Face-width of Pfaffian Braces
and Polyhex Graphs on Surfaces

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Abstract

A graph G with a perfect matching is Pfaffian if it admits an orientation D such that every central cycle C (i.e. C is of even size and $G - V(C)$ has a perfect matching) has an odd number of edges oriented in either direction of the cycle. It is known that the number of perfect matchings of a Pfaffian graph can be computed in polynomial time. In this paper, we show that every embedding of a Pfaffian brace (i.e. 2-extendable bipartite graph) on a surface with a positive genus has face-width at most 3. Further, we study Pfaffian cubic braces and obtain a characterization of Pfaffian polyhex graphs: a polyhex graph is Pfaffian if and only if it is either non-bipartite or isomorphic to the cube, or the Heawood graph, or the Cartesian product $C_k \times K_2$ for even integers $k \geqslant 6$.

Keywords: Pfaffian orientation; perfect matching; polyhex graphs; embedding

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching* of G is a set M of independent edges such that every vertex of G is incident with exactly one edge in M. The number of perfect matchings of a graph has applications in statistic physics and quantum chemistry and usually is difficult to compute (cf. [6, 13]). A subgraph H of a graph G is *central* if $G - V(H)$ has a perfect matching. If G has a perfect matching, a central cycle C must be of even size. In other words, G has a perfect matching M such that C is M-alternating (i.e. the edges of C alternate between M and $E(G) \backslash M$). Let

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D be an orientation of G. An even cycle C is *oddly oriented* if C has an odd number of edges directed in either direction of the cycle. An orientation of G is *Pfaffian* if every central cycle of G is oddly oriented. A graph G is *Pfaffian* if it has a Pfaffian orientation. It is known that the number of perfect matchings of a Pfaffian graph can be obtained by computing the determinant of the skew adjacency matrix of its Pfaffian orientation D (cf. [13]).

The Pfaffian orientation was first introduced by Kasteleyn [6] for solving 2-dimensional Ising problem, and Kasteleyn [6] showed that every planar graph admits a Pfaffian orientation. The problem of characterizing Pfaffian bipartite graphs is related to many interesting problems, such as the Pólya permanent problem, the sign-nonsingular matrix problem, etc, and gets a lot of attention ([11, 14, 21]). Little [11] proved that a bipartite graph is Pfaffian if and only if it does not contain an even-subdivision of $K_{3,3}$ as a central subgraph. However, Little's characterization does not provide a polynomial time algorithm to determine Pfaffian bipartite graphs.

A connected graph G is *k-extendable* ($|V(G)| \geqslant 2k+2$) if G has k independent edges and any k independent edges are contained in a perfect matching of G. A 2-extendable bipartite graph is called a *brace*. A k-extendable graph is $(k-1)$-extendable and $(k+1)$-connected [19, 13]. So a brace is 3-connected. A graph G is *bicritical* if, for any two vertices u and v of G, the subgraph $G-\{u,v\}$, obtained from G by deleting u and v, has a perfect matching. A bicritical graph with more than two vertices is 1-extendable and non-bipartite. A *brick* is a 3-connected bicritical graph. By Lovász's tight-cut decomposition [12, 13], every 1-extendable graph can be reduced to a list of braces and bricks. Vazirani and Yannakakis [27] showed that a graph G is Pfaffian if and only if all braces and bricks generated from tight-cut decomposition are Pfaffian. Robertson, Seymour and Thomas [21], and independently McCuaig [14] obtained an elegant characterization of Pfaffian braces, which leads to a polynomial time algorithm to determine whether a given braces is Pfaffian or not. The characterization problem remains open for non-bipartite graphs and graphs on surfaces with positive genera. Readers may refer to the survey of Thomas [24] on the Pfaffian orientations of graphs.

Let G be a graph and Π an embedding of G on a surface Σ. A closed simple curve ℓ in Σ is *contractible* if $\Sigma - \ell$ has precisely two components and one of them is homeomorphic to an open disk. The *face-width* (or *representativity*) of Π is the maximum integer k such that every non-contractible simple closed curve in Σ intersects the graph at least k points (see [15]), denoted by $\mathrm{fw}(G,\Pi)$. For convenience, assume that a plane graph has face-width infinity. Robertson and Vitray [22], and independently Thomassen [25], showed that a planar graph embedded on a surface Σ with a positive genus has the face-width at most 2.

A *polyhex graph* is a cubic graph cellularly embedded on a surface such that every face is bounded by a hexagon. By Euler's formula, the surface could be only the torus and the Klein bottle. Polyhex graphs have been studied as surface tilings [16, 26], and are also considered as possible generalizations of fullerenes in chemistry and material science [2, 8, 9]. A detailed classification of polyhex graphs was given by Thomassen [26]. A bipartite polyhex graph is a cubic brace [29]. In [5], P.E. John tried using the Pfaffian

method to enumerate the perfect matchings of polyhex graphs on the torus. However, not all polyhex graphs on torus are Pfaffian.

In this paper, we study Pfaffian graphs embedded on surfaces with positive genera. The paper is organized as follows. In Section 2, we show that every embedding of a Pfaffian brace embedded on a surface with a positive genus has face-width at most 3. It is natural to ask whether a Pfaffian brick has bounded face-width. In the end of this paper, we give a negative answer to this question via non-bipartite polyhex graphs. In Section 3, we obtain some structure properties of Pfaffian cubic braces, which are applied to characterize Pfaffian polyhex graphs. In Section 4, the construction of polyhex graphs is presented and polyhex graphs with small face-width have been determined. In Section 5, we completely characterize Pfaffian polyhex graphs. We show that a polyhex graph is Pfaffian if and only if it is either non-bipartite or isomorphic to the cube, or the Heawood graph, or the Cartesian product $C_k \times K_2$ for even integer $k \geqslant 6$.

2 Face-width of Pfaffian braces on surfaces

Let G be a graph *cellularly* embedded on a surface Σ (each face is homeomorphic to an open disk in the plane). For a face f of G, the boundary of f is often represented by a closed walk of G, denoted by ∂f, also called a *facial walk*. An embedding Π is a *strong embedding* if every facial walk is a cycle. The *face-width* fw(G, Π) of an embedding Π of G on a surface Σ is the smallest number k such that there exist k faces whose union contains a non-contractible curve of Σ. Undefined notations and concepts are referred to [15].

The following result presents an important property of a planar graph embedded on a surface Σ with genus $g(\Sigma) > 0$.

Theorem 1 ([22, 25]). *Let G be a planar 3-connected graph. Then every embedding of G on a surface Σ with genus $g(\Sigma) > 0$ has face-width at most 2.*

Kasteleyn [7] showed that every planar graph is Pfaffian. But the above result does not hold for all Pfaffian graphs. For example, the Heawood graph, a Pfaffian brace (a Pfaffian orientation is shown in Figure 1), admits an embedding on the torus with face-width 3 (see Lemma 2.3 and Lemma 4.7).
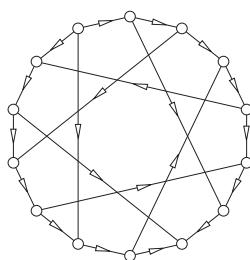

Figure 1: The Heawood graph with a Pfaffian orientation.

Lemma 2 (Prop. 5.5.12 on Page 150 in [15])**.** *Let G be a graph and Π be an embedding of G on a surface Σ. Then the following conditions are equivalent:*
(1) $\mathrm{fw}(G) \geqslant 3$ and G is 3-connected;
(2) All facial walks of Π are cycles and any two of them are either disjoint or their intersection is just a vertex or an edge.

Lemma 3. *Every embedding of the Heawood graph on a surface Σ has face-width at most 3.*

Proof. Let Π be an embedding of the Heawood graph G on a surface Σ. Suppose to the contrary that $\mathrm{fw}(G,\Pi) = k \geqslant 4$. Then there exist k faces $f_0, f_1, f_2, \ldots, f_{k-1}$ such that their union contains a non-contractible closed curve ℓ of Σ such that k is minimum. Since G is 3-connected, every ∂f_i is a cycle by Lemma 2. By the minimality of k and Lemma 2, it follows that f_i and f_j share an edge if and only if $|j - i| \equiv 1 \pmod{k}$. In other words, $|V(\partial f_i \cap \partial f_j)| \geqslant 2$ if $|i - j| \equiv 1 \pmod{k}$, and $|V(\partial f_i \cap \partial f_j)| = 0$ otherwise. Note that $|V(\partial f_i)| \geqslant 6$ since G has the girth six. So

$$14 = |V(G)| \geqslant |V(\bigcup_{i=0}^{k-1} \partial f_i)| = \sum_{i=0}^{k-1} V(\partial f_i) - \sum_{i=0}^{k-1} |V(\partial f_i \cap \partial f_{i+1})| \geqslant 6k - 2k = 4k \geqslant 16,$$

a contradiction. This completes the proof. $\qquad\square$

The following result shows that the above result holds for all Pfaffian braces.

Theorem 4. *Let G be a Pfaffian brace. Then every embedding of G on a surface Σ with $g(\Sigma) > 0$ has face-width at most 3.*

In order to prove Theorem 4, we need the characterization of Pfaffian braces obtained by Robertson, Thomas and Seymour [21], and independently by McCuaig [14]. Let G_0 be a graph and C a central 4-cycle of G_0; that means, $G_0 - V(C)$ has a perfect matching. Let G_1, G_2 and G_3 be three subgraphs of G_0 such that $G_1 \cup G_2 \cup G_3 = G_0$, and for distinct $i, j \in \{1, 2, 3\}$, $G_i \cap G_j = C$ and $V(G_i) - V(C) \neq \emptyset$. A graph G is a *tri-sum* of G_1, G_2 and G_3 if it is obtained from G_0 by deleting some edges (possibly none) of C. For example, Figure 2 shows a cubic brace generated from three copies of the cube by the tri-sum operation along the cycle C illustrated by dash lines (all edges of C are deleted in the operation). By a result of McCuaig (Lemma 19 on page 36 in [14]), every tri-sum of three braces is a new brace. The following is a variant of the statement of the characterization of Pfaffian braces obtained in [14, 21].

Theorem 5 ([21, 14], Theorem 4.2 in [24])**.** *A bipartite graph is a Pfaffian brace if and only if it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated application of the tri-sum operation.*

Let G be a graph, and G_1 and G_2 be two disjoint subgraphs of G. Denote the set of edges joining vertices of G_1 and vertices of G_2 by $E(G_1, G_2)$. For a vertex w of G, let $N(w)$ be the set of all neighbors of w in G. The following technical lemma will be used to prove Theorem 4.
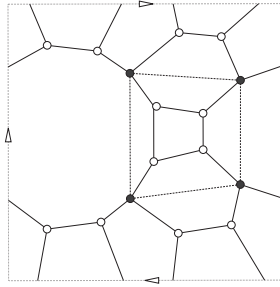
Figure 2: A cubic brace on the Klein bottle generated from three copies of the cube by the tri-sum operation (edges of the cycle C are illustrated by dash lines).

Lemma 6. *Let G be a 3-connected graph and W be a vertex-cut of size 4. If $G-W$ can be decomposed into three disjoint graphs G_1, G_2 and G_3 such that for distinct $i, j \in \{1, 2, 3\}$ and $w \in W$, $E(G_i, G_j) = \emptyset$ and $N(w) \cap V(G_i) \neq \emptyset$, then every embedding of G on a surface with a positive genus has face-width at most 3.*

Proof. Let Π be an embedding of G on a surface Σ with $g(\Sigma) > 0$. Suppose on the contrary that $\mathrm{fw}(G, \Pi) > 3$.

Let $W = \{v_1, v_2, v_3, v_4\}$ and $E(W)$ be the set of all edges with two end-vertices in W. For any $v_i \in W$, let $v_i v_{i,0}, \ldots, v_i v_{i,k_i}$ be the edges incident with v_i in clockwise direction in some small neighborhood of v_i homeomorphic to an open disc. Note that $N(v_i) \cap V(G_\alpha) \neq \emptyset$ for every $\alpha \in \{1, 2, 3\}$. Let $f_{i,\alpha}$ be a face containing $v_i v_{i,\alpha_i}$ and $v_i v_{i,\alpha_i+1}$ such that $v_{i,\alpha_i} \in V(G_\alpha)$ and $v_{i,\alpha_i+1} \notin V(G_\alpha)$. For an edge $v_i v_{i,k} \in E(W)$, let f be the face containing $v_i v_{i,k}$ and $v_i v_{i,k+1}$. By the definition of faces $f_{i,\alpha}$ and the ordering of edges incident with v_i, $f \neq f_{i,\alpha}$ for any $\alpha \in \{1, 2, 3\}$ because $v_{i,k} \notin G - W$. So the intersection of two distinct faces $f_{i,\alpha}$ and $f_{i,\beta}$ does not contain edges of $E(W)$ incident with v_i.

Claim 1. *The intersection of two distinct faces $f_{i,\alpha}$ and $f_{i,\beta}$ does not contain another vertex from W different from v_i.*

Proof of Claim 1. Note that G is 3-connected and $\mathrm{fw}(G, \Pi) > 3$. By Lemma 2, the intersection of $f_{i,\alpha}$ and $f_{i,\beta}$ is either v_i or an edge $v_i v_{i,k}$. Since $f_{i,\alpha} \cap f_{i,\beta}$ does not contain an edge from $E(W)$ incident with v_i, it follows that $v_{i,k} \notin W$. So Claim 1 holds.

Claim 2. *Every $f_{i,\alpha}$ contains precisely two vertices of W.*

Proof of Claim 2. First, we show that every $f_{i,\alpha}$ contains at least two vertices of W. By Lemma 2, each $f_{i,\alpha}$ is bounded by a cycle. Note that $v_{i,\alpha_i} \in V(G_\alpha)$, $v_{i,\alpha_i+1} \notin V(G_\alpha)$. If $v_{i,\alpha_i+1} \in W$, then $f_{i,\alpha}$ contains two vertices from W. So suppose that $v_{i,\alpha_i+1} \notin W$. Hence, $\partial f_{i,\alpha} - W$ consists of at least two components: one contains v_{i,α_i} and the other contains v_{i,α_i+1}. These two components are joined by vertices from W because $E(G_\alpha, G_\beta) = \emptyset$ for distinct $\alpha, \beta \in \{1, 2, 3\}$. So $f_{i,\alpha}$ contains at least two vertices from W.

Since $N(v_i) \cap V(G_\alpha) \neq \emptyset$, there are at least three distinct faces $f_{i,\alpha}$ ($\alpha \in \{1, 2, 3\}$) incident with v_i. If one of them contains three vertices from W, then there is a pair of

vertices which are contained by two distinct faces $f_{i,\alpha}$ and $f_{i,\beta}$, contradicting Claim 1. So Claim 2 follows.

By Claim 1 and Claim 2, we further have the following conclusion:

Any two vertices of W are contained by some face $f_{i,\alpha}$, and every face $f_{i,\alpha}$ contains precisely a pair vertices of W.

Without loss of generality, assume that $f_{1,1}$ contains v_1 and v_2, and $f_{1,2}$ contains v_1 and v_4, and $f_{1,3}$ contains v_1 and v_3 (relabeling G_α for $\alpha = 1, 2, 3$ if necessary). Then, assume that v_2 and v_4 are contained by $f_{2,\alpha}$, v_2 and v_3 are contained by $f_{2,\beta}$ with $\beta \neq \alpha$, v_3 and v_4 are contained by $f_{3,\mu}$ for some $\mu \in \{1, 2, 3\}$. (For example, see Figure 3 (left). The shadow parts illustrate the regions of Σ containing vertices and edges from only $G_\alpha - W$ for some $\alpha \in \{1, 2, 3\}$, or one edge from $E(W)$.) Then each of $f_{1,1} \cup f_{2,\beta} \cup f_{1,3}$, $f_{1,1} \cup f_{2,\alpha} \cup f_{1,2}$, $f_{1,2} \cup f_{3,\mu} \cup f_{1,3}$ and $f_{2,\alpha} \cup f_{2,\beta} \cup f_{3,\mu}$ contains a closed curve which intersects G at three vertices of W. Denote these closed curves by ℓ_1, ℓ_2, ℓ_3 and ℓ_4.
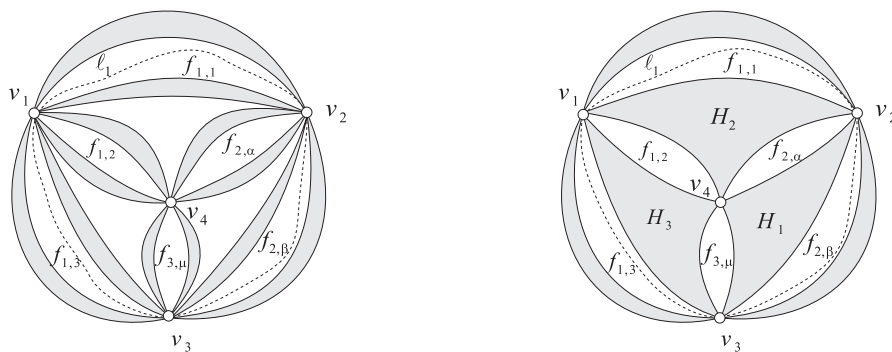

Figure 3: Illustration for the proof of Theorem 4.

Claim 3. *At least one of the closed curves ℓ_i $(i = 1, \ldots, 4)$ is non-contractible.*

Proof of Claim 3. If not, suppose that all ℓ_i $(i = 1, \ldots, 4)$ are contractible. Then each ℓ_i separates Σ into two regions, and at least one of them is homeomorphic to an open disc, denoted by D_i.

First, suppose that v_4 lies on D_1. Let $R_1 \subseteq D_1$ be the region bounded by faces $f_{1,1}$, $f_{1,2}$ and $f_{2,\alpha}$.

By the definition of $f_{1,2}$, R_1 contains a vertex from G_2. Let H_2 be the subgraph of G (not including v_1, v_2 and v_4) embedded in R_1. By Claim 1, every face of the embedding of the subgraph H_2 together with v_1, v_2 and v_4 inherited from the embdding Π of G in the region R_1 contains at most one vertex from v_1, v_2 and v_4. So $H_2 \subseteq G_2$. Similarly, the subgraph H_3 (not including v_1, v_3 and v_4) in the region $R_2 \subseteq D_1$ bounded by faces $f_{1,2}$, $f_{1,3}$ and $f_{3,\mu}$ is a subgraph of G_3. Further, the subgraph H_1 (not including v_2, v_3 and v_4) in the region $R_3 \subseteq D_1$ bounded by faces $f_{2,\alpha}$, $f_{2,\beta}$ and $f_{3,\mu}$ is a subgraph of G_1 (see Figure 3 (right)).

Let G' be the subgraph (including v_1, v_2 and v_3) in $\Sigma - D_1$. If $V(G') = \{v_1, v_2, v_3\}$, then G is embedded in an open disc of Σ. Hence $\mathrm{fw}(G, \Pi) = 0$ because $g(\Sigma) > 0$.

So suppose that G' contains at least one vertex w different from v_1, v_2 and v_3. Let $H'_\alpha = G' \cap G_\alpha$. Then all vertices v_1, v_2 and v_3 have a neighbor in H'_α since G is 3-connected. If two of H'_1, H'_2 and H'_3 are not empty, then there exists a face $f_{1,\gamma}$ for some $\gamma \in \{1, 2, 3\}$ which contains a vertex $v_{1,\gamma_i} \in H'_\gamma$ and a vertex $v_{1,\gamma_i+1} \notin H'_\gamma$. By Claim 1, $f_{1,\gamma}$ contains two vertices from W. Then there are two distinct faces, one $f_{1,\gamma}$ and another from $f_{1,1}, f_{1,2}, f_{1,3}$ contains the same pair of vertices from W, a contradiction. So only one of H'_1, H'_2 and H'_3 is not empty. Without loss of generality, assume $H'_1 \neq \emptyset$ and hence $G' - W \subset G_1$.

Then $G_2 = H_2$. Note that $N(v_3) \cap V(G_2) = \emptyset$, contradicting that $N(v_i) \cap V(G_\alpha) \neq \emptyset$ for any $v_i \in W$ and any $\alpha \in \{1, 2, 3\}$. The contradiction implies that v_4 does not lie on D_1.

By symmetry, the region D_i ($i \in \{1, 2, 3, 4\}$) does not contain the vertex of W which is not on ℓ_i. Then Σ must be the sphere since it is formed by pasting the four disc D_i along the four closed curves ℓ_i, contradicting $g(\Sigma) > 0$. This completes the proof of Claim 3.

Claim 3 implies that $\mathrm{fw}(G, \Pi) \leqslant 3$, which contradicts the assumption that $\mathrm{fw}(G, \Pi) > 3$. The contradiction completes the proof. $\qquad\square$

A minimal vertex-cut W of a connected graph G is a *tri-cut* if $G - W$ has at least three components. For any vertex w of a tri-cut W and any component G_i of $G - W$, $N(w) \cap V(G_i) \neq \emptyset$. As a direct corollary of Lemma 6, we have the following result.

Theorem 7. *Let G be a 3-connected graph with a tri-cut of size 4. Then every embedding of G on a surface with a positive genus has face-width at most 3.*

Now we are ready to prove Theorem 4.

Proof of Theorem 4: Let G be a Pfaffian brace. If G is the Heawood graph or a planar brace, the result follows from Lemma 3 and Theorem 1. Otherwise, by Theorem 5, G is generated from planar Pfaffian braces by applying the tri-sum operations. Assume that G is generated from Pfaffian braces G_1, G_2 and G_3 by the tri-sum operation along a central cycle $C = v_1 v_2 v_3 v_4 v_1$. Since each G_i is 3-connected and bipartite, $N(v_j) \cap (V(G_i) \backslash V(C)) \neq \emptyset$ for every $v_j \in V(C)$. Note that G itself is a brace and hence 3-connected. So G and $W = V(C)$ satisfy the conditions of Lemma 6. Hence every embedding of G on a surface with a positive genus has face-width at most 3 by Lemma 6. $\qquad\square$

3 Pfaffian cubic braces

In order to characterize Pfaffian polyhex graphs, we need more structure properties of Pfaffian cubic braces. The following result is a construction for Pfaffian cubic braces which follows from Theorem 5.

Theorem 8. *A cubic bipartite graph is a Pfaffian brace if and only if it is isomorphic to the Heawood graph, or it can be obtained from planar cubic braces by repeated application of the tri-sum operation.*

Proof. If G is a non-planar Pfaffian cubic brace different from the Heawood graph, then by Theorem 5, G is generated from planar braces by tri-sum operations. We may assume that G is generated from three Pfaffian braces G_1, G_2 and G_3 along a 4-cycle C by the tri-sum operation. Since each G_i is 3-connected and G is cubic, each vertex of C has precisely one neighbor in each $G_i - V(C)$. So every G_i is cubic. The other direction follows directly from Theorem 5. □

A set S of edges of a connected graph G is a *cyclic edge-cut* if $G-S$ has two components, each of which contains a cycle. A connected graph G is *cyclically k-edge-connected* if every cyclic edge-cut of G has at least k edges. The *cyclic edge-connectivity* of G is the maximum integer k such that G is cyclically k-edge-connected, denoted by $c\lambda(G)$.

Theorem 9 ([3, 4]). *Let G be a cubic bipartite graph. Then G is a brace if and only if it is cyclically 4-edge-connected.*

Theorem 10. *Let G be a Pfaffian cubic brace different from the Heawood graph. Then $c\lambda(G) = 4$. Furthermore, for any 4-cycle C of G, $E(C, G - V(C))$ is a cyclic 4-edge-cut.*

Proof. First we prove the following claim:

Claim: *If G has a 4-cycle C, then $E' := E(C, G - V(C))$ is a cyclic 4-edge-cut.*

Proof of Claim: It is obvious that $|E'| = 4$. Note that, the only cubic bipartite graph with 6 vertices is $K_{3,3}$, which is not Pfaffian. So G has at least $n \geqslant 8$ vertices. Since G is 3-connected, $G - V(C)$ is connected. It is easily seen that $G - V(C)$ has $n - 4$ vertices and $3n/2 - 8$ edges. Note that $3n/2 - 8 \geqslant n - 4$ because $n \geqslant 8$. So $G - V(C)$ has a cycle. Hence E' is a cyclic 4-edge-cut and the claim holds.

If G is planar, by Euler's formula, G must contain a 4-cycle since G is bipartite. The claim implies that G contains a cyclic 4-edge cut. If G is non-planar, then G is generated from Pfaffian cubic braces G_1, G_2, and G_3 by the tri-sum along a 4-cycle C by Theorem 8. Since G is not the Heawood graph, the claim implies that each $E(C, G_i - V(C))$ is a cyclic 4-edge-cut of G_i, which is also a cyclic 4-edge-cut of G since each $G_i - V(C)$ $(i \in \{1, 2, 3\})$ contains a cycle. So $c\lambda(G) \leqslant 4$. Hence the theorem follows from Theorem 9. □

Let G be a 3-connected graph with a tri-cut W of size 4. Then W is called an *ideal tri-cut* if W is an independent set of G and $G - W$ has exactly three components G'_1, G'_2 and G'_3 such that each G'_i has at least four vertices, and each $E(W, G'_i)$ is a matching covering all vertices of W.

Proposition 11. *Let G be a Pfaffian cubic brace. If G is neither planar nor the Heawood graph, then G has an ideal tri-cut W.*

Proof. By Theorem 8, G is generated from three Pfaffian cubic braces G_1, G_2 and G_3 by a tri-sum operation along a 4-cycle C. Let $W := V(C)$. Then W is an independent set of G. Since every G_i is 3-connected and Pfaffian, $G_i - V(C)$ is connected and has at least four vertices. Since each vertex of W has a neighbor in every $G_i - W$, it follows that W is a minimal vertex-cut of G. It remains to show that each $E(W, G_i - W)$ is a matching covering all vertices of W. This holds because $E(W, G_i - W)$ is a minimum cyclic edge-cut of G_i by Theorem 10. □

4 Construction of polyhex graphs

A polyhex graph is a cubic graph embedded on a surface such that every face is a hexagon, a cycle of length 6. The embedding of a polyhex graph is a strong embedding. By Euler's formula, the surfaces can be only the torus and the Klein bottle [2].

Let G_1 and G_2 be two polyhex graphs isomorphic to each other. An isomorphism σ from G_1 to G_2 is *hexagon-preserving* if h is a hexagon of G_1 if and only if $\sigma(h)$ is a hexagon of G_2. An *isomorphism* from a graph G to itself is called an *automorphism*. A graph G is *vertex-transitive* if, for any two vertices $v_1, v_2 \in V(G)$, there exists an automorphism σ such that $\sigma(v_1) = v_2$. A polyhex graph G is *hexagon-transitive* if for any two hexagons h_1 and h_2, there exists a hexagon-preserving automorphism σ such that $\sigma(h_1) = h_2$.

Let R be a rectangle of a 2-dimensional Euclidean plane $\mathbb{R}^2$ such that $R = \{(x, y) : 0 \leqslant x \leqslant k, 0 \leqslant y \leqslant q\}$, where k and q are positive integers. Let $v_{i,j}$ be a vertex corresponding to the point (i, j) where i and j are non-negative integers. A rectangular hexagon lattice $L(k, q)$ is a graph on R consisting of all vertices $v_{i,j}$ in R and edges in $\{v_{i,j}v_{i+1,j} | 0 \leqslant j \leqslant q, 0 \leqslant i \leqslant k - 1\} \cup \{v_{i,j}v_{i,j+1} | 0 \leqslant i \leqslant k, 0 \leqslant j \leqslant q - 1 \text{ and } i \equiv j \mod 2\}$. For example, $L(6, 4)$ is shown in Figure 4.
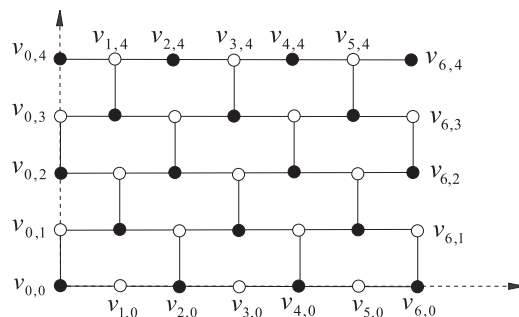

Figure 4: A rectangular hexagon lattice $L(6, 4)$.

The vertices $v_{i,j}$ of $L(k, q)$ are colored in black if $i + j \equiv 0 \pmod 2$ and white if $i + j \equiv 1 \pmod 2$ (see Figure 4). For even k, a polyhex tube $L'(k, q)$ is obtained from $L(k, q)$ by identifying the vertex $v_{0,j}$ with the vertex $v_{k,j}$ for every $j \in \{0, 1, \ldots, q\}$. So both $L(k, q)$ and $L'(k, q)$ are bipartite graphs. The cycle $v_{0,i}v_{1,i} \cdots v_{k,i}$ is called the *i-th*

layer, denoted by L_i. Let $h_{i,j}$ denote the hexagon with the center at $(2i + \alpha_j, j + \frac{1}{2})$ where

$$\alpha_j := \begin{cases} 0, & \text{if } j \equiv 1 \pmod 2; \\ 1, & \text{if } j \equiv 0 \pmod 2. \end{cases}$$

Equivalently,

$$h_{i,j} := \begin{cases} v_{2i-1,j}v_{2i,j}v_{2i+1,j}v_{2i+1,j+1}v_{2i,j+1}v_{2i-1,j+1}v_{2i-1,j}, & \text{if } j \equiv 1 \pmod 2; \\ v_{2i,j}v_{2i+1,j}v_{2i+2,j}v_{2i+2,j+1}v_{2i+1,j+1}v_{2i,j+1}v_{2i,j}, & \text{if } j \equiv 0 \pmod 2. \end{cases}$$

Altschuler [1] showed that polyhex graphs on torus, denoted by $T(k, q, t)$, can be constructed in the following way: From $L'(k, q)$ (k even) identify the vertices $v_{i,0}$ and $v_{i+q+2t,q}$ where $0 \leqslant t \leqslant k/2 - 1$ and the first subscript is always modulo k; that is, $v_{i,0}$ is connected to $v_{i+q+2t,q-1}$ by an edge for each odd i. For example, see Figure 5.
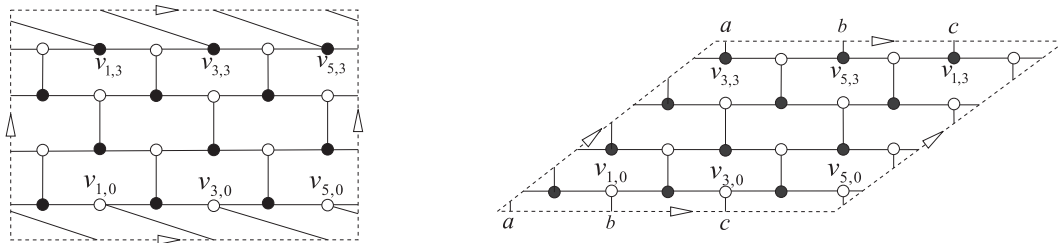

Figure 5: Representations for the polyhex graph $T(6, 4, 0)$ on the torus.

Since a polyhex graph on the torus is a strong embedding, we have the following result.

Theorem 12. *A polyhex graph on the torus is isomorphic to $T(k, q, t)$ for $k \equiv 0 \pmod 2$ and $(k, q, t) \notin \{(2, q, t), (4, 1, t), (k, 1, 0), (k, 1, k/2 - 1) | k, q \in \mathbb{Z}^+, 0 \leqslant t \leqslant k/2 - 1\}$.*

Thomassen [26] classified polyhex graphs into seven types: two on the torus, five on the Klein bottle. According to Thomassen's constructions, Li et al. [10] reclassified the polyhex graphs on the Klein bottle into the following two types:

- Bipartite polyhex $K_e(k, q)(q \geqslant 2, k \geqslant 4$ is even): From $L'(k, q)$, identify $v_{i,0}$ with $v_{k-i,q}$ if q is even, and $v_{i,0}$ with $v_{k-i-1,q}$ if q is odd.

- Non-bipartite polyhex $K_o(k, q)$ ($q \geqslant 2$ is even, and $k \geqslant 3$): From $L(k, q)$ first identify $v_{i,0}$ with $v_{i,q}$; then identify $v_{0,j}$ with $v_{k,q-1-j}$ if k is even, and $v_{0,j}$ with $v_{k,q-j}$ if k is odd.

Theorem 13 ([10]). *A polyhex graph on the Klein bottle is isomorphic to either $K_e(k, q)$ ($q \geqslant 2$, even $k \geqslant 4$) or $K_o(k, q)$ (even $q \geqslant 2$, $k \geqslant 3$).*

Figure 6: Polyhex graphs $K_e(6,4)$ (left) and $K_o(6,4)$ (right) on the Klein bottle.

Polyhex graphs contain many interesting graphs. For example, $T(14,1,2)$ is isomorphic to the Heawood graph, both $T(8,1,1)$ and $K_e(4,2)$ are isomorphic to the cube, and $T(6,1,1)$ is isomorphic to $K_{3,3}$.

Proposition 14 ([30]). *There is a hexagon-preserving isomorphism between $T(k,q,t)$ and $T(k,q,t')$ where $t' \equiv (k-2q-2t)/2 \pmod{k/2}$.*

Proposition 15 ([23]). *Every polyhex graph on the torus is vertex-transitive and hexagon-transitive.*

Theorem 16 ([29]). *Every bipartite polyhex graph is a brace.*

By Theorems 9 and 16, we have the following result.

Corollary 17. *Every bipartite polyhex graph is cyclically 4-edge-connected.*

Lemma 18. *Let G be a polyhex graph on the torus. Then $\mathrm{fw}(G) = 2$ if and only if G is isomorphic to $T(k,q,t)$ where integers k,q,t satisfy $(k,q,t) \in \{(4,q,t)|q \geqslant 2\}$ or $(k,q,t) \in \{(k,2,t)|k \geqslant 6, t \in \{k/2-2, k/2-1, 0\}\}$ or $(k,q,t) \in \{(k,1,t)|k \geqslant 6, k/4-1 \leqslant t \leqslant k/4\}$.*

Proof. Let G be a polyhex graph on the torus. Then G is isomorphic to some $T(k,q,t)$ by Theorem 12. It follows that $\mathrm{fw}(G) = 2$ if and only if G has two distinct hexagons h_1 and h_2 which intersect at two edges. By Proposition 15, without loss of generality, let h_1 be the hexagon $h_{0,0}$ and let $h_2 = h_{x,y}$ with $0 \leqslant y \leqslant q-1$. Since h_1 and h_2 intersect at two edges, $y = 0$ or 1.

If $y = 1$, then $q = 2$. It follows that $x = 1$ or $k - 1$ since $k > 4$. Hence $t \in \{k/2-2, k/2-1, 0\}$. On the other hand, a polyhex $T(k,2,t)$ with $t \in \{k/2-2, k/2-1, 0\}$ does have face-width 2.

If $y = 0$, then either $k = 4$ or $q = 1$. If $k = 4$, then $q \geqslant 2$ by Theorem 12, and further $\mathrm{fw}(T(4,q,t)) = 2$. The lemma holds. So suppose $q = 1$. Then $v_{i,0} = v_{i+1+2t,1}$ for each i. As h_1 and h_2 intersect at two edges, $\{v_{0,0}, v_{2,0}\} \cap V(h_{x,0}) \neq \emptyset$ and $\{v_{2x,0}, v_{2x+2,0}\} \cap V(h_{0,0}) \neq \emptyset$. Note that $\{v_{0,0}, v_{2,0}\} \cap V(h_{x,0}) \neq \emptyset$ implies

$$2x - 1 \leqslant 1 + 2t \leqslant 2x + 1, \tag{1}$$

and $\{v_{2x,0}, v_{2x+2,0}\} \cap V(h_{0,0}) \neq \emptyset$ implies

$$k - 1 \leqslant 2x + 1 + 2t \leqslant k + 1. \qquad (2)$$

Combining inequalities (1) and (2), we have $k/4 - 1 \leqslant t \leqslant k/4$. On the other hand, if $k/4 - 1 \leqslant t \leqslant k/4$ and $k \geqslant 6$, then $h_{0,0}$ and $h_{\lfloor k/4 \rfloor,0}$ intersect at two edges. So $T(k, 1, t)$ with $k/4 - 1 \leqslant t \leqslant k/4$ and $k \geqslant 6$ has face-width 2. $\qquad \square$

By Lemmas 3 and 18, the Heawood graph has an embedding $T(14, 1, 2)$ on the torus with face-width 3.

Theorem 19. *Let G be a polyhex graph on the Klein bottle. Then* $\mathrm{fw}(G) = \min\{\lceil k/2 \rceil, q\}$.

Proof. Let G be a polyhex graph on the Klein bottle. By Theorem 13, G is isomorphic to either $K_e(k, q)$ or $K_o(k, q)$.

Let $\mathcal{H}$ be a set of hexagons of G such that $\bigcup_{h_{x,y} \in \mathcal{H}} h_{x,y}$ contains a non-contractible curve ℓ and $\mathcal{H}$ is minimum. Then $\mathrm{fw}(G) = |\mathcal{H}|$.

First, assume that G is isomorphic to $K_e(k, q)$ (k is even). Let H_i be the graph consisting of all hexagons $h_{x,y}$ with $y = i$. If $\mathcal{H} \cap H_i \neq \emptyset$ for all $i \in \mathbb{Z}_q$, then $|\mathcal{H}| \geqslant q$. So suppose that $\mathcal{H} \cap H_t = \emptyset$ for some $t \in \mathbb{Z}_q$. Let $E_t = \{v_{i,t} v_{i,t+1} | i + t \equiv 0 \pmod 2, i \in \mathbb{Z}_k\}$. All hexagons of $K_e(k, q) - E_t$ induce a tube. Let $R_{j,t}$ be the graph consisting of all hexagons $h_{m,y}$ with $y \in \mathbb{Z}_q$, where

$$m = \begin{cases} j & \text{if } 0 \leqslant y \leqslant t; \\ k/2 - j & \text{if } t + 1 \leqslant y \leqslant q - 1. \end{cases}$$

Denote $E^j = \{e | e = h_{m,y} \cap h_{m,y+1} \text{ and } y, y+1 \in \mathbb{Z}_q \setminus \{t\}\} \cup \{e_1, e_2\}$ where $e_1 = h_{j,t-1} \cap h_{j,t}$ and $e_2 = h_{k/2-j,t} \cap h_{k/2-j,t+1}$. Note that $E^j \subset E(R_j)$. Then the union of all hexagons of $K_e(k, q) - E_t - E^j$ does not contain a non-contractible closed curve. So $\mathcal{H} \cap R_{j,t} \neq \emptyset$ for any $j \in \mathbb{Z}_{k/2}$. Hence $|\mathcal{H}| \geqslant k/2$. It follows that $\mathrm{fw}(G) = |\mathcal{H}| \geqslant \min\{q, k/2\}$.

On the other hand, each H_i and $R_{\lceil k/4 \rceil,t}$ of $K_e(k, q)$ contains a non-contratible closed curve. Note that H_i has $k/2$ hexagons and $R_{\lceil k/4 \rceil,t}$ has q hexagons. It follows that $\mathrm{fw}(G) \leqslant \min\{q, k/2\}$. Hence $\mathrm{fw}(G) = \min\{q, k/2\}$.

In the following, assume that G is isomorphic to $K_o(k, q)$ (q is even). Let R_i be the graph consisting of all hexagons $h_{x,y}$ with $x = i$ and $y \in \mathbb{Z}_q$. If $\mathcal{H} \cap R_i \neq \emptyset$ for all integer $i \in [0, \lceil k/2 \rceil]$, then $|\mathcal{H}| \geqslant \lceil k/2 \rceil$. So suppose that $\mathcal{H} \cap R_t = \emptyset$ for some $t \in \mathbb{Z}_{\lceil k/2 \rceil}$. Denote $E^t = \{v_{2t,j} v_{2t+1,j} | j \in \mathbb{Z}_q\}$. All hexagons of $K_o(k, q) - E^t$ induce a tube. Let H_j be the graph consisting of all hexagons $h_{x,m}$ with $x \in \mathbb{Z}_{\lceil k/2 \rceil \setminus \{t\}}$, where

$$m = \begin{cases} j & \text{if } 0 \leqslant x \leqslant t; \\ q - 1 - j & \text{if } t + 1 \leqslant x \leqslant k/2 - 1 \text{ and } k \text{ is even}; \\ q - j & \text{if } t + 1 \leqslant x \leqslant \lceil k/2 \rceil - 1 \text{ and } k \text{ is odd}; \end{cases}$$

Let $E_j = \{v_{i,m} v_{i,m+1} | i + m \equiv 0 \pmod 2, i \in \mathbb{Z}_k\}$ where m is defined as above. Then the union of all hexagons of $K_o(k, q) - E^t - E_j$ does not contain a non-contractible closed

curve. Note that the edges in E_j are contained by only hexagons from H_j. So $\mathcal{H} \cap H_j \neq \emptyset$ for any $j \in \mathbb{Z}_q$. Hence $|\mathcal{H}| \geqslant q$. It follows that $\mathrm{fw}(G) = |\mathcal{H}| \geqslant \min\{q, \lceil k/2 \rceil\}$.

On the other hand, both $H_{q/2}$ and R_i of $K_o(k, q)$ contain a non-contractible closed curve. Note that $H_{q/2}$ has $\lceil \frac{k}{2} \rceil$ hexagons, and $R_{\lceil k/2 \rceil - 1}$ contains q hexagons. So $\mathrm{fw}(G) \leqslant \min\{\lceil \frac{k}{2} \rceil, q\}$. This completes the proof. $\square$

Lemma 20. *Let G be a polyhex graph on the Klein bottle. Then $\mathrm{fw}(G) = 2$ if and only if G is isomorphic to either $K_e(k, q)$ with $k = 4$ or $q = 2$, or $K_o(k, q)$ with $3 \leqslant k \leqslant 4$ or $q = 2$.*

Proof. By Theorem 19, $\mathrm{fw}(G) = 2$ if and only if $\min\{\lceil \frac{k}{2} \rceil, q\} = 2$. It follows that $3 \leqslant k \leqslant 4$ or $q = 2$.

By Theorem 13, G is isomorphic to either $K_e(k, q)$ with $k = 4$ or $q = 2$, or $K_o(k, q)$ with $3 \leqslant k \leqslant 4$ or $q = 2$. $\square$

5 Pfaffian polyhex graphs

In this section, we characterize Pfaffian polyhex graphs.

5.1 Polyhex graphs on the torus

Let G be a polyhex graph on the torus. If G is planar, then it is Pfaffian. Now assume that G is neither planar nor the Heawood graph. Proposition 11 implies that G must contain an ideal tri-cut if it is Pfaffian.

Lemma 21. *Let G be a polyhex graph on the torus. Then G does not contain an ideal tri-cut.*

Proof. Let G be a polyhex graph on the torus. By Theorem 12, G can be represented as $T(k, q, t)$ for some suitable triple of integers (k, q, t). Suppose on the contrary that G has an ideal tri-cut W. Then $|W| = 4$ and $G - W$ has exactly three components, denoted by G_1, G_2 and G_3, each of which has at least four vertices.

First suppose $q \geqslant 2$. If $T(k, q, t)$ has a layer L_i containing at least three vertices of W. Then $k \geqslant 6$ as W is independent in G. Since $q \geqslant 2$, any other layer L_j $(j \neq i)$ contains at most one vertex in W. So all $L_j - W$ with $j \in \mathbb{Z}_q \backslash \{i\}$ are contained in a common component of $G - W$. Note that at least one vertex of every component of $L_i - W$ has a neighbor in either $L_{i+1} - W$ or $L_{i-1} - W$. Hence $G - W$ is connected, a contradiction. So every layer L_i has at most two vertices in W. That implies that $L_i - W$ has at most two paths as components. So $G - W$ has at most two components, also a contradiction.

Now suppose $q = 1$. By Lemma 15, let $W = \{v_{i_0,0}, v_{i_1,0}, v_{i_2,0}, v_{i_3,0}\}$ with $0 = i_0 < i_1 < i_2 < i_3 \leqslant 2k - 2$. Then $L_0 - W$ has four paths P_0, P_1, P_2 and P_3 such that $v_{i_\alpha,0}$ joins P_α and $P_{\alpha+1}$ with $\alpha, \alpha + 1 \in \mathbb{Z}_4$. Since $G - W$ has three components, precisely two paths from P_α's $(\alpha \in \mathbb{Z}_4)$ belong to a common component of $G - W$. Note that every vertex of W has exactly one neighbor in each component G_i. So P_α and $P_{\alpha+1}$ can not belong

to the same component of $G - W$. On the other hand, every component has at least 4 vertices. So every P_α has at least two vertices. We may assume that P_1 and P_3 belong to the same component of $G - W$. Then both P_0 and P_2 are components of $G - W$.

Let $v_{m,0} \in V(P_2)$ be adjacent to $v_{0,0} \in W$. Then $m = k - (1 + 2t)$ since $v_{m,0} = v_{m+1+2t,1} = v_{0,1}$. Note that $v_{m+2,0} = v_{k+1-2t,0}$ is adjacent to $v_{2,0} \in V(P_1)$. Since P_1 and P_2 belong to different components of $G - W$, $v_{m+2,0} \notin V(P_2)$. Since each $E(W, G_i)$ is a matching and P_1 and P_3 belong to the same component of $G - W$, $v_{m+2,0} \neq v_{i_2,0}$ and thus $v_{m+2,0} \in V(P_3)$. Then $v_{m+1,0} = v_{i_2,0}$, and $v_{m-2,0} \in V(P_2)$ is adjacent to $v_{k-2,0} \in V(P_0)$, which contradicts that P_0 and P_2 belong to different components of $G - W$. $\qquad\square$

By Proposition 11, Theorem 12 and Lemma 21, we immediately have the following result.

Corollary 22. *Let G be a polyhex graph on the torus. Then G is Pfaffian if and only if it is planar or isomorphic to the Heawood graph $T(14, 1, 2)$.*

By Theorem 1, planar polyhex graphs embedded on the torus must have face-width two. Using Kuratowski's Theorem that a graph is planar if and only if it contains no K_5 and $K_{3,3}$ as minor, the following lemma characterizes planar polyhex graphs on the torus.

Lemma 23. *A polyhex graph on the torus is planar if and only if it is isomorphic to either $T(4, 2, t)$, or $T(8, 1, t)$ or $T(k, 2, k/2 - 1)$.*

Proof. We can see that both $T(4, 2, t)$ and $T(8, 1, t)$ are isomorphic to the cube Q_3 and hence are planar. For $T(k, 2, k/2 - 1)$, it is isomorphic to $C_k \times K_2$ (a plane embedding shown in Figure 7).
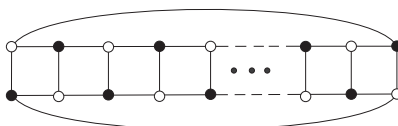

Figure 7: A plane embedding of $T(k, 2, k/2 - 1)$.

Next we will show that the other polyhex graphs on the torus are not planar. By Theorem 1 and Lemma 18, it suffices to show that $T(4, q, t)$ ($q \geqslant 3$), and $T(k, 2, t)$ ($k \geqslant 6$, $t = 0$ or $k/2 - 2$), and $T(k, 1, t)$ ($k \geqslant 6$, $k \neq 8$ and $k/4 - 1 \leqslant t \leqslant k/4$) are all non-planar.

Note that $T(4, q, t)$ ($q \geqslant 3$) is non-planar by Kuratowski's Theorem since it contains a subdivision of $K_{3,3}$ as a subgraph (see the subgraphs induced by thick lines in Figure 8).
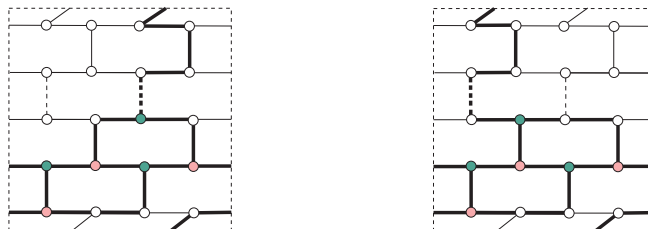

Figure 8: $K_{3,3}$-subdivisions in $T(4, q, t)$ with $q \geqslant 3$.

By Proposition 14, $T(k, 2, 0)$ is isomorphic to $T(k, 2, k/2 - 2)$. Since $T(k, 2, 0)$ ($k \geqslant 3$) contains a $K_{3,3}$-minor (see Figure 9), both $T(k, 2, 0)$ and $T(k, 2, k/2 - 2)$ are non-planar.
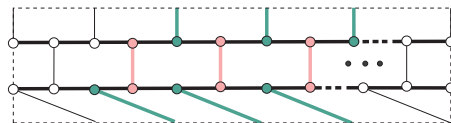

Figure 9: A $K_{3,3}$-minor in $T(k, 2, 0)$ ($k \geqslant 6$).

Now consider $T(k, 1, t)$ ($k \geqslant 6$, $k \neq 8$ and $k/4 - 1 \leqslant t \leqslant k/4$). Since $T(6, 1, 1)$ is isomorphic to $K_{3,3}$, it is thus non-planar. For $T(k, 1, t)$ with $k \geqslant 12$ and $k/4 - 1 \leqslant t \leqslant k/4$, it contains a subdivision of $K_{3,3}$ as a subgraph induced by edges $v_{0,0}v_{k-1-2t,0}$, $v_{2,0}v_{k+1-2t}$, $v_{k-1,0}v_{v-3-2t,0}$ and all edges of L_0 (see Figure 10 (left)). If $k = 10$, $t = 2$. So $T(10, 1, 2)$ contains a subdivision of $K_{3,3}$ as shown in Figure 10 (right). $\qquad\square$
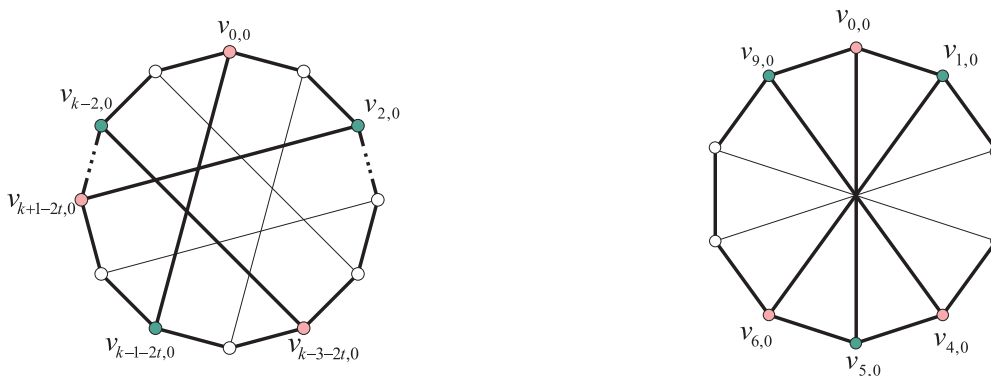

Figure 10: $K_{3,3}$-subdivisions in $T(k, 1, t)$ ($k \geqslant 12$ and $k/4 - 1 \leqslant t \leqslant k/4$ (left)) and $T(10, 1, 2)$ (right).

By Corollary 22 and Lemma 23, we have the following result which characterizes Pfaffian polyhex graphs on the torus.

Theorem 24. *Let G be a polyhex graph on the torus. Then G has a Pfaffian orientation if and only if G is isomorphic to either the Heawood graph, or the cube Q_3, or $C_k \times K_2$ for even $k \geqslant 4$.*

5.2   Polyhex graphs on the Klein bottle

First, we consider bipartite polyhex graphs on the Klein bottle.

Lemma 25. *Let G be a bipartite polyhex graph on the Klein bottle. Then G does not contain an ideal tri-cut.*

Proof. By Theorem 13, we have that G is isomorphic to $K_e(k,q)$ with $q \geqslant 2$. An analogous argument as the proof of Theorem 21 in case $q \geqslant 2$ shows that the lemma is true. $\square$

It can be seen that the Heawood graph can not be a polyhex graph on the Klein bottle. So, by Lemma 25 and Proposition 11, a Pfaffian bipartite polyhex graph on the Klein bottle must be planar.

Lemma 26. *A bipartite polyhex graph G on the Klein bottle is planar if and only if it is $K_e(4,2)$.*

Proof. Let G be a planar bipartite polyhex graph on the Klein bottle. By Theorem 1 and Lemma 20, we may assume that G is isomorphic to $K_e(4,q)$ $(q \geqslant 2)$ or $K_e(k,2)$ $(k \geqslant 4)$.

Since $K_e(k,2)$ $(k \geqslant 6)$ contains a subdivision of $K_{3,3}$ (see Figure 11), it is non-planar. Note that $K_e(4,q)$ is isomorphic to $T(4,q,1)$. Since $T(4,q,1)$ with $q \geqslant 3$ contains a subdivision of $K_{3,3}$, it is non-planar. The polyhex graph $K_e(4,2)$ is isomorphic to the cube and hence planar. Hence G is $K_e(4,2)$. $\square$

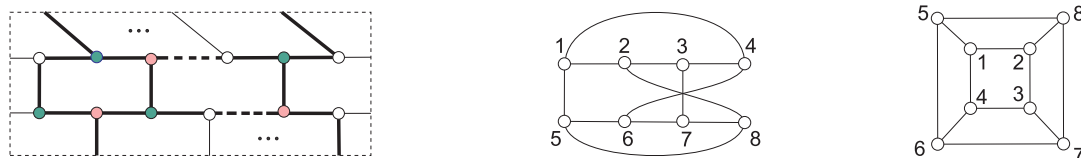

Figure 11: A $K_{3,3}$-subdivision in $K_e(k,2)$ and $K_e(4,2)$ is isomorphic to Q_3.

By Proposition 11, Theorem 8 and Lemmas 25 and 26, we have the following characterization of Pfaffian bipartite polyhex graphs on the Klein bottle.

Theorem 27. *Let G be a bipartite polyhex graphs on the Klein bottle. Then G is Pfaffian if and only if it is isomorphic to the cube.*

In the following, let G be a non-bipartite polyhex graphs on the Klein bottle. A cycle C of G is *1-sided* if its tubular neighborhood is homeomorphic to a Möbius strip, and *2-sided*, otherwise. An embedding of a graph G in the Klein bottle is *cross-cap-odd* if every non-separating cycle C of G has odd size if and only if it is 1-sided.

Lemma 28 ([17])**.** *Every graph that admits a cross-cap-odd embedding in the Klein bottle is Pfaffian.*

Lemma 29. *The embedding of a non-bipartite polyhex graph $K_o(k,q)$ is a cross-cap-odd embedding.*

Proof. For $K_o(k,q)$, let

$$E_0 := \begin{cases} \{v_{0,i}v_{k-1,q-1-i} | i \in \mathbb{Z}_k\}, & \text{if } k \equiv 0 \pmod 2; \\ \{v_{0,i}v_{k-1,q-2-i} | i \in \mathbb{Z}_k\}, & \text{otherwise.} \end{cases}$$
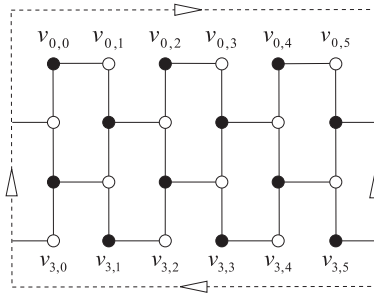
Figure 12: The tube $T_{4,6}$ obtained from $K_o(4,6)$ by deleting edges $v_{0,i}v_{3,5-i}$ for $i \in \mathbb{Z}_6$.

The subgraph obtained from $K_o(k,q)$ by deleting all edges in E_0, together with its hexagonal faces, is a tube, denoted by $T_{k,q}$. Conversely, $K_o(k,q)$ can be generated from the tube $T_{k,q}$ by adding edges in E_0.

Note that $T_{k,q}$ is a bipartite graph since a proper 2-coloring of $L(k,q)$ is also a proper 2-coloring of $T_{k,q}$. Any edge in E_0 joins two vertices with the same color. Now for any cycle C of $K_o(k,q)$, if C is also a cycle of $T_{k,q}$, then C is 2-sided and has even length. If C is not a cycle of $T_{k,q}$, then $E(C) \cap E_0 \neq \emptyset$. Let $\delta := |E(C) \cap E_0|$. Contract every edge in $E_0 \cap E(C)$ in C to a single vertex, and let C' be the new cycle. Then C' has even length since the vertices alternate in color along C'. Hence $|E(C)| \equiv \delta \pmod 2$.

Note that the tubular neighborhood of C is homeomorphic to a Möbius strip if and only if $\delta = |E(C) \cap E_0| \equiv 1 \pmod 2$. So C is 1-sided if and only if $|E(C)| \equiv \delta = |E(C) \cap E_0| \equiv 1 \pmod 2$. It follows that $K_o(k,q)$ is a cross-cap-odd embedding. $\qquad\square$

By Lemmas 28 and 29, the following result follows immediately.

Theorem 30. *Every non-bipartite polyhex graph on the Klein bottle is Pfaffian.*

In [10], it has been shown that $K_o(k,q)$ is 2-extendable if and only if $k \geqslant 4$ and $q \geqslant 5$. By a result of Lovász and Plummer (Theorem 5.5.23 on Page 206 in [13]) that a 2-extendable graph is either bicritical or elementary bipartitie, $K_o(k,q)$ is bicritical and hence a brick. By Theorem 19, the face-width $\mathrm{fw}(K_o(k,q)) = \min\{\lceil \frac{k}{2} \rceil, q\} \to \infty$ as $\min\{k,q\} \to \infty$. Hence we have the following remark.

Remark 1. Theorem 4 shows that a Pfaffian brace embedded on a surface Σ with $g(\Sigma) > 0$ has a small face-width. But the face-width of a Pfaffian brick on sufaces with a positive genus could be arbitrarily large.

Remark 2. In [18], Norine and Thomas conjectured that every Pfaffian cubic graph is 3-edge colorable. As shown in [28], every polyhex graph on the Klein bottle is Hamiltonian and hence 3-edge colorable. By Theorem 30, every Pfaffian polyhex graph is 3-edge colorable. Hence the conjecture of Norine and Thomas is true for Pfaffian polyhex graphs.

Acknowledgements

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