A Slight Improvement to the Colored Bárány's Theorem

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Abstract

Suppose $d+1$ absolutely continuous probability measures $m_0, \ldots, m_d$ on $\mathbb{R}^d$ are given. In this paper, we prove that there exists a point of $\mathbb{R}^d$ that belongs to the convex hull of $d+1$ points $v_0, \ldots, v_d$ with probability at least $\frac{2d}{(d+1)!(d+1)}$, where each point v_i is sampled independently according to probability measure m_i.

1 Introduction

Let $P \subset \mathbb{R}^d$ be a set of n points. Every $d+1$ of them span a simplex, for a total of $\binom{n}{d+1}$ simplices. The point selection problem asks for a point contained in as many simplices as possible. Boros and Füredi [BF84] showed for $d = 2$ that there always exists a point in $\mathbb{R}^2$ contained in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ simplices. A short and clever proof of this result was given by Bukh [Buk06]. Bárány [Bár82] generalized this result to higher dimensions:

Theorem 1 (Bárány [Bár82])**.** *There exists a point in $\mathbb{R}^d$ that is contained in at least $c_d\binom{n}{d+1} - O(n^d)$ simplices, where $c_d > 0$ is a constant depending only on the dimension d.*

This general result, the Bárány's theorem, is also known as the first selection lemma. We will henceforth denote by c_d the largest possible constant for which the Bárány's theorem holds true. Bukh, Matoušek and Nivasch [BMN10] used a specific construction called the stretched grid to prove that the constant $c_2 = \frac{2}{9}$ in the planar case found by Boros and Füredi [BF84] is the best possible. In fact, they proved that $c_d \leqslant \frac{d!}{(d+1)^d}$. On the other hand, Bárány's proof in [Bár82] implies that $c_d \geqslant (d+1)^{-d}$, and Wagner [Wag03] improved it to $c_d \geqslant \frac{d^2+1}{(d+1)^{d+1}}$.

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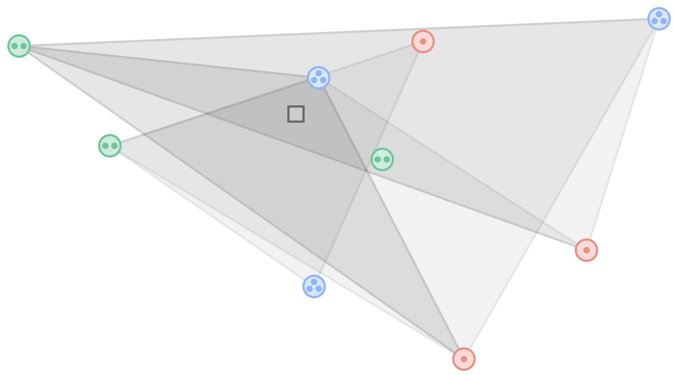

Figure 1: 3 red points, 3 green points and 3 blue points are placed in the plane. The point marked by a square is contained in 6 $(= \frac{2}{9} \cdot 3^3)$ colorful triangles.

Gromov [Gro10] further improved the lower bound on c_d by topological means. His method gives $c_d \geqslant \frac{2d}{(d+1)(d+1)!}$. Matoušek and Wagner [MW11] provided an exposition of the combinatorial component of Gromov's approach in a combinatorial language, while Karasev [Kar12] found a very elegant proof of Gromov's bound, which he described as a "decoded and refined" version of Gromov's proof.

The exact value of c_d has been the subject of ongoing research and is unknown, except for the planar case. Basit, Mustafa, Ray and Raza [BMRR10] and successively Matoušek and Wagner [MW11] improved the Bárány's theorem in $\mathbb{R}^3$. Král', Mach and Sereni [KMS12] used flag algebras from extremal combinatorics and managed to further improve the lower bound on c_3 to more than 0.07480, whereas the best upper bound known is 0.09375.

However, in this paper, we are concerned with a colored variant of the point selection problem. Let $P_0, \ldots, P_d$ be $d+1$ disjoint finite sets in $\mathbb{R}^d$. A *colorful simplex* is the convex hull of $d+1$ points each of which comes from a distinct P_i. For the colored point selection problem, we are concerned with the point(s) contained in many colorful simplices. Karasev proved:

Theorem 2 (Karasev [Kar12])**.** *Given a family of $d+1$ absolutely continuous probability measures $\mathbf{m} = (m_0, \ldots, m_d)$ on $\mathbb{R}^d$, an $\mathbf{m}$-simplex*[1] *is the convex hull of $d+1$ points $v_0, \ldots, v_d$ with each point v_i sampled independently according to probability measure m_i. There exists a point of $\mathbb{R}^d$ that is contained in an $\mathbf{m}$-simplex with probability $p_d \geqslant \frac{1}{(d+1)!}$. In addition, if two probability measures coincide, then the probability can be improved to $p_d \geqslant \frac{2d}{(d+1)(d+1)!}$.*

By a standard argument which we will provide immediately, a result on the colored point selection problem follows:

Corollary 3. *If $P_0, \ldots, P_d$ each contains n points, then there exists a point that is contained in at least $\frac{1}{(d+1)!} \cdot n^{d+1}$ colorful simplices.*

[1] An $\mathbf{m}$-simplex is actually a simplex-valued random variable.

Our result drops the additional assumption in theorem 2, hence improves corollary 3:

Main Theorem. *There is a point in $\mathbb{R}^d$ that belongs to an* $\mathbf{m}$*-simplex with probability* $p_d \geqslant \frac{2d}{(d+1)(d+1)!}$.

Corollary 4. *There exists a point that is contained in at least* $\frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ *colorful simplices.*

Proof of corollary 4 from the main theorem. Given $d+1$ sets $P_0, \dots, P_d$ in $\mathbb{R}^d$ each of which contains n points. Let $\Psi\colon \mathbb{R}^d \to \mathbb{R}$ be the *bump function* defined by $\Psi(x_1, \dots, x_d) = \prod_{i=1}^d \psi(x_i)$, where $\psi(x) = e^{-1/(1-x^2)}\mathbf{1}_{|x|<1}$, and set $\Psi_n(x_1, \dots, x_d) = n^d \Psi(nx_1, \dots, nx_d)$ for $n \in \mathbb{N}$. It is a standard fact that Ψ and Ψ_n are absolutely continuous probability measures supported on $[-1,1]^d$ and $[-1/n, 1/n]^d$ respectively.

For each $n \in \mathbb{N}$ and $0 \leqslant k \leqslant d$, define $m_k^{(n)}(x) := \frac{1}{n}\sum_{p \in P_k} \Psi_n(x-p)$ for $x \in \mathbb{R}^d$. Note that $m_k^{(n)}$ is an absolutely continuous probability measure supported on the Minkowski sum of P_k and $[-1/n, 1/n]^d$. Let $\mathbf{m}^{(n)}$ be the family of $d+1$ probability measures $m_0^{(n)}, \dots, m_d^{(n)}$. By the main theorem, there is a point $p^{(n)}$ of $\mathbb{R}^d$ that belongs to an $\mathbf{m}^{(n)}$-simplex with probability at least $\frac{2d}{(d+1)(d+1)!}$.

Because no point in a certain neighborhood of infinity is contained in any $\mathbf{m}^{(n)}$-simplex, the set $\{p^{(n)} : n \in \mathbb{N}\}$ is bounded, and consequently the set has a limit point p. Suppose p is contained in N colorful simplices. Let $\epsilon > 0$ be the distance from p to all the colorful simplices that do not contain p. Choose n large enough such that $1/n \ll \epsilon$ and $\left|p^{(n)} - p\right| \ll \epsilon$. By the choice of n, if p is not contained in a colorful simplex spanned by $v_0, \dots, v_d$, then $p^{(n)}$ is not contained the convex hull of $v'_0, \dots, v'_d$ for all $v'_i \in v_i + [-1/n, 1/n]^d$. This implies that the probability that $p^{(n)}$ is contained in an $\mathbf{m}^{(n)}$-simplex is at most $\frac{N}{n^{d+1}}$. Hence p is the desired point contained in $N \geqslant \frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ colorful simplices. □

Readers who are familiar with Karasev's work [Kar12] would notice that our proof of the main theorem heavily relies on his arguments. The author is deeply in debt to him.

2 Proof of the Main Theorem

In this section, we provide the proof of the main theorem. The topological terms in the proof are standard, and can be found in [Mat03]. In addition to the notion of an $\mathbf{m}$-simplex, in the proof, we will often refer to an $(m_k, \dots, m_d)$-face which means the convex hull of $d-k+1$ points $v_k, \dots, v_d$ with each point v_i sampled independently according to probability measure m_i. An $\mathbf{m}$-simplex and an $(m_k, \dots, m_d)$-face are both set-valued random variables.

Proof of the main theorem. To obtain a contradiction, we suppose that for any point v in $\mathbb{R}^d$, the probability that v belongs to an $\mathbf{m}$-simplex is less than $p_d := \frac{2d}{(d+1)(d+1)!}$. Since this probability, as a function of point v, is continuous and uniformly tends to 0 as v goes to infinity, there is an $\epsilon > 0$ such that v is contained in an $\mathbf{m}$-simples with probability at most $p_d - \epsilon$ for all v in $\mathbb{R}^d$.

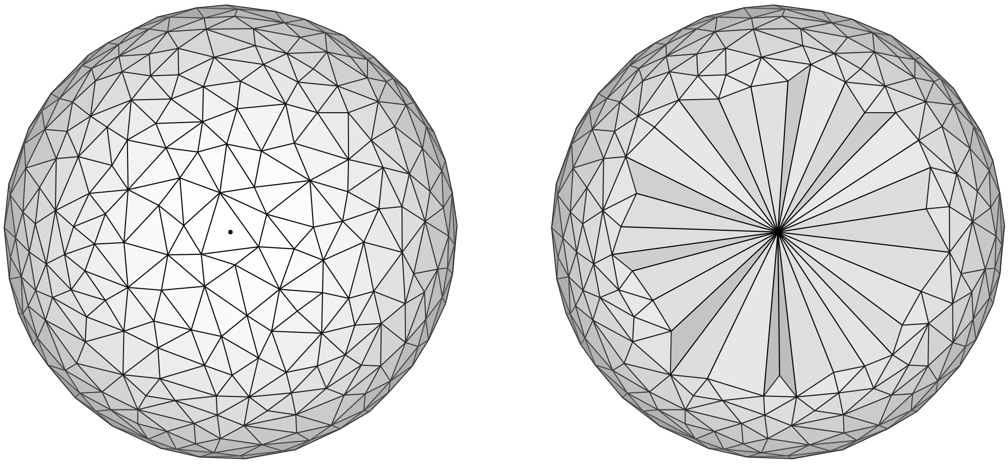

Figure 2: The bird's-eye view of a triangulation of S^2 with a 2-simplex containing ∞ and the cone over part of the triangulation.

Let $S^d := \mathbb{R}^d \cup \{\infty\}$ be the one-point compactification of the Euclidean space $\mathbb{R}^d$. Take $\delta = \epsilon/d$. Choose a finite triangulation[2] $\mathcal{T}$ of S^d with one of the d-simplices containing ∞ such that for $0 < k \leqslant d$, any k-face of $\mathcal{T}$ intersects an $(m_k, \ldots, m_d)$-face with probability less than δ and that the measure of any d-face of $\mathcal{T}$ under $(m_{d-1} + m_d)/2$ is less than δ. This can be done by taking a sufficiently fine triangulation of S^2 with one d-simplex having ∞ in its relative interior.

We use cone$(\cdot)$ as the cone functor[3] with apex O. A triangulation $\mathcal{T}$ of S^d naturally extends to a triangulation cone$(\mathcal{T})$ of cone(S^d). We denote the k-skeleton[4] of $\mathcal{T}$ and cone$(\mathcal{T})$ by $\mathcal{T}^{\leqslant k}$ and cone$(\mathcal{T})^{\leqslant k}$ respectively.

We are going to define a continuous map $f\colon \text{cone}(\mathcal{T})^{\leqslant d} \to S^d$. Put $f(x) = x$ for all $x \in S^d = ||\mathcal{T}|| \subset \left|\left|\text{cone}(\mathcal{T})^{\leqslant d}\right|\right|$, and set $f(O) = \infty$. We proceed to define f on cone(σ) for all the k-faces σ of $\mathcal{T}$ inductively on dimension k of σ while we maintain the property that the image of the boundary of cone(σ) under f, that is $f(\partial\text{cone}(\sigma))$, intersects an $(m_k, \ldots, m_d)$-face with probability at most $(k+1)!(p_d - \epsilon + k\delta)$. We say f is *economical* over a k-face σ of $\mathcal{T}^{\leqslant d-1}$ if f and σ satisfy the above property. Unlike Karasev [Kar12], our inductive construction of f follows the same pattern until $k = d-2$ instead of $d-1$. The main innovation of this proof is a different construction for $k = d-1$, which enables us to remove the additional assumption in theorem 2.

Note that for any 0-face σ in $\mathcal{T}$, $f(\partial\text{cone}(\sigma)) = f(\{\sigma, O\}) = \{\sigma, \infty\}$. According to the assumption at the beginning of the proof, $f(\partial\text{cone}(\sigma))$ intersects an $(m_0, \ldots, m_d)$-face,

[2]A triangulation $\mathcal{T}$ of a topological space X is a simplicial complex K, homeomorphic to X, together with a homeomorphism $h\colon ||\mathsf{K}|| \to X$. Since the finite triangulation of interest is an extension of the triangulation of a d-simplex X in $\mathbb{R}^d$ and h is an identity map, we will freely use topological notions such as "a k-face (as a subset of S^d)" instead of "the image of a k-face in K under h". With such abuse of language, we can avoid going back and forth between the simplicial complex and the topological space.

[3]The cone over a space X is the quotient space cone$(X) := (X \times [0,1]) / (X \times \{1\})$. The apex is the equivalence class $\{(x,1) : x \in X\}$.

[4]The k-skeleton of a simplicial complex Δ consists of all simplices of Δ of dimension at most k.

that is, an $\mathbf{m}$-simplex, with probability at most $p_d - \epsilon$. Therefore f is economical over 0-faces of $\mathcal{T}$. This finishes the first step.

Suppose f is already defined on $\mathrm{cone}(\mathcal{T})^{\leqslant k}$ and it is economical over k-faces of $\mathcal{T}$. We are going to extend the domain of f to $\mathrm{cone}(\mathcal{T})^{\leqslant k+1}$. Indeed, we only need to define f on $\mathrm{cone}(\sigma)$ for every k-face σ of $\mathcal{T}$.

Take any k-face σ of $\mathcal{T}$. Suppose convex hull of $v_k, \dots, v_d$, denoted by $\mathrm{conv}(v_k, \dots, v_d)$, is an $(m_k, \dots, m_d)$-face. Notice that the following statements are equivalent:

- $f(\partial\mathrm{cone}(\sigma))$ intersects $\mathrm{conv}(v_k, \dots, v_d)$;
- for some $v \in f(\partial\mathrm{cone}(\sigma))$, the ray with initial point v in the direction $\overrightarrow{v_k v}$ intersects $\mathrm{conv}(v_{k+1}, \dots, v_d)$.

We call the union of such rays the *shadow* of $f(\partial\mathrm{cone}(\sigma))$ centered at v_k. Since f is economical over σ, the probability for an $(m_k, \dots, m_d)$-face to meet $f(\partial\mathrm{cone}(\sigma))$ is at most $(k+1)!(p_d - \epsilon + k\delta)$, and so there exists $v_k^\sigma \in \mathbb{R}^d$ such that the shadow of $f(\partial\mathrm{cone}(\sigma))$ centered at v_k^σ intersects $\mathrm{conv}(v_{k+1}, \dots, v_d)$ with probability at most $(k+1)!(p_d - \epsilon + k\delta)$.

Now, we define f on $\mathrm{cone}(\sigma)$. First, let g be the homeomorphism from $\mathrm{cone}(\sigma)$ onto the cone over $\partial\mathrm{cone}(\sigma)$ with apex c such that g is an identity on $\partial\mathrm{cone}(\sigma)$. This can be done because $\mathrm{cone}(\sigma)$ is homeomorphic to a $(k+1)$-simplex Δ and it is easy to find a homeomorphism from Δ to $\mathrm{cone}(\partial\Delta)$ that keeps $\partial\Delta$ fixed.

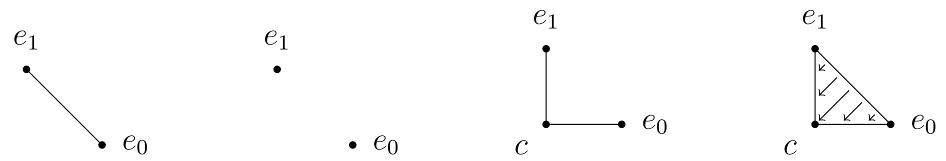

Figure 3: An illustration of an 1-simplex Δ, $\partial\Delta$, $\mathrm{cone}(\partial\Delta)$ and a homeomorphism from Δ to $\mathrm{cone}(\partial\Delta)$.

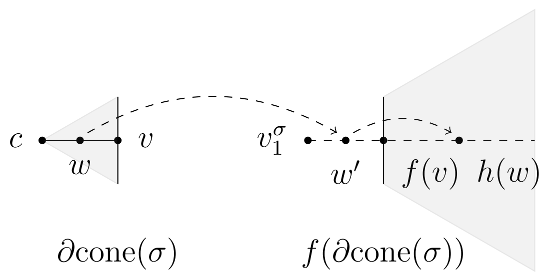

Figure 4: The illustration shows a cone over part of $\partial\mathrm{cone}(\sigma)$ with apex c and a point v on the boundary, and how a point w on the line segment $[v, c)$ are mapped under h.

Next, note that every point w in $\mathrm{cone}(\sigma)$ except c is on a line segment $[v, c)$ for a unique point v on $\partial\mathrm{cone}(\sigma)$. If $t = \overline{vw}/\overline{wc} \in [0, \infty)$, then put $h(w) = \overrightarrow{f(v)} + t \cdot \overrightarrow{v_k^\sigma f(v)}$. In addition, set $h(c) = \infty$. The function h maps $[v, c)$ onto $[f(v), v_k^\sigma)$ linearly and then takes the inversion centered at v_k^σ with radius $\overline{v_k^\sigma f(v)}$ so that $[f(v), v_k^\sigma)$ gets mapped onto the ray with the initial point $f(v)$ in the direction $\overrightarrow{v_k^\sigma f(v)}$. Evidently, h is a continuous map from $\operatorname{cone}(\partial\operatorname{cone}(\sigma))$ onto the shadow of $f(\partial\operatorname{cone})$ centered at v_k^σ that coincides with f on $\partial\operatorname{cone}(\sigma)$.

Define f on $\operatorname{cone}(\sigma)$ to be the composition of g and h:

$$\begin{array}{ccccc} \partial\operatorname{cone}(\sigma) & \xrightarrow{=} & \partial\operatorname{cone}(\sigma) & \xrightarrow{f} & f(\partial\operatorname{cone}(\sigma)) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{cone}(\sigma) & \xrightarrow{g} & \operatorname{cone}\left(\partial\operatorname{cone}(\sigma)\right) & \xrightarrow{h} & \text{the shadow of } f(\partial\operatorname{cone}(\sigma)) \text{ centered at } v_k^\sigma. \end{array}$$

According to the commutative diagram above, f is well-defined on $\operatorname{cone}(\sigma)$ in the sense that it is compatible with its definition on $\operatorname{cone}(\mathcal{T})^{\leqslant k}$. We use the phrase "fill in the boundary of $\operatorname{cone}(\sigma)$ against the center v_k^σ" to represent the above process that extends the domain of f from $\partial\operatorname{cone}(\sigma)$ to $\operatorname{cone}(\sigma)$.

To complete the inductive step, we must demonstrate that f is economical over $(k+1)$-faces of $\mathcal{T}$. Pick any $(k+1)$-face τ of $\mathcal{T}$. Let $\sigma_0, \dots, \sigma_{k+1}$ be the k-faces of τ. Observing that $f(\partial\operatorname{cone}(\tau)) = f(\tau \cup \operatorname{cone}(\partial\tau)) = \tau \cup f(\operatorname{cone}(\sigma_0)) \cup \ldots \cup f(\operatorname{cone}(\sigma_{k+1}))$ and that $f(\operatorname{cone}(\sigma_i))$ is the shadow of $f(\partial\operatorname{cone}(\sigma_i))$ centered at $v_k^{\sigma_i}$ which intersects an $(m_{k+1}, \dots, m_d)$-face with probability at most $(k+1)!(p_d - \epsilon + k\delta)$, we obtain that the probability for an $(m_{k+1}, \dots, m_d)$-face to intersect $f(\partial\operatorname{cone}(\tau))$ is dominated by $\delta + (k+2)(k+1)!(p_d - \epsilon + k\delta) \leqslant (k+2)!(p_d - \epsilon + (k+1)\delta)$.

We have so far defined a continuous map f on $\operatorname{cone}(\mathcal{T})^{\leqslant d-1}$ such that for any $(d-1)$-face σ of $\mathcal{T}$ the probability for an $(m_{d-1}m_d)$-face to intersect $D := f(\partial\operatorname{cone}(\sigma))$ is at most $d!(p_d - \epsilon + (d-1)\delta)$. We write $f(X)\text{mod}2 := \{y \in f(X) : |f^{-1}(y) \cap X| = 1 \pmod 2\}$ for the set of points in $f(X)$ whose fibers in X have an odd number of points. Set $\bar{m} := (m_{d-1} + m_d)/2$. We are going to define f on $\operatorname{cone}(\sigma)$ such that $\bar{m}\left(f(\operatorname{cone}(\sigma))\text{mod}2\right)$ is less than $\frac{1-\delta}{d+1}$.

Fix a point s in $\mathbb{R}^d \backslash D$. For any point t in $\mathbb{R}^d \backslash D$, if a generic piecewise linear path from s to t intersects with D an odd number of times, then put t in B, otherwise put it in A. Here the number of intersections of a piecewise linear path L and D might not be the cardinality of $L \cap D$. Instead, the number of intersections is precisely $\sum_{x \in L \cap D} |f^{-1}(x) \cap \partial\operatorname{cone}(\sigma)|$, that is, it takes the multiplicity into account. Thus we have partitioned $\mathbb{R}^d \backslash D$ into A and B such that any generic piecewise linear path from a point in A to a point in B meets D an odd number of times. Suppose $a := m_{d-1}(A)$, $b := m_d(A)$ and $x := \bar{m}(A) = (a+b)/2$. The probability that an $(m_{d-1}m_d)$-face intersects with D is at least $a(1-b) + (1-a)b$. Hence $a(1-b) + (1-a)b < d!(p_d - \epsilon + (d-1)\delta) < 2\left(\frac{1-\delta}{d+1}\right)\left(1 - \frac{1-\delta}{d+1}\right)$. Because $a(1-b) + (1-a)b = (a+b) - 2ab \geqslant (a+b) - (a+b)^2/2 = 2x(1-x)$, either x or $1-x$ is less than $\frac{1-\delta}{d+1}$. In other words, one of $\bar{m}(A)$ and $\bar{m}(B)$ is less than $\frac{1-\delta}{d+1}$. We may assume that $\bar{m}(B) < \frac{1-\delta}{d+1}$.

Fix a point $c \in A$. Again, we fill in the boundary of $\operatorname{cone}(\sigma)$ against the center c. For any generic point $x \in A$, the line segment $[c, x]$ intersects with D an even number of times. For every v on $\partial\operatorname{cone}(\sigma)$, the ray with the initial point $f(v)$ in the direction $\overrightarrow{cf(v)}$

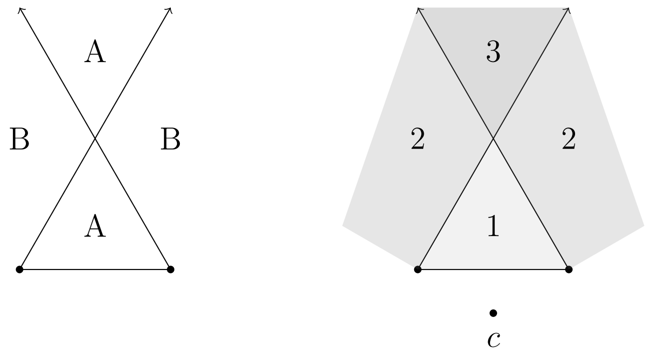

Figure 5: An illustration of the partition, the result of filling in against c, and $f(\mathrm{cone}(\sigma))\mathrm{mod}2$.

covers x once if and only if the line segment $[c, x]$ intersects with D at $f(v)$. Because $f(\mathrm{cone}(\sigma))$ is the union of such rays, the number of times that x is covered by $f(\mathrm{cone}(\sigma))$ is exactly the number of intersections between $[c, x]$ and D. This implies that x is not in $f(\mathrm{cone}(\sigma))\mathrm{mod}2$. Therefore $f(\mathrm{cone}(\sigma))\mathrm{mod}2$ is a subset of $B \cup D$ almost surely. Noticing that $\bar{m}(D) = 0$, the extension of f has the desired property $\bar{m}\left(f(\mathrm{cone}(\sigma))\mathrm{mod}2\right) < \frac{1-\delta}{d+1}$.

Pick any d-face τ of $\mathcal{T}$. Suppose the $(d-1)$-faces of τ are $\sigma_0, \ldots, \sigma_d$. By a parity argument, we have

$$\begin{aligned} f(\partial\mathrm{cone}(\tau))\mathrm{mod}2 &= [\tau \cup f(\mathrm{cone}(\sigma_0)) \cup \ldots \cup f(\mathrm{cone}(\sigma_d))]\,\mathrm{mod}2 \\ &\subset \tau \cup f(\mathrm{cone}(\sigma_0))\mathrm{mod}2 \cup \ldots \cup f(\mathrm{cone}(\sigma_d))\mathrm{mod}2. \end{aligned}$$

Therefore $\bar{m}\left(f(\partial\mathrm{cone}(\tau))\mathrm{mod}2\right)$ is less than $\delta + (d+1)\frac{1-\delta}{d+1} = 1$, and so the degree of f on $\partial\mathrm{cone}(\tau)$, denoted by $\deg\left(f, \partial\mathrm{cone}(\tau)\right)$, is even. Because

$$\sum_{\tau} \deg\left(f, \partial\mathrm{cone}(\tau)\right) = 2\sum_{\sigma} \deg\left(f, \mathrm{cone}(\sigma)\right) + \deg\left(f, \mathcal{T}\right) = \deg\left(f, \mathcal{T}\right) \pmod 2,$$

where the first sum and the second sum are over all d-faces and all $(d-1)$-faces of $\mathcal{T}$ respectively, we know that $\deg\left(f, \mathcal{T}\right)$ is even, which contradicts with the fact that f is identity on $\mathcal{T}$. ☐

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