) + \mu([\widehat{0}, 0]) = -(1 + (-1)) = 0,$$

the proof is complete. □

6 Length-generating functions

Recall that the *standard form* of an involution $\pi \in I_n$ is a product of transpositions of the form

$$\pi = (i_1, j_1)(i_2, j_2) \cdots (i_m, j_m), \quad (12)$$

where for all $1 \leq t \leq m$, $i_t < j_t$ and $i_1 < i_2 < \cdots < i_m$. We call the transpositions appearing in (12) as *arcs*. Using bijection (4) from the Introduction section, we identify the elements of PF_n as involutions in S_n . With this identification, let us denote by $I(n, k)$ the set of involutions of S_n having k arcs, and define its length generating function by

$$\mathbf{i}_q(n, k) := \sum_{\pi \in I(n, k)} q^{\ell_{PF_n}(\pi)}.$$

Recall also that the q -analog of a natural number $n \in \mathbb{N}$ is the polynomial $[n]_q = 1 + q + \cdots + q^{n-1}$.

Proposition 21. *For all $n \geq 2$ and $k \in \{2, \dots, n\}$, we have*

$$\mathbf{i}_q(n+1, k) = q^n \mathbf{i}_q(n, k) + [n]_q \mathbf{i}_q(n-1, k-1).$$

Proof. We begin with defining a bijection:

$$\Phi : I(n+1, k) \rightarrow I(n, k) \bigcup (\{2, 3, \dots, n, n+1\} \times I(n-1, k-1)).$$

Let π be an element of $I(n+1, k)$. If $\pi(1) = 1$, then we define $\Phi(\pi) = \sigma \in I(n, k)$ as follows: $\sigma(j) = \pi(j+1)$ for $j \in \{1, 2, \dots, n\}$. In other words, in matrix notation, σ is obtained from π by deleting its first row and its first column. Notice that if $\pi(1) = 1$, then $\ell_{PF_{n+1}}(\pi) = \ell_{PF_n}(\sigma) + n$, since when we delete the first zero row from the rank-control matrix, the parameter $\rho_{<}$ decreases by n , which is the number of zeros in this row in positions from 2 to $n+1$.

Suppose now that $\pi(1) = i \in \{2, 3, \dots, n+1\}$. In this case, we define $\Phi(\pi)$ to be the pair $\Phi(\pi) = (i, \sigma)$, where σ is the involution from $I(n-1, k-1)$ defined by

$$\sigma(j) = \begin{cases} \pi(j+1) & \text{if } j \in \{1, \dots, i-2\}, \\ \pi(j+2) & \text{if } j \in \{i-1, \dots, n-1\}. \end{cases}$$

In matrix notation, σ is obtained from π by deleting the first and the i -th rows of π , as well as deleting its first and i -th columns. In this case we have: $\ell_{PF_{n+1}}(\pi) = \ell_{PF_{n-1}}(\sigma) + i - 2$. To see this, notice that all the equalities in the upper triangular portion of $Rk(\pi)$ are carried into that of $Rk(\sigma)$ with additional $i - 2$ equalities arising from the 0's at the positions $(1, 2), (1, 3), \dots, (1, i-1)$ of π . Thus $\rho_{<}(\pi) = \rho_{<}(\sigma) + i - 1$. On the other hand, since the ranks of π and σ differ by 2, and their sizes differ by 2, by the formula (10), we see that

$$\begin{aligned} \ell_{PF_{n+1}}(\pi) &= \rho_{<}(\pi) - \frac{2(n+1) - rk(\pi)}{2} \\ &= \rho_{<}(\sigma) + i - 1 - \frac{2(n-1) - rk(\sigma) + 2}{2} \\ &= \ell_{PF_{n-1}}(\sigma) + i - 2. \end{aligned} \tag{13}$$

See Example 22 for an illustration.

Now, in the light of these observations, we derive the desired recurrence:

$$\begin{aligned}
i_q(n+1, k) &= \sum_{\pi \in I(n+1, k)} q^{\ell_{PF_{n+1}}(\pi)} \\
&= \sum_{\pi \in I(n+1, k), \pi(1)=1} q^{\ell_{PF_{n+1}}(\pi)} + \sum_{\pi \in I(n+1, k), \pi(1) \neq 1} q^{\ell_{PF_{n+1}}(\pi)} \\
&= \sum_{\sigma \in I(n, k)} q^{\ell_{PF_n}(\sigma)+n} + \sum_{i=2}^{n+1} \sum_{\sigma \in I(n-1, k-1)} q^{\ell_{PF_{n-1}}(\sigma)+i-2} \\
&= q^n \cdot \sum_{\sigma \in I(n, k)} q^{\ell_{PF_n}(\sigma)} + \sum_{i=2}^{n+1} q^{i-2} \cdot \sum_{\sigma \in I(n-1, k-1)} q^{\ell_{PF_{n-1}}(\sigma)} \\
&= q^n i_q(n, k) + (1 + q + q^2 + \cdots + q^{n-1}) i_q(n-1, k-1) \\
&= q^n i_q(n, k) + [n]_q i_q(n-1, k-1).
\end{aligned}$$

□

Example 22. Let us consider an example in order to understand (13). Consider

$$\begin{aligned}
\pi &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ with } Rk(\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 4 \end{pmatrix}. \\
\sigma = \Phi(\pi) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ with } Rk(\sigma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.
\end{aligned}$$

By using (10), it is easy to verify $\ell_{PF_6}(\pi) = 12 - 4 = 8$ and $\ell_{PF_4}(\sigma) = 8 - 3 = 5$.

6.1 An explicit formula for $i_q(n, k)$.

Let $\pi \in PF_n$ be a partial involution and let $\pi = (i_1, j_1)(i_2, j_2) \cdots (i_m, j_m)$ denote its standard form viewed as an involution in I_n via bijection (4). It follows from the proof of Proposition 6.2 of [8] that the following equality is true:

$$\ell_{PF_n}(\pi) = \rho_{<}(\pi) = \widetilde{inv}(\pi) + \sum_{a: \pi(a)=a} (n-a), \quad (14)$$

where $\widetilde{inv}(\pi)$ is the “modified inversion number,” which is equal to the number of inversions in the word $i_1 j_1 i_2 j_2 \cdots i_m j_m$.

Proposition 23. $i_q(2k, k) = [2k - 1]_q!!$.

Proof. By Proposition 21 we have

$$i_q(2k, k) = q^{2n-1} i_q(2k - 1, k) + [2k - 1]_q i_q(2k - 2, k - 1).$$

Since there are no involutions in S_{2k-1} which have k arcs (the maximal number of arcs for an involution in S_{2k-1} is $k - 1$), we have $i_q(2k - 1, k) = 0$ and therefore $i_q(2k, k) = [2k - 1]_q i_q(2k - 2, k - 1) = [2k - 1]_q i_q(2(k - 1), k - 1)$. Now, by induction we get $i_q(2k, k) = [2k - 1]_q!!$. $\square$

Proposition 24.

$$i_q(n, k) = q^{\binom{n-2k}{2}} \cdot \binom{n}{2k}_q \cdot [2k - 1]_q!!,$$

$$\text{where } \binom{n}{2k}_q = \frac{[n]_q!}{[2k]_q! [n-2k]_q!}.$$

Proof. Let π an element from $I(n, k)$. The involution $\pi \in S_n$ has k arcs, hence, it has $n - 2k$ fixed points. Thus, $n - 2k$ zero rows and columns in the corresponding partial fixed-point-free involution matrix. So, there is a natural bijection

$$\pi \leftrightarrow (\{i_1, \dots, i_{n-2k}\}, \sigma),$$

where $1 \leq i_1 < i_2 < \dots < i_{n-2k} \leq n$ are the fixed points of π and $\sigma \in I(2k, k)$ is the fixed point free involution of S_{2k} , whose partial fixed-point-free involution matrix is obtained from π by deleting zero rows and columns. Now, using formula (14) we have

$$\begin{aligned} i_q(n, k) &= \sum_{\pi \in I(n, k)} q^{\ell_{PF_n}(\pi)} \\ &= \sum_{\substack{(\{i_1, \dots, i_{n-2k}\}, \sigma): \\ 1 \leq i_1 < i_2 < \dots < i_{n-2k} \leq n, \sigma \in I(2k, k)}} q^{n-i_1+n-i_2+\dots+n-i_{n-2k}+\ell_{F_{2k}}(\sigma)} \\ &= \left(\sum_{1 \leq i_1 < i_2 < \dots < i_{n-2k} \leq n} q^{n-i_1+\dots+n-i_{n-2k}} \right) \cdot \left(\sum_{\sigma \in I(2k, k)} q^{\ell_{F_{2k}}(\sigma)} \right) \\ &= \left(\sum_{0 \leq j_1 < \dots < j_{n-2k} \leq n-1} q^{j_1+\dots+j_{n-2k}} \right) \cdot i_q(2k, k). \end{aligned} \tag{15}$$

To simplify (15), we use well known Gaussian identity (see [21], formula (1.87)):

$$\prod_{i=0}^{j-1} (1 + xq^i) = \sum_{k=0}^j x^k q^{\binom{k}{2}} \binom{j}{k}_q, \tag{16}$$

which is equivalent, by expanding the product, to

$$\sum_{0 \leq s_1 < s_2 < \dots < s_k \leq j-1} q^{\sum_{r=1}^k s_r} x^k = \sum_{k=0}^j x^k q^{\binom{k}{2}} \binom{j}{k}_q. \quad (17)$$

Replacing j by n , and comparing the coefficients of x^{n-2k} in (17), we obtain our desired formula

$$\mathbf{i}_q(n, k) = q^{\binom{n-2k}{2}} \cdot \binom{n}{2k}_q \cdot [2k-1]_q!!.$$

□

6.2 Length generating function of PF_n

Next, we look at the length generating function of PF_n more closely.

$$\mathbf{p}_q(n) := \sum_{\pi \in PF_n} q^{\ell_{PF_n}(\pi)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathbf{i}_q(n, k).$$

By a straightforward calculation we see that $\mathbf{p}_q(1) = 1$, $\mathbf{p}_q(2) = 1+q$, $\mathbf{p}_q(3) = 1+q+q^2+q^3$.

Proposition 25. *For all $n \geq 2$, we have*

$$\mathbf{p}_q(n+1) = q^n \mathbf{p}_q(n) + [n]_q \mathbf{p}_q(n-1).$$

Proof. Follows from Proposition 21. □

Example 26. It is easy to verify the following calculation from the Hasse diagram of PF_4 in Figure 2: $\mathbf{p}_q(4) = q^3 \mathbf{p}_q(3) + [3]_q \mathbf{p}_q(2) = 1 + 2q + 2q^2 + 2q^3 + q^4 + q^5 + q^6$.

6.3 Skew-symmetric matrices over $\mathbb{F}_q$

There is an interesting similarity between the rank generating function $\mathbf{i}_q(n, k)$ and the number of $\mathbb{F}_q$ -rational points of rank $2k$, $n \times n$ skew symmetric matrices, which we denote by Skew_n^{2k} . Here $\mathbb{F}_q$ is the finite field with q elements. It is well known that the number of $\mathbb{F}_q$ -rational points of the general linear group GL_n and the symplectic group Sp_n ($n = 2m$) are given by

$$|\text{GL}_n|_{\mathbb{F}_q} = q^{\binom{n}{2}} \prod_{i=1}^n (q^i - 1) \quad \text{and} \quad |\text{Sp}_{2m}|_{\mathbb{F}_q} = q^{m^2} \prod_{i=1}^m (q^{2i} - 1).$$

The group $G = \text{GL}_n$ acts Skew_n^{2k} transitively. A simple matrix computation shows that

$$|G_x|_{\mathbb{F}_q} = |\text{GL}_{n-2k}|_{\mathbb{F}_q} |\text{Sp}_{2k}|_{\mathbb{F}_q} |\text{Mat}_{n-2k, 2k}|_{\mathbb{F}_q},$$

where $\text{Mat}_{n-2k,2k}$ is the space of $2k \times (n - 2k)$ matrices. Thus,

$$|\text{Skew}_n^{2k}|_{\mathbb{F}_q} = |G/G_x|_{\mathbb{F}_q} = \frac{q^{\binom{n}{2}} \prod_{i=1}^n (q^i - 1)}{q^{k^2} \prod_{i=1}^k (q^{2i} - 1) q^{\binom{n-2k}{2}} \prod_{i=1}^{n-2k} (q^i - 1) q^{2k(n-2k)}},$$

which simplifies as follows

$$\begin{aligned} |\text{Skew}_n^{2k}|_{\mathbb{F}_q} &= q^{\binom{n}{2} - k^2 - \binom{n-2k}{2} - 2k(n-2k)} \frac{[n]!(q-1)^n}{(\prod_{i=1}^k [2i])(q-1)^k [n-2k]!(q-1)^{n-2k}} \\ &= q^{2\binom{k}{2}} \frac{[n]!(q-1)^k}{(\prod_{i=1}^k [2i])[n-2k]!} \\ &= q^{2\binom{k}{2}} (q-1)^k \binom{n}{2k}_q [2k-1]!! . \end{aligned}$$

In other words,

$$|\text{Skew}_n^{2k}|_{\mathbb{F}_q} = \mathbf{i}_q(n, k) q^{2\binom{k}{2} - \binom{n-2k}{2}} (q-1)^k .$$

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