On floors and ceilings of the k-Catalan arrangement

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Abstract

The set of dominant regions of the k-Catalan arrangement of a crystallographic root system Φ is a well-studied object enumerated by the Fuß-Catalan number $Cat^{(k)}(\Phi)$. It is natural to refine this enumeration by considering floors and ceilings of dominant regions. A conjecture of Armstrong states that counting dominant regions by their number of floors of a certain height gives the same distribution as counting dominant regions by their number of ceilings of the same height. We prove this conjecture using a bijection that provides even more refined enumerative information.

Keywords: Fuss-Catalan combinatorics; Catalan arrangement; Floors; Ceilings

1 Introduction

Let Φ be a crystallographic root system of rank n with simple system S, positive system Φ^+, and ambient vector space V. For background on root systems see [Hum90]. For k a positive integer, we define the *k-Catalan arrangement* of Φ as the hyperplane arrangement given by the hyperplanes $H_\alpha^r = \{x \in V \mid \langle x, \alpha \rangle = r\}$ for $\alpha \in \Phi$ and $r \in \{0, 1, \ldots, k\}$. The complement of this arrangement falls apart into connected components which we call the *regions* of the arrangement. Those regions R that have $\langle x, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$ and all $x \in R$ we call *dominant*. The number of dominant regions of the k-Catalan arrangement equals the Fuß-Catalan number $Cat^{(k)}(\Phi)$ [Ath04] of Φ. This number remains somewhat mysterious, in the sense that it also counts other objects in combinatorics, like the set of k-divisible noncrossing partitions $NC^{(k)}(\Phi)$ of Φ [Arm09, Theorem 3.5.3] and the number of facets of the k-generalised cluster complex $\Delta^{(k)}(\Phi)$ of Φ [FR05, Proposition 8.4], but no uniform proof of this fact is known, that is every known proof of this fact appeals to the classification of irreducible crystallographic root systems.

For a dominant region R of the k-Catalan arrangement, we call those hyperplanes that support a facet of R the *walls* of R. Those walls of R which do not contain the origin and have the origin on the same side as R we call the *ceilings* of R. The walls of R that do not contain the origin and separate R from the origin are called its *floors*. We say a hyperplane is of *height* r if it is of the form H_α^r for $\alpha \in \Phi^+$.

One reason why floors and ceilings of dominant regions are interesting is that they give a more refined enumeration of the dominant regions of the k-Catalan arrangement of Φ that corresponds to refined enumerations of other objects counted by the Fuß-Catalan number $Cat^{(k)}(\Phi)$. More precisely, the number of dominant regions in the k-Catalan arrangement of Φ that have exactly j floors of height k equals the Fuß-Narayana number $Nar^{(k)}(\Phi, j)$ [Ath05, Proposition 5.1] [Thi14, Theorem 1], which also counts the number of k-divisible noncrossing partitions of Φ of rank j [Arm09, Definition 3.5.4], as well as equalling the $(n-j)$-th entry of the h-vector of the k-generalised cluster complex $\Delta^{(k)}(\Phi)$ [FR05, Theorem 10.2]. Similarly, the number of bounded dominant regions of the k-Catalan arrangement of Φ that have exactly j ceilings of height k equals the $(n-j)$-th entry of the h-vector of the positive part of $\Delta^{(k)}(\Phi)$ [AT06, Conjecture 1.2] [Thi14, Corollary 5].

For the special case where Φ is of type A_{n-1}, more is known. For example, there is an explicit bijection between the set of dominant regions of the k-Catalan arrangement of Φ and the set of facets of the cluster complex of Φ [FKT13]. There is also an enumeration of those dominant regions that have a fixed hyperplane as a floor [FTV13]. In contrast to those results, all results in this paper are stated and proven uniformly for all crystallographic root systems without appeal to the classification.

If M is any set of hyperplanes of the k-Catalan arrangement, let $U(M)$ be the set of dominant regions R of the k-Catalan arrangement such that all hyperplanes in M are floors of R. Similarly, let $L(M)$ be the set of dominant regions R' of the k-Catalan arrangement such that all hyperplanes in M are ceilings of R'. Use the standard notation $[n] := \{1, 2, \ldots, n\}$. Then we have the following theorem.

Theorem 1.1. *For any set* $M = \{H_{\alpha_1}^{i_1}, H_{\alpha_2}^{i_2}, \ldots, H_{\alpha_m}^{i_m}\}$ *of* m *hyperplanes with* $i_j \in [k]$ *and* $\alpha_j \in \Phi^+$ *for all* $j \in [m]$*, there is an explicit bijection* Θ *from* $U(M)$ *to* $L(M)$*.*

See Figure 1 for an example. From this theorem, we obtain some enumerative corollaries. In particular, let $fl_r(l)$ be the number of dominant regions in the k-Catalan arrangement that have exactly l floors of height r, and let $cl_r(l)$ be the number of dominant regions that have exactly l ceilings of height r [Arm09, Definition 5.1.23]. We deduce the following conjecture of Armstrong.

Corollary 1.2 ([Arm09, Conjecture 5.1.24])**.** *We have* $fl_r(l) = cl_r(l)$ *for all* $1 \leqslant r \leqslant k$ *and* $0 \leqslant l \leqslant n$*.*

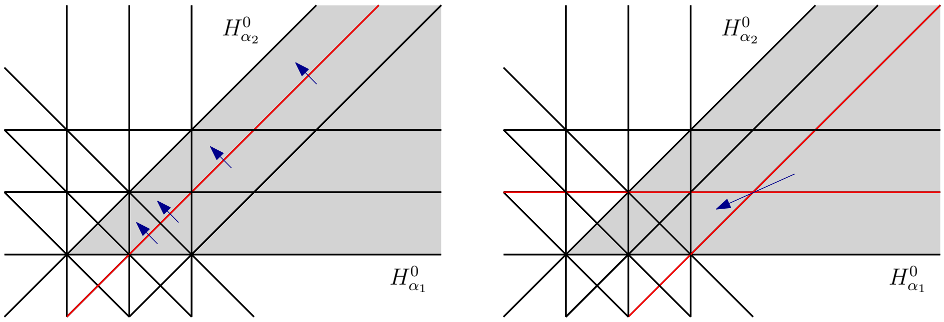

Figure 1: The bijection Θ for the 2-Catalan arrangement of the root system of type B_2, for $M = \{H^1_{\alpha_2}\}$ and for $M = \{H^1_{\alpha_1}, H^2_{\alpha_2}\}$. The dominant chamber is shaded in grey.

Specialising to the $k = 1$ case, we also give a geometric interpretation in terms of dominant regions of the Catalan arrangement of the Panyushev complement on ideals in the root poset of Φ.

2 Definitions

For this section and the next one, suppose that Φ is irreducible. Define the *affine Coxeter arrangement* of Φ as the union of all hyperplanes of the form $H^r_\alpha = \{x \in V \mid \langle x, \alpha\rangle = r\}$ for $\alpha \in \Phi$ and $r \in \mathbb{Z}$. Then the complement of this falls apart into connected components, all of which are congruent open n-simplices, called *alcoves*. The *affine Weyl group* W_a generated by all the reflections through hyperplanes of the form H^r_α for $\alpha \in \Phi$ and $r \in \mathbb{Z}$ is a Coxeter group, with generating set $S_a = \{s_0, s_1, \ldots, s_n\}$, where $s_1, \ldots, s_n$ are the reflections in the hyperplanes orthogonal to the simple roots of Φ and s_0 is the reflection in $H^1_{\tilde{\alpha}}$, where $\tilde{\alpha}$ is the highest root of Φ.

The group W_a acts simply transitively on the alcoves, so if we define the *fundamental alcove* as

$$A_\circ = \{x \in V \mid \langle x, \alpha_i\rangle > 0 \text{ for all } \alpha_i \in S, \langle x, \tilde{\alpha}\rangle < 1\},$$

then every alcove A can be written as $w(A_\circ)$ for a unique $w \in W_a$.

Clearly any alcove is contained in exactly one region R of the k-Catalan arrangement of Φ. For any alcove A in the affine Coxeter arrangement of Φ and $\alpha \in \Phi^+$, there exists a unique integer r with $r - 1 < \langle x, \alpha\rangle < r$ for all $x \in A$. We denote this integer by $r(A, \alpha)$.

Suppose that for each $\alpha \in \Phi^+$ we are given a positive integer r_α. The following is due to Shi [Shi87, Theorem 5.2].

Lemma 2.1 ([AT06, Lemma 2.3]). *There is an alcove A with $r(A, \alpha) = r_\alpha$ for all $\alpha \in \Phi^+$ if and only if $r_\alpha + r_\beta - 1 \leqslant r_{\alpha+\beta} \leqslant r_\alpha + r_\beta$ whenever $\alpha, \beta, \alpha + \beta \in \Phi^+$.*

Define a partial order on Φ^+ by

$$\alpha \leqslant \beta \text{ if and only if } \beta - \alpha \in \langle S \rangle_{\mathbb{N}},$$

that is, $\beta \geqslant \alpha$ if and only if $\beta - \alpha$ can be written as a linear combination of simple roots with nonnegative integer coefficients. The set of positive roots Φ^+ with this partial order is called the *root poset*. A subset $I \subseteq \Phi^+$ is called an *ideal* if for all $\alpha \in I$ and $\beta \leqslant \alpha$, also $\beta \in I$. A subset $J \subseteq \Phi^+$ is called an *order filter* if for all $\alpha \in J$ and $\beta \geqslant \alpha$, also $\beta \in J$.

Suppose $\mathcal{I} = (I_1, I_2, \ldots, I_k)$ is an ascending (multi)chain of k ideals in the root poset of Φ, that is $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k$. Setting $J_i = \Phi^+ \backslash I_i$ for $i \in [k]$ and $\mathcal{J} = (J_1, J_2, \ldots, J_k)$ gives us the corresponding descending chain of order filters. That is, we have $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_k$. The ascending chain of ideals $\mathcal{I}$ and the corresponding descending chain of order filters $\mathcal{J}$ are both called *geometric* if the following conditions are satisfied simultaneously.

1. $(I_i + I_j) \cap \Phi^+ \subseteq I_{i+j}$ for all $i, j \in \{0, 1, \ldots, k\}$ with $i + j \leqslant k$, and
2. $(J_i + J_j) \cap \Phi^+ \subseteq J_{i+j}$ for all $i, j \in \{0, 1, \ldots, k\}$.

Here we set $I_0 = \varnothing$, $J_0 = \Phi^+$, and $J_i = J_k$ for $i > k$. We call $\mathcal{I}$ and $\mathcal{J}$ *positive* if $S \subseteq I_k$, or equivalently $S \cap J_k = \varnothing$.

Let R be a dominant region of the k-Catalan arrangement of Φ. Let us define $\theta(R) = (I_1, I_2, \ldots, I_k)$ and $\phi(R) = (J_1, J_2, \ldots, J_k)$, where

$$I_i = \{\alpha \in \Phi^+ \mid \langle x, \alpha \rangle < i \text{ for all } x \in R\} \text{ and}$$

$$J_i = \{\alpha \in \Phi^+ \mid \langle x, \alpha \rangle > i \text{ for all } x \in R\},$$

for $i \in \{0, 1, \ldots, k\}$. It is not difficult to verify that $\theta(R)$ is a geometric chain of ideals and that $\phi(R)$ is the corresponding geometric chain of order filters.

For a geometric chain of ideals $\mathcal{I} = (I_1, I_2, \ldots, I_k)$, and $\alpha \in \Phi^+$, we define

$$r_\alpha(\mathcal{I}) = \min\{r_1 + r_2 + \ldots + r_m \mid \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_m \text{ and } \alpha_i \in I_{r_i} \text{ for all } i \in [m]\},$$

where we set $r_\alpha(\mathcal{I}) = \infty$ if α cannot be written as a linear combination of elements in I_k. So $r_\alpha(\mathcal{I}) < \infty$ for all $\alpha \in \Phi^+$ if and only if $\mathcal{I}$ is positive.

For a geometric chain of order filters $\mathcal{J} = (J_1, J_2, \ldots, J_k)$, and $\alpha \in \Phi^+$, we define

$$k_\alpha(\mathcal{J}) = \max\{k_1 + k_2 + \ldots + k_m \mid \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_m \text{ and } \alpha_i \in J_{k_i} \text{ for all } i \in [m]\},$$

where $k_i \in \{0, 1, \ldots, k\}$ for all $i \in [m]$.

It turns out that ϕ is a bijection from the set of dominant regions of the k-Catalan arrangement of Φ to the set of geometric chains of k order filters in the root poset of

Φ [Ath05, Theorem 3.6]. Its inverse ψ is the map sending a geometric chain of order filters $\mathcal{J}$ to the region R of the k-Catalan arrangement containing the alcove A with $r(A,\alpha) = k_\alpha(\mathcal{J}) + 1$ for all $\alpha \in \Phi^+$. This alcove A is called the *minimal alcove* of R. Its floors are exactly the floors of R [Ath05, Theorem 3.11].

Thus the map θ is a bijection from dominant regions R of the k-Catalan arrangement to geometric chains of ideals $\mathcal{I}$. It restricts to a bijection between bounded dominant regions of the k-Catalan arrangement and positive geometric chains of ideals. The inverse of this restriction maps a positive geometric chain of ideals $\mathcal{I}$ to the bounded dominant region R in the k-Catalan arrangement containing the alcove B with $r(B,\alpha) = r_\alpha(\mathcal{I})$ for all $\alpha \in \Phi^+$ [AT06, Theorem 3.6]. This alcove B is called the *maximal alcove* of R. Its ceilings are exactly the ceilings of R [AT06, Theorem 3.11].

We call $\alpha \in \Phi^+$ a *rank r indecomposable element* [Ath05, Definition 3.8] of a geometric chain of order filters $\mathcal{J} = (J_1, J_2, \ldots, J_k)$ if $\alpha \in J_r$ and

1. $k_\alpha(\mathcal{J}) = r$,
2. $\alpha \notin J_i + J_j$ for $i + j = r$ and
3. if $k_{\alpha+\beta}(\mathcal{J}) = t \leqslant k$ for some $\beta \in \Phi^+$ then $\beta \in J_{t-r}$.

We have that H_α^r is a floor of R if and only if α is a rank r indecomposable element of the geometric chain of order filters $\mathcal{J} = \phi(R)$ [Ath05, Theorem 3.11].

We call $\alpha \in \Phi^+$ a *rank r indecomposable element* [AT06, Definition 3.8] of a geometric chain of ideals $\mathcal{I} = (I_1, I_2, \ldots, I_k)$ if $\alpha \in I_r$ and

1. $r_\alpha(\mathcal{I}) = r$,
2. $\alpha \notin I_i + I_j$ for $i + j = r$ and
3. if $r_{\alpha+\beta}(\mathcal{I}) = t \leqslant k$ for some $\beta \in \Phi^+$ then $\beta \in I_{t-r}$.

We will soon see that H_α^r is a ceiling of R if and only if α is a rank r indecomposable element of the geometric chain of ideals $\mathcal{I} = \theta(R)$.

3 Lemmas

Our aim for this rather technical section is to prove the following theorem.

Theorem 3.1. *Let R be a dominant region in the k-Catalan arrangement of Φ, $\mathcal{I} = \theta(R)$ and $\alpha \in \Phi^+$. Then R contains an alcove B such that for all $r \in [k]$ the following are equivalent:*

1. *H_α^r is a ceiling of R,*

2. *α is a rank r indecomposable element of $\mathcal{I}$, and*

3. *H_α^r is a ceiling of B.*

It is already known that Theorem 3.1 holds for bounded dominant regions [AT06, Theorem 3.11]. In that case, we may take the alcove B to be the maximal alcove of the bounded region R.

Our approach to proving Theorem 3.1 is to note that when a region R of the k-Catalan arrangement is subdivided into regions of the $(k+1)$-Catalan arrangement by hyperplanes of the form H_α^{k+1} for $\alpha \in \Phi^+$, at least one of the resulting regions is bounded. We find a region $\underline{R}$ of the $(k+1)$-Catalan arrangement which, among the bounded regions of the $(k+1)$-Catalan arrangement that are contained in R, is the one furthest away from the origin. We call the maximal alcove B of $\underline{R}$ the *pseudomaximal* alcove of R. It equals the maximal alcove of R if R is bounded. The alcove $B \subseteq R$ will be seen to satisfy the assertion of Theorem 3.1. Instead of working directly with the dominant regions of the k- and $(k+1)$-Catalan arrangements, we usually phrase our results in terms of the corresponding geometric chains of ideals.

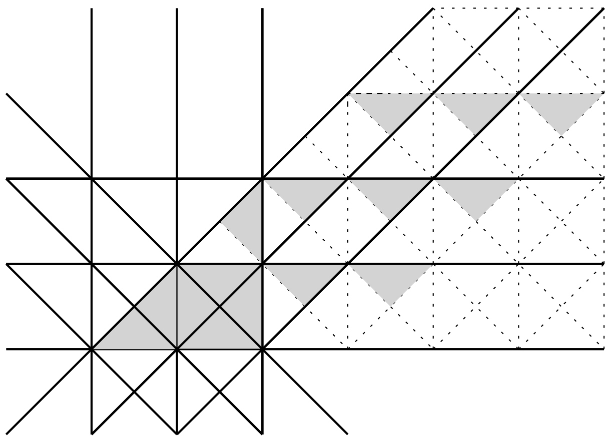

Figure 2: The dominant regions of the 2-Catalan arrangement of the root system of type B_2 together with their pseudomaximal alcove, shaded in grey.

We require the following lemmas:

Lemma 3.2 ([Ath05, Lemma 2.1 (ii)]). *If $\alpha_1, \alpha_2, \ldots, \alpha_r \in \Phi$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_r = \alpha \in \Phi$, then $\alpha_1 = \alpha$ or there exists i with $2 \leqslant i \leqslant r$ such that $\alpha_1 + \alpha_i \in \Phi \cup \{0\}$.*

Lemma 3.3 ([AT06, Lemma 3.2]). *For $\alpha \in \Phi^+$ and $r_\alpha(\mathcal{I}) = r \leqslant k$, we have that $\alpha \in I_r$.*

Lemma 3.4 ([AT06, Lemma 3.10]). *Suppose α is an indecomposable element of $\mathcal{I}$. Then*

1. $r_\alpha(\mathcal{I}) = r_\beta(\mathcal{I}) + r_\gamma(\mathcal{I}) - 1$ *if* $\alpha = \beta + \gamma$ *for* $\beta, \gamma \in \Phi^+$ *and*

2. $r_\alpha(\mathcal{I}) + r_\beta(\mathcal{I}) = r_{\alpha+\beta}(\mathcal{I})$ *if* $\beta, \alpha + \beta \in \Phi^+$.

Lemma 3.5. *If* $\alpha, \beta, \gamma \in \Phi^+$, $\beta + \gamma \in \Phi^+$ *and* $\alpha \leqslant \beta + \gamma$, *then* $\alpha \leqslant \beta$ *or* $\alpha \leqslant \gamma$ *or* $\alpha = \beta' + \gamma'$ *with* $\beta', \gamma' \in \Phi^+$, $\beta' \leqslant \beta$ *and* $\gamma' \leqslant \gamma$.

Proof. Let $\alpha = \beta + \gamma - \sum_{j \in J} \alpha_j$ with $\alpha_j \in S$ for all $j \in J$. We proceed by induction on $|J|$. If $|J| = 0$, we are done. If $|J| = 1$, we have that $\alpha = -\alpha_i + \beta + \gamma$ for some $\alpha_i \in S$. Thus by Lemma 3.2, we have either $\alpha = -\alpha_i$ (a contradiction), or $\beta' = \beta - \alpha_i \in \Phi \cup \{0\}$ or $\gamma' = \gamma - \alpha_i \in \Phi \cup \{0\}$. Notice that if $\beta' \neq 0$, then $\beta' \in \Phi^+$, and similarly for γ'. So if $\beta' \in \Phi^+$ we may write $\alpha = \beta' + \gamma$ and otherwise we have $\gamma' \in \Phi^+$ and thus $\alpha = \beta + \gamma'$ as required.

If $|J| > 1$, we have $\alpha + \sum_{j \in J} \alpha_j = \beta + \gamma$, so by Lemma 3.2, either $\alpha = \beta + \gamma$, so we are done, or $\alpha + \alpha_j \in \Phi \cup \{0\}$ for some $j \in J$. In the latter case we even have $\alpha + \alpha_j \in \Phi^+$. By induction hypothesis, $\alpha + \alpha_j \leqslant \beta$ or $\alpha + \alpha_j \leqslant \gamma$ or $\alpha + \alpha_j = \beta' + \gamma'$ with $\beta', \gamma' \in \Phi^+$, $\beta' \leqslant \beta$ and $\gamma' \leqslant \gamma$. In the first two cases, we are done. In the latter case, we have $\alpha = -\alpha_j + \beta' + \gamma'$, so we proceed as in the $|J| = 1$ case. □

We are now ready to define the bounded dominant region $\underline{R}$ of the $(k+1)$-Catalan arrangement in terms of the corresponding geometric chain of $k+1$ ideals $\underline{\mathcal{I}}$. For a geometric chain of ideals $\mathcal{I} = (I_1, I_2, \ldots, I_k)$, let $\underline{I}_i = I_i$ for all $i \in [k]$ and let $\underline{I}_{k+1} = \bigcup_{i+j=k+1}((I_i + I_j) \cap \Phi^+) \cup I_k \cup S$. By Lemma 3.5, $\underline{I}_{k+1}$ is an ideal. Define $\underline{\mathcal{I}} = (\underline{I}_1, \ldots, \underline{I}_{k+1})$.

Lemma 3.6. *If* $\mathcal{I} = (I_1, I_2, \ldots, I_k)$ *is a geometric chain of* k *ideals in the root poset of* Φ, *then* $\underline{\mathcal{I}}$ *is a positive geometric chain of* $k+1$ *ideals. The bounded dominant region* $\underline{R} = \theta^{-1}(\underline{\mathcal{I}})$ *of the* $(k+1)$*-Catalan arrangement of* Φ *is contained in the region* $R = \theta^{-1}(\mathcal{I})$ *of the* k*-Catalan arrangement.*

Proof. By construction, $\underline{\mathcal{I}}$ is an ascending chain of ideals. If $i + j \leqslant k$, we have that $(\underline{I}_i + \underline{I}_j) \cap \Phi^+ = (I_i + I_j) \cap \Phi^+ \subseteq I_{i+j} = \underline{I}_{i+j}$ as $\mathcal{I}$ is geometric. If $i + j = k + 1$ with $i, j \neq 0$ (otherwise the result is trivial) we have that $(\underline{I}_i + \underline{I}_j) \cap \Phi^+ = (I_i + I_j) \cap \Phi^+ \subseteq \bigcup_{i+j=k+1}((I_i + I_j) \cap \Phi^+) \subseteq \underline{I}_{i+j}$.

Let $\mathcal{J} = (J_1, J_2, \ldots, J_k)$ be the geometric chain of order filters corresponding to the geometric chain of ideals $\mathcal{I}$. Define $\underline{\mathcal{J}}$ similarly. We need to verify that $(\underline{J}_i + \underline{J}_j) \cap \Phi^+ \subseteq \underline{J}_{i+j}$ for all $i, j \in [k+1]$.

Suppose first that $i + j \leqslant k$. Then $(\underline{J}_i + \underline{J}_j) \cap \Phi^+ = (J_i + J_j) \cap \Phi^+ \subseteq J_{i+j} = \underline{J}_{i+j}$ since $\mathcal{J}$ is geometric.

Suppose next that $i + j = k + 1$. Take any region R' of the $(k+1)$-Catalan arrangement that is contained in R. Let $\theta(R') = \mathcal{I}' = (I'_1, I'_2, \ldots, I'_{k+1})$ be the geometric chain of ideals corresponding to R' and let $\mathcal{J}' = (J'_1, J'_2, \ldots, J'_{k+1})$ be the corresponding geometric chain of order filters. Then R and R' are on the same side of each hyperplane of the k-Catalan arrangement. Thus $I'_l = I_l$ and $J'_l = J_l$ for $l \in [k]$. Thus we have $\underline{I}_{k+1} = \bigcup_{i+j=k+1}((I_i+I_j)\cap\Phi^+)\cup I_k \cup S = \bigcup_{i+j=k+1}((I'_i+I'_j)\cap\Phi^+)\cup I'_k\cup S \subseteq I'_{k+1}\cup S$ since $\mathcal{I}'$ is geometric. Since $\mathcal{J}'$ is geometric, we have $(\underline{J}_i+\underline{J}_j)\cap\Phi^+ = (J'_i+J'_j)\cap\Phi^+ \subseteq J'_{i+j} = J'_{k+1}$. The sum of two positive roots is never a simple root, so we even have $(\underline{J}_i+\underline{J}_j)\cap\Phi^+ \subseteq J'_{k+1}\backslash S$. But $J'_{k+1}\backslash S \subseteq \underline{J}_{k+1}$, as $\underline{I}_{k+1} \subseteq I'_{k+1}\cup S$. Thus $(\underline{J}_i+\underline{J}_j)\cap\Phi^+ \subseteq \underline{J}_{i+j}$.

Lastly, in the case where $i+j > k+1$, we have $\underline{J}_j \subseteq \underline{J}_{k+1-i}$, so that $(\underline{J}_i+\underline{J}_j)\cap\Phi^+ \subseteq (\underline{J}_i+\underline{J}_{k+1-i})\cap\Phi^+ \subseteq \underline{J}_{k+1} = \underline{J}_{i+j}$.

Thus the chain of ideals $\underline{\mathcal{I}}$ is geometric. It is also clearly positive, so $\underline{R} = \theta^{-1}(\underline{\mathcal{I}})$ is bounded. Since $\underline{I}_i = I_i$ for $i \in [k]$, $\underline{R}$ and R are on the same side of each hyperplane of the k-Catalan arrangement, so $\underline{R}$ is contained in R. □

For a geometric chain of k ideals $\mathcal{I} = (I_1, I_2, \ldots, I_k)$, define $\mathsf{supp}(\mathcal{I}) = I_k\cap S$. In particular, $\mathsf{supp}(\mathcal{I}) = S$ if and only if $\mathcal{I}$ is positive.

Lemma 3.7. *If $\alpha \in \langle\mathsf{supp}(\mathcal{I})\rangle_{\mathbb{N}}$, then $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$. In particular, if $r_\alpha(\underline{\mathcal{I}}) \leqslant k$, then $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$.*

Proof. First note that $\alpha \in \langle\mathsf{supp}(\mathcal{I})\rangle_{\mathbb{N}}$ implies that $r_\alpha(\mathcal{I}) < \infty$. So may write $\alpha = \alpha_1+\alpha_2+\ldots+\alpha_m$ with $\alpha_i \in I_{r_i}$ for $i \in [m]$ and $r_1+r_2+\ldots+r_m = r_\alpha(\mathcal{I})$. Since $\alpha_i \in I_{r_i} = \underline{I}_{r_i}$ this implies that $r_\alpha(\underline{\mathcal{I}}) \leqslant r_\alpha(\mathcal{I})$.

We may write $\alpha = \alpha_1+\alpha_2+\ldots+\alpha_m$ with $\alpha_i \in \underline{I}_{r_i}$ for $i\in[m]$ and $r_1+r_2+\ldots+r_m = r_\alpha(\underline{\mathcal{I}})$. We wish to show that $r_\alpha(\mathcal{I}) \leqslant r_\alpha(\underline{\mathcal{I}})$. Thus we seek to write $\alpha = \alpha'_1+\alpha'_2+\ldots+\alpha'_l$ with $\alpha'_i \in I_{r'_i}$ for $i \in [l]$ and $r'_1+r'_2+\ldots+r'_l = r_\alpha(\underline{\mathcal{I}})$. If $r_p = k+1$ for some $p \in [m]$, then $\alpha_p \in \underline{I}_{k+1} = \bigcup_{i+j=k+1}((I_i+I_j)\cap\Phi^+)\cup I_k\cup S$. If $\alpha_p \in I_k = \underline{I}_k$, we get a contradiction with the minimality of $r_\alpha(\underline{I})$. If $\alpha_p \in S$, then since $\alpha_p \in \langle\mathsf{supp}(\mathcal{I})\rangle_{\mathbb{N}}$, we have that $\alpha_p \in \mathsf{supp}(\mathcal{I}) \subseteq I_k$, again a contradiction. So $\alpha_p \in \bigcup_{i+j=k+1}((I_i+I_j)\cap\Phi^+)$. Thus write $\alpha_p = \beta_p+\beta'_p$, where $\beta_p \in I_i$ and $\beta'_p \in I_j$ for some i,j with $i+j=k+1$. So in the sum $\alpha = \alpha_1+\alpha_2+\ldots+\alpha_m$ replace each α_p with $r_p = k+1$ with $\beta_p+\beta'_p$ to obtain (after renaming) $\alpha = \alpha'_1+\alpha'_2+\ldots+\alpha'_l$ with $\alpha'_i \in I_{r'_i}$ for $i\in[l]$ and $r'_1+r'_2+\ldots+r'_l = r_\alpha(\underline{\mathcal{I}})$, as required.

If $r_\alpha(\underline{\mathcal{I}}) = r \leqslant k$, then $\alpha \in I_r \subseteq I_k$ by Lemma 3.3, so $\alpha \in \langle\mathsf{supp}(\mathcal{I})\rangle_{\mathbb{N}}$ and thus $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$. □

For R a dominant region of the k-Catalan arrangement, define the *pseudomaximal* alcove of R to be the maximal alcove of $\underline{R}$. This term is justified by the following proposition.

Proposition 3.8. *If R is a bounded dominant region of the k-Catalan arrangement, its pseudomaximal alcove is equal to its maximal alcove.*

Proof. Let A and B be the maximal and pseudomaximal alcoves of R respectively. If $\mathcal{I} = \theta(R)$, then $r(\alpha, A) = r_\alpha(\mathcal{I})$ for all $\alpha \in \Phi^+$. Since B is the maximal alcove of $\underline{R}$, we have $r(\alpha, B) = r_\alpha(\underline{\mathcal{I}})$ for all $\alpha \in \Phi^+$. Now $\mathcal{I}$ is positive since R is bounded, so $\mathsf{supp}(\mathcal{I}) = S$. Thus $r_\alpha(\mathcal{I}) = r_\alpha(\underline{\mathcal{I}})$ for all $\alpha \in \Phi^+$ by Lemma 3.7. So $r(\alpha, A) = r(\alpha, B)$ for all $\alpha \in \Phi^+$ and therefore $A = B$. □

Lemma 3.9. *Let R be a region of the k-Catalan arrangement of Φ, let be B be its pseudomaximal alcove and let $t \leqslant k$ be a positive integer. If $\langle x_0, \alpha\rangle > t$ for some $x_0 \in R$, then $\langle x, \alpha\rangle > t$ for all $x \in B$.*

Proof. Let $\mathcal{I} = \theta(R)$. Since $r(B, \alpha) = r_\alpha(\underline{\mathcal{I}})$ for all $\alpha \in \Phi^+$, it suffices to show that $r_\alpha(\underline{\mathcal{I}}) > t$. If $r_\alpha(\underline{\mathcal{I}}) > k$ this is immediate, so we may assume that $r_\alpha(\underline{\mathcal{I}}) \leqslant k$. Thus we have $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I})$ by Lemma 3.7. Write $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_m$, with $\alpha_i \in I_{r_i}$ for all $i \in [m]$ and $r_1 + r_2 + \ldots + r_m = r_\alpha(\mathcal{I})$. Then $\langle x, \alpha_i\rangle < r_i$ for all $i \in [m]$ and $x \in R$, so $\langle x, \alpha\rangle < r_\alpha(\mathcal{I})$ for all $x \in R$. So if $\langle x_0, \alpha\rangle > t$ for some $x_0 \in R$, then $r_\alpha(\mathcal{I}) > \langle x_0, \alpha\rangle > t$, so $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I}) > t$. □

Lemma 3.10. *If α is a rank r indecomposable element of $\mathcal{I}$, then α is a rank r indecomposable element of $\underline{\mathcal{I}}$.*

Proof. Let α be a rank r indecomposable element of $\mathcal{I}$. Then $\alpha \in I_r = \underline{I}_r$, and $r_\alpha(\underline{\mathcal{I}}) = r_\alpha(\mathcal{I}) = r$ by Lemma 3.7. We have that $\alpha \notin I_i + I_j = \underline{I}_i + \underline{I}_j$ for $i + j = r$. If $r_{\alpha+\beta}(\underline{\mathcal{I}}) = t \leqslant k+1$, then $\alpha+\beta \in \underline{I}_t$ by Lemma 3.3. So if $t \leqslant k$, we have $r_{\alpha+\beta}(\mathcal{I}) = r_{\alpha+\beta}(\underline{\mathcal{I}})$ by Lemma 3.7. If $t = k+1$, then $\alpha + \beta \in I_k$ or $\alpha + \beta \in \bigcup_{i+j=k+1}((I_i + I_j) \cap \Phi^+)$, since $\alpha + \beta \notin S$. Either way, $\alpha + \beta \in \langle I_k\rangle_{\mathbb{N}}$ so $r_{\alpha+\beta}(\mathcal{I}) = r_{\alpha+\beta}(\underline{\mathcal{I}})$ by Lemma 3.7. Thus we have $r_\alpha(\mathcal{I}) + r_\beta(\mathcal{I}) = r_{\alpha+\beta}(\mathcal{I}) = r_{\alpha+\beta}(\underline{\mathcal{I}}) = t$ using Lemma 3.4. So $r_\beta(\mathcal{I}) = t - r_\alpha(\mathcal{I}) = t - r$, so $\beta \in I_{t-r} = \underline{I}_{t-r}$ by Lemma 3.3. Thus α is a rank r indecomposable element of $\underline{\mathcal{I}}$. □

Lemma 3.11. *If $\alpha \in \Phi^+$ and H_α^r is a ceiling of a dominant region R of the k-Catalan arrangement, then α is a rank r indecomposable element of $\mathcal{I} = \theta(R)$.*

Proof. Since the origin and R are on the same side of H_α^r, we have that $\langle x, \alpha\rangle < r$ for all $x \in R$, so $\alpha \in I_r$ and thus $r_\alpha(\mathcal{I}) \leqslant r$. But if $r_\alpha(\mathcal{I}) = i < r$, then $\alpha \in I_i$ by Lemma 3.3, so $\langle x, \alpha\rangle < i \leqslant r-1$ for all $x \in R$. So H_α^r is not a wall of R, a contradiction. Thus $r_\alpha(\mathcal{I}) = r$.

If $\alpha = \beta + \gamma$ for $\beta \in I_i$ and $\gamma \in I_j$ with $i + j = r$, then the fact that $\langle x, \alpha\rangle < r$ for all $x \in R$ is a consequence of $\langle x, \beta\rangle < i$ and $\langle x, \gamma\rangle < j$ for all $x \in R$, so H_α^r does not support a facet of R. So $\alpha \notin I_i + I_j$ for $i + j = r$.

If $r_{\alpha+\beta}(\mathcal{I}) = t \leqslant k$, then $\alpha + \beta \in I_t$ by Lemma 3.3, so $\langle x, \alpha + \beta\rangle < t$ for all x in R. If also $\langle x, \beta\rangle > t - r$ for all $x \in R$, then $\langle x, \alpha\rangle < r$ for all $x \in R$ is a consequence of these, so H_α^r does not support a facet of R. So $\langle x, \beta\rangle < t - r$ for all $x \in R$, so $\beta \in I_{t-r}$.

Thus α is a rank r indecomposable element of $\mathcal{I}$. □

Proof of Theorem 3.1. We take B to be the pseudomaximal alcove of R, that is the maximal alcove of $\underline{R}$. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

The statement that (1) $\Rightarrow$ (2) is Lemma 3.11.

For (2) $\Rightarrow$ (3), suppose α is a rank r indecomposable element of $\mathcal{I}$. Then by Lemma 3.10, α is also a rank r indecomposable element of $\underline{\mathcal{I}}$. So by Lemma 3.4, we have $r_\alpha(\underline{\mathcal{I}}) = r_\beta(\underline{\mathcal{I}}) + r_\gamma(\underline{\mathcal{I}}) - 1$ if $\alpha = \beta + \gamma$ for $\beta, \gamma \in \Phi^+$, and also $r_\alpha(\underline{\mathcal{I}}) + r_\beta(\underline{\mathcal{I}}) = r_{\alpha+\beta}(\underline{\mathcal{I}})$ if $\beta, \alpha + \beta \in \Phi^+$. Thus there exists an alcove B' with $r(B', \beta) = r_\beta(\underline{\mathcal{I}})$ for $\beta \neq \alpha$ and $r(B', \alpha) = r_\alpha(\underline{\mathcal{I}}) + 1$ by Lemma 2.1. Since $r(B, \beta) = r_\beta(\underline{\mathcal{I}})$ for all $\beta \in \Phi^+$, this means that B' and B are on the same side of each hyperplane of the affine Coxeter arrangement, except for $H_\alpha^{r_\alpha(\mathcal{I})} = H_\alpha^r$. Thus H_α^r is a wall of B. Since H_α^r does not separate B from the origin, it is a ceiling of B.

For (3) $\Rightarrow$ (1), suppose H_α^r is a ceiling of B. Let B' be the alcove which is the reflection of B in the hyperplane H_α^r. Then $\langle x, \alpha \rangle > r$ for all $x \in B'$, so by Lemma 3.9 the alcove B' is not contained in R. Thus H_α^r is a wall of R. It does not separate R from the origin, so it is a ceiling of R. This completes the proof. ☐

4 Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let us at first suppose that Φ is an irreducible crystallographic root system of rank n. For $m = 0$, the statement is immediate. Suppose that $0 < m \leqslant n$.

To define the bijection Θ, let $R \in U(M)$ and let A be the minimal alcove of R. The reflections $s_{\alpha_1}^{i_1}, \ldots, s_{\alpha_m}^{i_m}$ in the hyperplanes $H_{\alpha_1}^{i_1}, \ldots, H_{\alpha_m}^{i_m}$ are reflections in facets of the alcove $A = w(A_\circ)$, so the set $S' = \{s_{\alpha_1}^{i_1}, \ldots, s_{\alpha_m}^{i_m}\}$ equals wJw^{-1} for some $J \subset S_a$ and $w \in W_a$. Thus the reflection group W' generated by S' is a proper parabolic subgroup of W_a. In particular, it is finite. With respect to the finite reflection group W', the alcove A is contained in the dominant Weyl chamber, that is the set

$$C = \{x \in V \mid \langle x, \alpha_j \rangle > i_j \text{ for all } j \in [m]\}.$$

So if w_0' is the longest element of W' with respect to the generating set S', the alcove $A' = w_0'(A)$ is contained in the Weyl chamber

$$w_0'(C) = \{x \in V \mid \langle x, \alpha_j \rangle < i_j \text{ for all } j \in [m]\}$$

of W', so it is on the other side of all the hyperplanes $H_{\alpha_1}^{i_1}, \ldots, H_{\alpha_m}^{i_m}$. A' is an alcove, so it is contained in some region R'. Set $\Theta(R) = R'$.

Claim 1. *The region R' is dominant and all hyperplanes in M are ceilings of R', that is $R' \in L(M)$, so Θ is well-defined.*

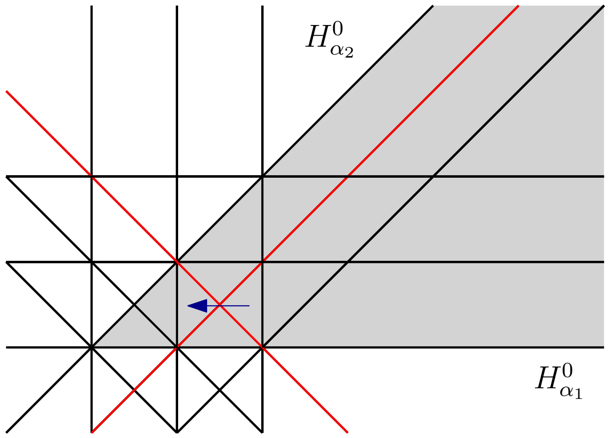

Figure 3: The bijection Θ for the 2-Catalan arrangement of the root system of type B_2 with $M = \{H^1_{\alpha_2}, H^2_{2\alpha_1+\alpha_2}\}$.

Proof. The origin is contained in the Weyl chamber $w'_0(C)$ of W'. Thus no reflection in W' fixes the origin. We can write $A' = w'_0(A)$ as $t_r \cdots t_1(A)$ where $t_i \in W'$ is a reflection in a facet of $t_{i-1} \cdots t_1(A)$ for all $i \in [r]$. In fact, if $w'_0 = s'_1 \cdots s'_r$ with $s'_i \in S'$ for all $i \in [r]$ is a reduced expression for w'_0 in W', we can take $t_i = s'_1 \cdots s'_{i-1} s'_i s'_{i-1} \cdots s'_1$. So $t_i \cdots t_1(A)$ and $t_{i-1} \cdots t_1(A)$ are on the same side of every hyperplane in the affine Coxeter arrangement of Φ except for the reflecting hyperplane of t_i. Since t_i does not fix the origin, if $t_{i-1} \cdots t_1(A)$ is dominant, then so is $t_i \cdots t_1(A)$. Thus by induction on i, the alcove A' is dominant, so R' is dominant.

Consider the Coxeter arrangement of W', which is the hyperplane arrangement given by the reflecting hyperplanes of all the reflections in W'. The action of W' on V restricts to an action on the set of these hyperplanes. Since $H^{i_1}_{\alpha_1}, \ldots, H^{i_m}_{\alpha_m}$ support facets of A, $w'_0(H^{i_1}_{\alpha_1}), \ldots, w'_0(H^{i_m}_{\alpha_m})$ support facets of $A' = w'_0(A)$. Now the set $\{w'_0(H^{i_1}_{\alpha_1}), \ldots, w'_0(H^{i_m}_{\alpha_m})\}$ is the set of walls of $w'_0(C)$ in the Coxeter arrangement of W', so it equals the set $M = \{H^{i_1}_{\alpha_1}, \ldots, H^{i_m}_{\alpha_m}\}$. Since all hyperplanes in M are floors of A, and A' is on the other side of each of them, they are all ceilings of A'. Thus they are ceilings of R'. □

We show that Θ is a bijection by exhibiting its inverse Ψ, a map from $L(M)$ to $U(M)$. Suppose $R' \in L(M)$. Let B be the alcove in R' given by Theorem 3.1. Let R'' be the region that contains $B' = w'_0(B)$. Similarly to the proof of Claim 1, we have that $R'' \in U(M)$. So let $\Psi(R') = R''$.

Claim 2. *The maps Θ and Ψ are inverse to each other, so Θ is a bijection.*

Proof. Suppose $R \in U(M)$, $R' = \Theta(R)$ and $R'' = \Psi(R')$. Use the same notation as above for the alcoves A, A', B and B'. Suppose for contradiction that $R'' \neq R$. Then there is a hyperplane $H = H^r_\alpha$ of the k-Catalan arrangement that separates R and R''. So H separates A and B'. Now A and B' are in the dominant Weyl chamber of W', so they are on the same side of each reflecting hyperplane of W'. Thus H is not a reflecting hyperplane of W'. Now we may write A' as $t_r \cdots t_1(A)$, where $t_i \in W'$ is a reflection in a facet of $t_{i-1} \cdots t_1(A)$ for all $i \in [r]$. So $t_i \cdots t_1(A)$ and $t_{i-1} \cdots t_1(A)$ are on the same side of every hyperplane in the affine Coxeter arrangement, except for the reflecting hyperplane of t_i, which cannot be H. Thus by induction on i, the alcove A' is on the same side of H as A. Similarly B is on the same side of H as B'. So A' and B are on different sides of H, a contradiction, as they are contained in the same region, namely R'. Thus $\Psi(\Theta(R)) = R'' = R$, so $\Psi \circ \Theta = id$. Similarly $\Theta \circ \Psi = id$, so Θ and Ψ are inverse to each other, so Θ is a bijection. □

For any dominant alcove, at least one of its $n+1$ facets must either be a floor or contain the origin, and at least one must be a ceiling. So it has at most n ceilings and at most n floors. So any dominant region R of the k-Catalan arrangement has at most n ceilings and at most n floors. Thus if $m > n$, both $U(M)$ and $L(M)$ are empty. This completes the proof in the case where Φ is irreducible.

Now suppose Φ is reducible, say $\Phi = \Phi_1 \amalg \Phi_2$ with $\Phi_1 \perp \Phi_2$. So $V = V_1 \oplus V_2$ with $V_1 = \langle \Phi_1 \rangle$ and $V_2 = \langle \Phi_2 \rangle$, and $V_1 \perp V_2$. Then the regions of the k-Catalan arrangement of Φ are precisely the sets of the form $R_1 \oplus R_2$ where R_i is a region of the k-Catalan arrangement of Φ_i for $i = 1, 2$. The region $R_1 \oplus R_2$ is dominant if and only if R_1 and R_2 are both dominant. A hyperplane H_α^r is a floor of $R_1 \oplus R_2$ if and only if H_α^r is a floor of R_i for some $i = 1, 2$. The same holds for ceilings. Say $M = M_1 \amalg M_2$ with $H_{\alpha_j}^{i_j} \in M_i$ if $\alpha_j \in \Phi_i$ for $j \in [m]$ and $i = 1, 2$. Assume the theorem holds for Φ_1 and Φ_2, giving us bijections Θ_1 and Θ_2 for Φ_1 together with M_1 and Φ_2 together with M_2 respectively. Then $\Theta(R_1 \oplus R_2) = \Theta_1(R_1) \oplus \Theta_2(R_2)$ gives the required bijection for Φ together with M. This completes the proof by induction on the number of irreducible components of Φ. □

5 Corollaries

We deduce some enumerative corollaries of Theorem 1.1. For any set M of hyperplanes of the k-Catalan arrangement, let $U_=(M)$ be the set of dominant regions R of the k-Catalan arrangement such that the floors of R are exactly the hyperplanes in M, and let $L_=(M)$ be the set of dominant regions R' of the k-Catalan arrangement such that the ceilings of R' are exactly the hyperplanes in M.

Corollary 5.1. *For any set $M = \{H_{\alpha_1}^{i_1}, H_{\alpha_2}^{i_2}, \ldots, H_{\alpha_m}^{i_m}\}$ of m hyperplanes with $i_j \in [k]$ and $\alpha_j \in \Phi^+$ for all $j \in [m]$, we have that $|U_=(M)| = |L_=(M)|$.*

Proof. This follows from Theorem 1.1 by an application of the Principle of Inclusion and Exclusion. □

Corollary 5.2. *For any tuple $(a_1, a_2, \ldots, a_k)$ of nonnegative integers, the number of dominant regions R that have exactly a_j floors of height j for all $j \in [k]$ is the same as the number of dominant regions R' that have exactly a_j ceilings of height j for all $j \in [k]$.*

Proof. Sum Corollary 5.1 over all sets M containing exactly a_j hyperplanes of height j for all $j \in [k]$. $\square$

Proof of Corollary 1.2. Set $a_r = l$ and sum Corollary 5.2 over all choices of a_j for all $j \neq r$. $\square$

6 The Panyushev complement

In the special case where $k = 1$, a geometric chain of ideals $\mathcal{I}$ is simply the single ideal I_1, similarly a geometric chain of order filters $\mathcal{J}$ is just the single order filter J_1. The indecomposable elements of an ideal I are then just its maximal elements [AT06, Lemma 3.9]. The indecomposable elements of an order filter J are just its minimal elements [Ath05, Lemma 3.9] [Thi14, Lemma 1].

There is a natural bijection between ideals and antichains of any poset that sends an ideal to the set of its maximal elements. Similarly, there is a natural bijection between order filters and antichains that sends an order filter to the set of its minimal elements.

So for an ideal I in the root poset of Φ, we define the Panyushev complement $\mathbf{Pan}(I)$ as the ideal generated by the minimal elements of the order filter $J = \Phi^+ \backslash I$. From the above considerations, this is a bijection from the set of order ideals of the root poset of Φ to itself.

For a region R of the Catalan arrangement, let

$$CL(R) = \{\alpha \in \Phi^+ \mid H_\alpha^1 \text{ is a ceiling of } R\} \text{ and}$$

$$FL(R) = \{\alpha \in \Phi^+ \mid H_\alpha^1 \text{ is a floor of } R\}.$$

Since a region R in the Catalan arrangement corresponds to a unique ideal $I = \theta(R)$, which corresponds uniquely to the set of its maximal elements, which equals $CL(R)$ by Theorem 3.1, the map $CL : R \mapsto CL(R)$ gives a bijection from the set of dominant regions in the Catalan arrangement to the set of antichains in the root poset. That the same holds for the map $FL : R \mapsto FL(R)$ follows from an analogous argument that can already be deduced from [Ath05, Theorem 3.11].

Theorem 6.1. *For an ideal I in the root poset of Φ, the region $\theta^{-1}(\mathbf{Pan}(I))$ is the unique region of the Catalan arrangement of Φ whose ceilings are exactly the floors of the region $\theta^{-1}(I)$.*

Proof. The set $CL(\theta^{-1}(\mathbf{Pan}(I)))$ is the set of maximal elements of $\mathbf{Pan}(I)$, which equals the set of minimal elements of $J = \Phi^+ \backslash I$, which equals $FL(\theta^{-1}(I))$. Since CL is a bijection, $\theta^{-1}(\mathbf{Pan}(I))$ is the only region R' with $CL(R') = FL(\theta^{-1}(I))$. $\square$

We could rephrase Theorem 6.1 as $\mathbf{Pan} = \theta \circ CL^{-1} \circ FL \circ \theta^{-1}$. The fact that the Panyushev complement has a natural interpretation in terms of the dominant regions of the Catalan arrangement may serve to explain why it seems to be of particular interest for root posets.

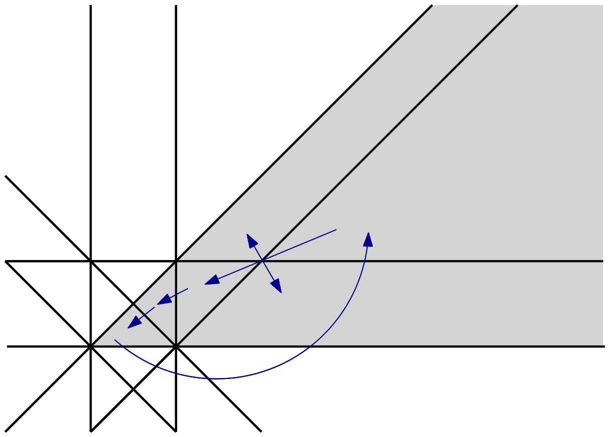

Figure 4: The action of $\theta^{-1} \circ \mathbf{Pan} \circ \theta = CL^{-1} \circ FL$ on the dominant regions of the Catalan arrangement of the root system of type B_2.

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