The Ramsey numbers of paths versus wheels: a complete solution

Binlong Li[1,2] Bo Ning[1]

[1]Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi 710072, P.R. China

[2]European Centre of Excellence NTIS
306 14 Plzeň, Czech Republic

libinlong@mail.nwpu.edu.cn, ningbo_math84@mail.nwpu.edu.cn

Submitted: Dec 14, 2013; Accepted: Nov 14, 2014; Published: Nov 27, 2014
Mathematics Subject Classifications: 05C55, 05D10

Abstract

Let G_1 and G_2 be two given graphs. The Ramsey number $R(G_1, G_2)$ is the least integer r such that for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2. We denote by P_n the path on n vertices and W_m the wheel on $m+1$ vertices. Chen et al. and Zhang determined the values of $R(P_n, W_m)$ when $m \leqslant n+1$ and when $n+2 \leqslant m \leqslant 2n$, respectively. In this paper we determine all the values of $R(P_n, W_m)$ for the left case $m \geqslant 2n+1$. Together with Chen et al.'s and Zhang's results, we give a complete solution to the problem of determining the Ramsey numbers of paths versus wheels.

Keywords: Ramsey number; Path; Wheel

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here, and consider finite simple graphs only.

Let G be a graph. We denote by $\nu(G)$ the order of G, by $\delta(G)$ the minimum degree of G, and by $\omega(G)$ the component number of G. We denote by P_n and C_n the path and cycle on n vertices, respectively. The *wheel* on $n+1$ vertices, denoted by W_n, is the graph obtained by joining a vertex to each vertex of a C_n.

Let G_1 and G_2 be two graphs. The *Ramsey number* $R(G_1, G_2)$, is defined as the least integer r such that for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2, where $\overline{G}$ is the complement of G. If G_1 and G_2 are both complete, then $R(G_1, G_2)$

is the classical Ramsey number $r(\nu(G_1), \nu(G_2))$. Otherwise, $R(G_1, G_2)$ is usually called the *generalized Ramsey number*.

In 1967, Gerencsér and Gyárfás [9] computed the Ramsey numbers of all path-path pairs, and gave the first generalized Ramsey number formula. (In fact, this question of determining Ramsey numbers of paths versus paths appeared in a paper of Erdős [5] in 1947, and the right upper bound was also determined there.) After that, Faudree et al. [8] determined the Ramsey numbers of paths versus cycles. We list these results as bellow, both of them will be used in this paper.

Theorem 1 (Gerencsér and Gyárfás [9]). *If $m \geqslant n \geqslant 2$, then*

$$R(P_n, P_m) = m + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Theorem 2 (Faudree et al. [8]). *If $n \geqslant 2$ and $m \geqslant 3$, then*

$$R(P_n, C_m) = \begin{cases} 2n-1, & \text{for } n \geqslant m \text{ and } m \text{ is odd;} \\ n + m/2 - 1, & \text{for } n \geqslant m \text{ and } m \text{ is even;} \\ \max\{m + \lfloor n/2 \rfloor - 1, 2n-1\}, & \text{for } m > n \text{ and } m \text{ is odd;} \\ m + \lfloor n/2 \rfloor - 1, & \text{for } m > n \text{ and } m \text{ is even.} \end{cases}$$

Recently, graph theorists have begun to investigate the Ramsey numbers of paths versus wheels. Baskoro and Surahmat [1] conjectured the values of $R(P_n, W_m)$ when $n \geqslant m-1$, and got some partial results. Chen et al. [3] completely determined the values of $R(P_n, W_m)$ when $n \geqslant m-1$. Salman and Broersma [11] further generalized Chen et al.'s result. Zhang [12] firstly obtained all the values of $R(P_n, W_m)$ when $n+2 \leqslant m \leqslant 2n$. We list the results of Chen et al.'s and Zhang's in the following.

Theorem 3 (Chen et al. [3]). *If $3 \leqslant m \leqslant n+1$, then*

$$R(P_n, W_m) = \begin{cases} 3n-2, & m \text{ is odd;} \\ 2n-1, & m \text{ is even.} \end{cases}$$

Theorem 4 (Zhang [12]). *If $n+2 \leqslant m \leqslant 2n$, then*

$$R(P_n, W_m) = \begin{cases} 3n-2, & m \text{ is odd;} \\ m+n-2, & m \text{ is even.} \end{cases}$$

For the case $m \geqslant 2n+1$, some upper bounds and lower bounds of $R(P_n, W_m)$ were given [11, 12]. Furthermore, for some n, m, the exact values of $R(P_n, W_m)$ were also determined in [11, 12].

In this paper we will prove the following formula, which can be used to determine all the values of $R(P_n, W_m)$ for the left case $m \geqslant 2n+1$.

Theorem 5. *If $n \geqslant 2$ and $m \geqslant 2n+1$, then*

$$R(P_n, W_m) = \begin{cases} (n-1) \cdot \beta + 1, & \alpha \leqslant \gamma; \\ \lfloor (m-1)/\beta \rfloor + m, & \alpha > \gamma, \end{cases}$$

where

$$\alpha = \frac{m-1}{n-1},\ \beta = \lceil \alpha \rceil \text{ and } \gamma = \frac{\beta^2}{\beta+1}.$$

Together with Theorems 3 and 4, we give a complete solution to the problem of determining the Ramsey numbers of paths versus wheels.

2 Preliminaries

Before our proof we will first list one result due to Zhang [12] and give some additional terminology and notation. Second, we will prove a series of lemmas which support our proof of the main theorem.

The following result is a rewriting of two corollaries in [12]. It helps us to deal with the cases $n = 3, 4$ in our proof.

Theorem 6 (Zhang [12]). *If $n \geqslant 3$ and $m \geqslant 2n+1$, then*

$$R(P_n, W_m) = \begin{cases} m+n-1, & \text{if } m \equiv 1 \mod (n-1); \\ m+n-2, & \text{if } m \equiv 0, 2 \mod (n-1). \end{cases}$$

For integers s, t, the *interval* $[s, t]$ is the set of integers i with $s \leqslant i \leqslant t$. Note that if $s > t$, then $[s, t] = \emptyset$. Let X be a subset of $\mathbb{N}$. We set $\mathcal{L}(X) = \{\sum_{i=1}^{k} x_i : x_i \in X, k \in \mathbb{N}\}$, and suppose $0 \in \mathcal{L}(X)$ for any set X. Note that if $1 \in X$, then $\mathcal{L}(X) = \mathbb{N}$. For an interval $[s, t]$, we use $\mathcal{L}[s, t]$ instead of $\mathcal{L}([s, t])$.

In the following of the paper, n always denotes an integer at least 2 and m an integer at least 3. We denote by $\mathrm{par}(n)$ the parity of n, i.e., $\mathrm{par}(n) = \lceil n/2 \rceil - \lfloor n/2 \rfloor$.

For integers n, m, let $t(n, m)$ be the values of $R(P_n, W_m)$ defined in Theorem 5, that is,

$$t(n,m) = \begin{cases} (n-1)\cdot\beta + 1, & \alpha \leqslant \gamma; \\ \lfloor (m-1)/\beta \rfloor + m, & \alpha > \gamma, \end{cases}$$

where

$$\alpha = \frac{m-1}{n-1}, \beta = \lceil \alpha \rceil \text{ and } \gamma = \frac{\beta^2}{\beta+1}.$$

Lemma 1. *If $m \geqslant 2n+1$, then $t(n, m) = \min\{t : t \notin \mathcal{L}[t-m+1, n-1]\}$.*

Proof. Set $T = \{t : t \in \mathcal{L}[t-m+1, n-1]\}$. Note that if $t \in T$, then $t-1 \in T$. So it is sufficient to prove that $t(n, m) = \max(T) + 1$.

Note that

$$\begin{aligned} t \in T &\Leftrightarrow t \in \mathcal{L}[t-m+1, n-1] \\ &\Leftrightarrow t \in [k(t-m+1), k(n-1)], \text{ for some integer } k \\ &\Leftrightarrow t \leqslant \frac{k}{k-1}(m-1) \text{ and } t \leqslant k(n-1), \text{ for some integer } k \end{aligned}$$

$$\Leftrightarrow t \leqslant k(n-1) \text{ for some integer } k < \alpha+1, \text{ or}$$
$$t \leqslant \left\lfloor \frac{m-1}{k-1} \right\rfloor + m - 1, \text{ for some integer } k \geqslant \alpha+1.$$

This implies that

$$T = \{t : t \leqslant k(n-1), k \leqslant \beta\} \cup \left\{t : t \leqslant \left\lfloor \frac{m-1}{k-1} \right\rfloor + m - 1, k \geqslant \beta+1\right\}.$$

Thus

$$\begin{aligned} \max(T) &= \max\left\{(n-1)\beta, \left\lfloor \frac{m-1}{\beta} \right\rfloor + m - 1\right\} \\ &= \begin{cases} (n-1)\cdot\beta, & \alpha \leqslant \gamma; \\ \lfloor (m-1)/\beta \rfloor + m - 1, & \alpha > \gamma. \end{cases} \end{aligned}$$

We conclude that $t(n,m) = \max(T) + 1$. □

Lemma 2. *Let p be an integer, and G be a graph on at least three vertices.*

(1) *If G is 2-connected and $\delta(G) \geqslant \lceil p/2 \rceil$, then G contains a cycle of order at least $\min\{\nu(G), p\}$.*

(2) *If $x \in V(G)$, G is connected and $d(v) \geqslant p-1$ for every vertex $v \in V(G)\backslash\{x\}$, then G contains a path from x of order at least p.*

(3) *If $x, y \in V(G)$, $G + xy$ is 2-connected and $d(v) \geqslant p-1$ for every vertex $v \in V(G)\backslash\{x,y\}$, then G contains a path from x to y of order at least p.*

(4) *If $x, y \in V(G)$, $G + xy$ is 2-connected and $d(v) \geqslant \lceil p/2 \rceil$ for every vertex $v \in V(G)\backslash\{x,y\}$, then G contains a path from x of order at least $\min\{\nu(G), p\}$.*

(5) *If G is connected and $\delta(G) \geqslant \lfloor p/2 \rfloor$, then G contains a path of order at least $\min\{\nu(G), p\}$.*

(6) *If $x \in V(G)$, G is connected, and $d_{G-x}(v) \geqslant p-2$ for every vertex $v \in V(G)\backslash\{x\}$, then G contains a path from x of order at least p.*

(7) *If $x \in V(G)$, G is 2-connected and $d_{G-x}(v) \geqslant \lfloor p/2 \rfloor$ for every vertex $v \in V(G)\backslash\{x\}$, then G contains a path from x of order at least $\min\{\nu(G), p\}$.*

Proof. The assertions (1), (2) and (3) are results of Dirac [4], Erdős and Gallai [6], respectively. Now we prove the other assertions.

(4) Let $G' = G + xy$. Since every two nonadjacent vertices of G' contain one with degree at least $\lceil p/2 \rceil$, by Fan's theorem [7], G' contains a cycle C with order at least $\min\{\nu(G), p\}$. If C does not contain the added edge xy, then C is a cycle of G and G contains a path from x of order at least $\min\{\nu(G), p\}$; if C contains the added edge xy, then $P = C - xy$ is a path of G from x of order at least $\min\{\nu(G), p\}$.

(5) We add a new vertex x and join x to every vertex of G. We denote the resulting graph as G'. Thus every vertex in $V(G')$ has degree at least $\lfloor p/2 \rfloor + 1 = \lceil (p+1)/2 \rceil$. By (1), G' contains a cycle C of order at least $\min\{\nu(G'), p+1\}$, and $P = C - x$ is a path of G of order at least $\min\{\nu(G), p\}$.

(6) Let H be a component of $G-x$, and let x' be a neighbor of x in H. Note that every vertex in H has degree at least $p-2$ in H. By (2), H contains a path P from x' of order at least $p-1$. Thus $P'=xx'P$ is a path of G from x of order at least p.

(7) Let $G'=G-x$. If G' contains a vertex with degree 1, then $p\leqslant 3$ and the assertion is trivially true. Now we assume that $\delta(G')\geqslant 2$.

We first assume that G' is 2-connected. By (1), G' contains a cycle C of order at least $\min\{\nu(G'),p-\mathrm{par}(p)\}$. Let P be a path from x to C, let x' be the end-vertex of P on C, and let x'' be a neighbor of x' on C. Then $P'=P\cup C-x'x''$ (with the obvious meaning) is a path from x of order at least $\min\{\nu(G),p\}$.

Now we assume that G' is separable. Then every end-block of G' is 2-connected. Let B be an end-block of G', and b be the cut-vertex of G' contained in B. Since G is 2-connected, x is adjacent to some vertex, say x', in $B-b$. By (3), B contains a path P from x' to b of order at least $\lfloor p/2\rfloor+1$, and by (2), $G'-(B-b)$ contains a path P' from b of order at least $\lfloor p/2\rfloor+1$. Thus $P''=xx'PbP'$ is a path from x of order at least p. ☐

Lemma 3. *If G is a disconnected graph such that*

(1) *$m\leqslant\nu(G)$; and*

(2) *every component of G has order at most $\lfloor m/2\rfloor$,*

then $\overline{G}$ contains a C_m.

Proof. Let G' be an induced subgraph of G of order m. Clearly every component of G' has order at most $\lfloor m/2\rfloor$. Thus every vertex of G' has degree at least $\lceil m/2\rceil$ in $\overline{G'}$. By Lemma 2, $\overline{G'}$ contains a C_m. ☐

Lemma 4. *Let G be a graph.*

(1) *If $n\leqslant\nu(G)\leqslant\lfloor 3n/2\rfloor-2$ and G contains no P_n, then $\overline{G}$ contains a path of order $2\nu(G)+3-2n$.*

(2) *If $\nu(G)\geqslant\lfloor 3n/2\rfloor-1$ and G contains no P_n, then $\overline{G}$ contains a path of order $\nu(G)+1-\lfloor n/2\rfloor$.*

(3) *If $n\geqslant 4$ is even, $\nu(G)\geqslant 3n/2-1$, and G contains no C_n then $\overline{G}$ contains a path of order $\nu(G)+1-n/2$.*

Proof. The lemma can be deduced by Theorems 1 and 2. ☐

Lemma 5. *Let G_1 and G_2 be two disjoint graphs. If*

(1) *$\overline{G_1}$ contains a path of order $p\geqslant 2$; and*

(2) *$m\leqslant\min\{2\nu(G_1),\nu(G_1)+\nu(G_2),p+2\nu(G_2)-1\}$,*

then $\overline{G_1\cup G_2}$ contains a C_m.

Proof. We first assume that $\nu(G_2)\geqslant\lfloor m/2\rfloor$. If m is even, then $\nu(G_1)\geqslant m/2$ and $\nu(G_2)\geqslant m/2$. Let $x_1,x_2,\ldots,x_k$ be $k=m/2$ vertices in G_1, and let $y_1,y_2,\ldots,y_k$ be k vertices in G_2. Then $C=x_1y_1x_2y_2\cdots x_ky_kx_1$ is a C_m in $\overline{G_1\cup G_2}$. If m is odd, then then $\nu(G_1)\geqslant(m+1)/2$ and $\nu(G_2)\geqslant(m-1)/2$. Note that G_1 has two nonadjacent vertices. Let $x_1,x_2,\ldots,x_k$ be $k=(m+1)/2$ vertices in G_1 such that $x_1x_k\notin E(G_1)$, and let $y_1, y_2, \ldots, y_{k-1}$ be $k-1$ vertices in G_2. Then $C = x_1y_1x_2y_2\cdots x_{k-1}y_{k-1}x_kx_1$ is a C_m in $\overline{G_1 \cup G_2}$.

Now we assume that $\nu(G_2) \leqslant \lfloor m/2 \rfloor - 1$. Let $V(G_2) = \{y_1, y_2, \cdots, y_k\}$, where $k = \nu(G_2)$. Since $2 \leqslant m+1-2k \leqslant p$, $\overline{G_1}$ contains a path P of order $m+1-2k$. Let s, t be the two end-vertices of P. Note that $\nu(G_1) - \nu(P) \geqslant m-k-m-1+2k = k-1$. Let $x_1, x_2, \ldots, x_{k-1}$ be $k-1$ vertices in $V(G_1 - P)$. Then $C = sy_1x_1y_2x_2\cdots x_{k-1}y_ktP$ is a C_m in $\overline{G_1 \cup G_2}$. □

Lemma 6. *Suppose $m \geqslant 2n+1$. Let G be a disconnected graph containing no P_n. If*

(1) *$m \leqslant \nu(G)$; and*

(2) *the order sum of every $\omega(G)-1$ components in G is at least $m + \lfloor n/2 \rfloor - \nu(G)$,*

then $\overline{G}$ contains a C_m.

Proof. If every component of G has order at most $\lfloor m/2 \rfloor$, then we are done by Lemma 3. Now we assume that there is a component H with order at least $\lfloor m/2 \rfloor + 1$.

Let $G_1 = H$, and $G_2 = G - H$. Note that $m \leqslant 2\nu(G_1)$, $m \leqslant \nu(G) = \nu(G_1) + \nu(G_2)$ and $\nu(G_2) \geqslant m + \lfloor n/2 \rfloor - \nu(G)$.

Note that $\nu(G_1) \geqslant \lfloor m/2 \rfloor + 1 \geqslant n$. If $\nu(G_1) \leqslant \lfloor 3n/2 \rfloor - 2$, then by Lemma 4, $\overline{G_1}$ contains a path of order $p = 2\nu(G_1) + 3 - 2n$. Since

$$\begin{aligned} p + 2\nu(G_2) - 1 &= 2\nu(G_1) + 3 - 2n + 2\nu(G_2) - 1 \\ &= 2\nu(G) + 2 - 2n \\ &\geqslant 2m + 2 - 2n \\ &\geqslant m, \end{aligned}$$

by Lemma 5, $\overline{G}$ contains a C_m. If $\nu(G_1) \geqslant \lfloor 3n/2 \rfloor - 1$, then by Lemma 4, $\overline{G_1}$ contains a path of order $p = \nu(G_1) + 1 - \lfloor n/2 \rfloor$. Since

$$\begin{aligned} p + 2\nu(G_2) - 1 &= \nu(G_1) + 1 - \left\lfloor \frac{n}{2} \right\rfloor + 2\nu(G_2) - 1 \\ &= \nu(G) + \nu(G_2) - \left\lfloor \frac{n}{2} \right\rfloor \\ &\geqslant \nu(G) + m + \left\lfloor \frac{n}{2} \right\rfloor - \nu(G) - \left\lfloor \frac{n}{2} \right\rfloor \\ &= m, \end{aligned}$$

by Lemma 5, $\overline{G}$ contains a C_m. □

Lemma 7. *Let G be a graph, X an independent set of G, $R = G - X$. If*

(1) *$|X| \geqslant 3$;*

(2) *every component of R is joined to at most one vertex in X;*

(3) *$\overline{R}$ contains a path of order $p \geqslant 2$; and*

(4) *$m \leqslant \min\{\nu(G), p + 2|X| - 3\}$,*

then $\overline{G}$ contains a C_m.

Proof. Let P be a path in $\overline{R}$ with the largest order. Clearly $\nu(P) \geqslant p$.

If $\nu(P) \geqslant m-1$, then let P' be a subpath of P of order $m-1$. Let s, t be the two end-vertices of P'. Since each of s and t is adjacent to at most one vertex in X and $|X| \geqslant 3$, there is a vertex x in X nonadjacent to both s and t. Thus $C = sxtP'$ is a C_m in $\overline{G}$. Now we assume that $\nu(P) \leqslant m-2$.

Let s, t be the two end-vertices of P. If P contains all vertices in R, then $\nu(P) = \nu(R)$. Let x be a vertex in X nonadjacent to s, and x' be a vertex in $X \backslash \{x\}$ nonadjacent to t. Note that $|X| = \nu(G) - \nu(R) \geqslant m - \nu(P)$. Let $x_1, x_2, \ldots, x_k$ be $k = m - \nu(P)$ vertices in X such that $x_1 = x$ and $x_k = x'$, then $C = sx_1x_2 \cdots x_k tP$ is a C_m in $\overline{G}$. Now we assume that $V(R) \backslash V(P) \neq \emptyset$.

Let $U = V(R - P)$. Note that each of s, t is adjacent to every vertex in U, and this implies that $U \cup \{s, t\}$ is contained in a component of R. Thus $U \cup \{s, t\}$ is joined to at most one vertex in X. Let y be the vertex in X that is joined to $U \cup \{s, t\}$. If such a vertex does not exist, then let y be any one vertex in X.

Note that $m - \nu(P) \leqslant m - p \leqslant 2|X| - 3$. If $m - \nu(P)$ is odd, then $|X| \geqslant (m - \nu(P) + 1)/2 + 1$. Let $x_1, x_2, \ldots, x_k$ be $k = (m - \nu(P) + 1)/2$ vertices in $X \backslash \{y\}$, and let $u_1, \ldots, u_{k-1}$ be $k-1$ vertices in $U \cup X \backslash \{x_1, x_2, \ldots, x_k\}$. Then $C = sx_1u_1x_2u_2 \cdots x_{k-1}u_{k-1}x_k tP$ is a C_m in $\overline{G}$. If $m - \nu(P)$ is even, then $m - \nu(P) \leqslant 2|X| - 4$ and $|X| \geqslant (m - \nu(P))/2 + 2$. Let $x_1, x_2, \ldots, x_k$ be $k = (m - \nu(P))/2 + 1$ vertices in $X \backslash \{y\}$, and let $u_1, \ldots, u_{k-2}$ be $k-2$ vertices in $U \cup X \backslash \{x_1, x_2, \ldots, x_k\}$. Then $C = sx_1u_1x_2u_2 \cdots x_{k-2}u_{k-2}\, x_{k-1}x_k tP$ is a C_m in $\overline{G}$. □

Lemma 8. *Let G be a graph, X_1, X_2 two independent sets of G (possibly joint), $X = X_1 \cup X_2$, $R = G - X$. If*

(1) $|X_1| = |X_2| \geqslant 3$, $|X_1 \backslash X_2| = |X_2 \backslash X_1| \geqslant 2$;

(2) *every component of R is joined to at most one vertex in X_i, $i = 1, 2$;*

(3) *$\overline{R}$ contains a path of order $p \geqslant 2$; and*

(4) $m \leqslant \min\{\nu(G), p + 2|X| - 5\}$,

then $\overline{G}$ contains a C_m.

Proof. We first define an *adjustable segment* of a cycle C. If $X_1 \cap X_2 = \emptyset$, then let $x_1, x_1', x_1'' \in X_1$, $x_2, x_2', x_2'' \in X_2$ and $u \in V(R)$, and we call a subpath A an adjustable segment of C with the center u if one of the following is true:

(1) $A = x_1x_1'ux_2'x_2$ with $x_1'', x_2'' \notin V(C)$;

(2) $A = x_1x_1'x_1''ux_2'x_2$ with $x_2'' \notin V(C)$;

(3) $A = x_1x_1'ux_2''x_2'x_2$ with $x_1'' \notin V(C)$; or

(4) $A = x_1x_1'x_1''ux_2''x_2'x_2$.

If $X_1 \cap X_2 \neq \emptyset$, then let $x_1, x_1' \in X_1 \backslash X_2$, $x_2, x_2' \in X_2 \backslash X_1$ and $x \in X_1 \cap X_2$, and we call a subpath A an adjustable segment of C with the center x if one of the following is true:

(1) $A = x_1xx_2$ with $x_1', x_2' \notin V(C)$;

(2) $A = x_1x_1'xx_2$ with $x_2' \notin V(C)$;

(3) $A = x_1xx_2'x_2$ with $x_1' \notin V(C)$; or

(4) $A = x_1x_1'xx_2'x_2$.

If $X_1 \cap X_2 \neq \emptyset$, then let P be a path in $\overline{R}$ with the largest order; if $X_1 \cap X_2 = \emptyset$, then let P be a non-Hamilton path in $\overline{R}$ with the largest order.

If $\nu(P) \geqslant m-5$, then let P' be a subpath of P of order $m-5$ and s,t be the two end-vertices of P'. If $X_1 \cap X_2 \neq \emptyset$, then let x be a vertex in $X_1 \cap X_2$, x_1 a vertex in $X_1 \backslash X_2$ nonadjacent to s, x_1' a vertex in $X_1 \backslash (X_2 \cup \{x_1\})$, x_2 a vertex in $X_2 \backslash X_1$ nonadjacent to t and x_2' a vertex in $X_2 \backslash (X_1 \cup \{x_2\})$. Then $C = sx_1x_1'xx_2'x_2tP'$ is a C_m in $\overline{G}$. If $X_1 \cap X_2 = \emptyset$, then let u be a vertex in $V(R-P')$, x_1 a vertex in X_1 nonadjacent to s, x_1' a vertex in $X_1 \backslash \{x_1\}$ nonadjacent to u, x_2 a vertex in X_2 nonadjacent to t and x_2' a vertex in $X_2 \backslash \{x_2\}$ nonadjacent to u. Then $C = sx_1x_1'ux_2'x_2tP'$ is a C_m in $\overline{G}$.

Now we assume that $\nu(P) \leqslant m-6$. By a similar argument in the analysis above, we can get a cycle C in $\overline{G}$ of order at least $\nu(P)+5$ such that

(a) C contains P as a subpath;

(b) C contains an adjustable segment A (with end-vertices x_1, x_2);

(c) every edge of C has a vertex in R, unless it is an edge in A.

Now we choose a cycle C in $\overline{G}$ satisfying (a)(b)(c) with order as large as possible but at most m. If $\nu(C) = m$, then we are done. So we assume that $\nu(C) \leqslant m-1$. We claim that $V(R) \subset V(C)$. Assume the contrary. Let v be a vertex in $U = V(R) \backslash V(C)$.

If ($X_1 \cap X_2 = \emptyset$ and) $A = x_1x_1'ux_2'x_2$ with $x_1'' \in X_1 \backslash V(C)$, $x_2'' \in X_2 \backslash V(C)$, then $C' = C - x_1x_1' \cup x_1x_1''x_1'$ is a required cycle with order $\nu(C)+1$, a contradiction. Using the same analysis, we can conclude that $A = x_1x_1'x_1''ux_2''x_2'x_2$ (if $X_1 \cap X_2 = \emptyset$) or $A = x_1x_1'xx_2'x_2$ (if $X_1 \cap X_2 \neq \emptyset$).

If $X_1 \cap X_2 \neq \emptyset$, then P is a longest path of $\overline{R}$; if $X_1 \cap X_2 = \emptyset$, then noting that $u, v \in V(R-P)$, P is a longest path of $\overline{R}$ as well. Thus $\nu(P) \geqslant p$ and $U \cup \{s,t\}$ is contained in a component of R. If there is a vertex y in X that is joined to $U \cup \{s,t\}$, then we use y instead of the vertex x_1', x_2' or x in C, for the case $y \in X_1 \backslash X_2$, $y \in X_2 \backslash X_1$, or $y \in X_1 \cap X_2$, respectively. Thus we assume that every vertex in $X \backslash \{x_1', x_2', x\}$ is not joined to $U \cup \{s,t\}$.

If every vertex in X is in $V(C)$, then noting that there are at most 5 vertices in X each of which has a successor on C such that it is not in U, we have

$$\nu(C) \geqslant \nu(P) + |X| + (|X|-5) \geqslant p + 2|X| - 5 \geqslant m,$$

a contradiction. So we assume that there is a vertex x' in X which is not in C. Let v' be the predecessor of x_1 in C. Clearly $v' \in U \cup \{s,t\}$. Then $C' = v'x'vx_1x_1''\overrightarrow{C}[x_1'', v']$ (if $X_1 \cap X_2 = \emptyset$) or $C' = v'x'vx_1x\overrightarrow{C}[x, v']$ (if $X_1 \cap X_2 \neq \emptyset$) is a required cycle of order $\nu(C)+1$, a contradiction. Thus as we claimed, every vertex in R is in C. This implies that C is a cycle in $\overline{G}$ satisfying

(d) there is an edge $x_1x_1' \in E(C)$ such that $x_1, x_1' \in X_1$;

(e) there is an edge $x_2x_2' \in E(C)$ such that $x_2, x_2' \in X_2$;

(f) $V(R) \subset V(C)$.

Now we choose a cycle C in $\overline{G}$ satisfying (d)(e)(f) with order as large as possible but at most m. If $\nu(C) = m$, then we are done. So we assume that $\nu(C) \leqslant m - 1$. If every vertex in X is in C, then

$$\nu(C) = \nu(R) + |X| \geqslant m,$$

a contradiction. So we assume that there is a vertex x' in X which is not in C. If $x' \in X_1$, then $C' = C - x_1x_1' \cup x_1x'x_1'$ is a required cycle of order $\nu(C) + 1$; if $x' \in X_2$, then $C' = C - x_2x_2' \cup x_2x'x_2'$ is a required cycle of order $\nu(C) + 1$, a contradiction.

Thus the lemma holds. □

The proof of the next lemma is similar as the proof of Lemma 8, but more involved.

Lemma 9. *Let G be a graph, R be an induced subgraph of G, X_1, X_2 two independent sets of $G - R$ (possibly joint), $X = X_1 \cup X_2$. If*

(1) $|X_1| = |X_2| \geqslant 3$, $|X_1 \backslash X_2| = |X_2 \backslash X_1| \geqslant 2$;

(2) *every component of R has order at least 2;*

(3) *every component of R is joined to at most one vertex in X_i, $i = 1, 2$;*

(4) *for any component H of R, there are at least q vertices in $G - R$ each of which is either in X or not joined to H;*

(5) *$\overline{R}$ contains a path of order $p \geqslant 2$; and*

(6) $m \leqslant \min\{\lceil 3\nu(R)/2 \rceil + 4, \nu(R) + q - 1, p + 2q - 5\}$,

then $\overline{G}$ contains a C_m.

Proof. We use the concept of an adjustable segment defined in Lemma 8. If $X_1 \cap X_2 \neq \emptyset$, then let P be a path in $\overline{R}$ with the largest order; if $X_1 \cap X_2 = \emptyset$, then let P be a non-Hamilton path in $\overline{R}$ with the largest order.

If $\nu(P) \geqslant m - 5$, then similar as in Lemma 8, we can find a C_m in $\overline{G}$. Thus we assume that $\nu(P) \leqslant m - 6$. By a similar argument as in Lemma 8, we can get a cycle C in $\overline{G}$ of order at least $\nu(P) + 5$ such that

(a) C contains P as a subpath;

(b) C contains an adjustable segment A (with end-vertices x_1, x_2);

(c) every edge of C has a vertex in R, unless it is an edge in A.

Now we choose a cycle C in $\overline{G}$ satisfying (a)(b)(c) with order as large as possible but at most m. If $\nu(C) = m$, then we are done. So we assume that $\nu(C) \leqslant m - 1$. We claim that $V(R) \subset V(C)$. Assume the contrary. Let v be a vertex in $U = V(R - C)$.

Using the same analysis in Lemma 8, we can conclude that $A = x_1x_1'x_1''ux_2''x_2'x_2$ (if $X_1 \cap X_2 = \emptyset$) or $A = x_1x_1'xx_2'x_2$ (if $X_1 \cap X_2 \neq \emptyset$) and P is a longest path of $\overline{R}$. Thus $\nu(P) \geqslant p$ and $U \cup \{s, t\}$ is contained in a common component of R. Furthermore, we can assume that every vertex in $X \backslash \{x_1', x_2', x\}$ is not joined to $U \cup \{s, t\}$.

Let W be the union of X and the set of vertices in $G - R$ that are not joined to $U \cup \{s, t\}$. Then $|W| \geqslant q$. If every vertex in W is in $V(C)$, then noting that there are at most 5 vertices in W each of which has a successor on C such that it is not in U, we have

$$\nu(C) \geqslant \nu(P) + |W| + (|W| - 5) \geqslant p + 2q - 5 \geqslant m,$$

a contradiction. So we assume that there is a vertex w in W that is not in $V(C)$. Let v' be the predecessor of x_1 in C. Clearly $v' \in U \cup \{s,t\}$. Then $C' = v'wvx_1x_1''\overrightarrow{C}[x_1'', v']$ (if $X_1 \cap X_2 = \emptyset$) or $C' = v'wvx_1x\overrightarrow{C}[x, v']$ (if $X_1 \cap X_2 \neq \emptyset$) is a required cycle of order $\nu(C)+1$, a contradiction. Thus as we claimed, every vertex in R is in C. This implies C satisfies (b)(c) and

(d) $V(R) \subset V(C)$.

Now we choose a cycle C in $\overline{G}$ satisfying (b)(c)(d) with order as large as possible but at most m. If $\nu(C) = m$, then we are done. So we assume that $\nu(C) \leqslant m-1$. By a similar argument as above, we can conclude that $A = x_1x_1'x_1''ux_2''x_2'x_2$ (if $X_1 \cap X_2 = \emptyset$) or $A = x_1x_1'xx_2'x_2$ (if $X_1 \cap X_2 \neq \emptyset$).

We claim that there are two vertices u_1, u_2 in C such that u_1, u_2 are in a common component of R and $u_1^+, u_2^+ \in V(R)$. Assume the contrary. Note that every component of R has at least 2 vertices, there is at most one vertex in a component, such that it has a successor on C in R, and there are 4 vertices of C (in the adjusted segment) each of which is not a successor of some vertex in R. Thus

$$\nu(C) \geqslant \nu(R) + \left\lceil \frac{\nu(R)}{2} \right\rceil + 4 = \left\lceil \frac{3\nu(R)}{2} \right\rceil + 4 \geqslant m,$$

a contradiction. Thus as we claimed, there are two edges $u_1u_1^+, u_2u_2^+$ such that u_1, u_2 are in a common component of R and $u_1^+, u_2^+ \in V(R)$.

If there is a vertex y in $X \backslash V(C)$ that is joined to $\{u_1, u_2\}$, then we use y instead of the vertex x_1', x_2' or x in C. Thus we assume that every vertex in $X \backslash V(C)$ is not joined to $\{u_1, u_2\}$. Let W be the union of X and the set of vertices in $G - R$ that are not joined to $\{u_1, u_2\}$. Then $|W| \geqslant q$. If every vertex in W is in C, then

$$\nu(C) \geqslant \nu(R) + |W| \geqslant \nu(R) + q \geqslant m,$$

a contradiction. Thus we assume that there is a vertex w in W that is not in C.

If u_1^+, u_2^+ are in distinct components of R, then $C' = u_1wu_2\overleftarrow{C}[u_2, u_1^+]u_1^+u_2^+\overrightarrow{C}[u_2^+, u_1]$ is a required cycle with order $\nu(C)+1$. Now we assume that u_1^+, u_2^+ are in a common component of R.

If there is a vertex y' in $X \backslash \{w\}$ that is joined to $\{u_1^+, u_2^+\}$, then we use y' instead of the vertex x_1', x_2' or x in C. Thus we assume that every vertex in $X \backslash V(C) \backslash \{w\}$ is not joined to $\{u_1^+, u_2^+\}$.

Let W' be the union of X and the set of vertices in $G - R$ that are not joined to $\{u_1^+, u_2^+\}$. Then $|W'| \geqslant q$. If every vertex in $W' \backslash \{w\}$ is in C, then

$$\nu(C) \geqslant \nu(R) + |W'| - 1 \geqslant \nu(R) + q - 1 \geqslant m,$$

a contradiction. Thus we assume that there is a vertex w' in $W \backslash \{w\}$ that is not in C. Let $C' = u_1wu_2\overleftarrow{C}[u_2, u_1^+]u_1^+w'u_2^+\overrightarrow{C}[u_2^+, u_1]$. Then $C'' = C' - x_1x_1'x_1'' \cup x_1x_1''$ (if $X_1 \cap X_2 = \emptyset$) or $C'' = C' - x_1x_1'x \cup x_1x$ (if $X_1 \cap X_2 \neq \emptyset$) is a required cycle of order $\nu(C)+1$, a contradiction. □

3 Proof of Theorem 5

The case of $n = 2$ is trivial. For the case of $n = 3$ or $n = 4$, we are done by Theorem 6. Thus in the following we will assume that $n \geqslant 5$.

By Lemma 1, $t(n, m) = \min\{t : t \notin \mathcal{L}[t - m + 1, n - 1]\}$. Let $t = t(n, m)$. Thus $t - 1 \in \mathcal{L}[t - m, n - 1]$. Let $t - 1 = \sum_{i=1}^{k} t_i$, where $t_i \in [t - m, n - 1]$, $1 \leqslant i \leqslant k$. Let G be a graph with k components $H_1, \ldots, H_k$ such that H_i is a clique on t_i vertices. Note that G contains no P_n since every component of G has less than n vertices; and $\overline{G}$ contains no W_m since every vertex of G has less than m nonadjacent vertices. Thus G is a graph on $t - 1$ vertices such that G contains no P_n and $\overline{G}$ contains no W_m. This implies that $R(P_n, W_m) \geqslant t$.

Now we will prove that $R(P_n, W_m) \leqslant t$. Assume not. Let G be a graph on t vertices such that G contains no P_n and $\overline{G}$ contains no W_m.

Let $s = m + n - t$ (i.e., $\nu(G) = m + n - s$).

Claim 1. $1 \leqslant s \leqslant \lfloor (n + 5)/4 \rfloor$.

Proof. Let $t' = m+n-1$. Since $t'-m+1 = n$, $[t'-m+1, n-1] = \emptyset$, and $t' \notin \mathcal{L}(\emptyset) = \{0\}$, we have $t \leqslant t' = m + n - 1$. This implies that $s \geqslant 1$ (and $t - m + 1 \leqslant n$).

Now we prove that $s \leqslant (n + 5)/4$. By Lemma 1, $t \notin \mathcal{L}[t - m + 1, n - 1]$. Thus $t \notin [k(t - m + 1), k(n - 1)]$, for every k. That is, $t \in [k(n - 1) + 1, (k + 1)(t - m + 1) - 1]$, for some k, which implies

$$t \geqslant k(n - 1) + 1 \text{ and } t \geqslant \frac{k + 1}{k} m - 1,$$

for some k.

If $k \leqslant 2$, then we have $t \leqslant 3(t - m + 1) - 1$, and

$$t \geqslant \frac{k + 1}{k} m - 1 \geqslant \frac{3}{2} m - 1 > 3n - 1 \geqslant 3(t - m + 1) - 1,$$

a contradiction. Thus we assume that $k \geqslant 3$.

If $m \leqslant (k^2 n - k^2 + 2k)/(k + 1)$, then

$$\begin{aligned} s = m + n - t &\leqslant \frac{k^2 n - k^2 + 2k}{k + 1} + n - (k(n - 1) + 1) \\ &= \frac{n + 2k - 1}{k + 1} \leqslant \frac{n + 5}{4}. \end{aligned}$$

If $m > (k^2 n - k^2 + 2k)/(k + 1)$, then

$$\begin{aligned} s = m + n - t &\leqslant m + n - (\frac{k + 1}{k} m - 1) \\ &= n - \frac{m}{k} + 1 < n - \frac{k^2 n - k^2 + 2k}{k(k + 1)} + 1 \\ &= \frac{n + 2k - 1}{k + 1} \leqslant \frac{n + 5}{4}. \end{aligned}$$

Thus the claim holds. □

We list the possible values of s for $n \leqslant 16$.

n	5	6	7	8	9	10	11	12	13	14	15	16
$s \leqslant$	2	2	3	3	3	3	4	4	4	4	5	5

Table 1: The possible values of s for $n \leqslant 16$.

Claim 2. *Let v be an arbitrary vertex of G and G' be an induced subgraph of $G-v-N(v)$. Then $\overline{G'}$ contains no C_m.*

Proof. Otherwise, noting that v is nonadjacent to every vertex in the C_m, there will be a W_m in $\overline{G}$ (with the hub v). □

Claim 3. $\delta(G) \geqslant \lceil n/2 \rceil - s + 1$.

Proof. Assume the contrary. Let v be a vertex of G with $d(v) \leqslant \lceil n/2 \rceil - s$. Then $G' = G - v - N(v)$ has at least $m + \lfloor n/2 \rfloor - 1$ vertices. Since G' contains no P_n, by Theorem 2, $\overline{G'}$ contains a C_m (note that $m \geqslant 2n+1$), a contradiction to Claim 2. □

From Claims 1 and 3, one can see that $\delta(G) \geqslant 2$ (when $n \geqslant 5$).

Case 1. G is disconnected.

Case 1.1. Every component of G has order less than n.

Let H_i, $1 \leqslant i \leqslant k = \omega(G)$, be the components of G. Since $t \notin \mathcal{L}[t-m+1, n-1]$, there is a component, say H_1, with order at most $t-m$. Thus $\sum_{i=2}^{k} \nu(H_i) \geqslant m$. Since $\nu(H_i) \leqslant n-1 \leqslant \lfloor m/2 \rfloor$. By Lemma 3, $\overline{G-H_1}$ contains a C_m, a contradiction.

Case 1.2. There is a component of G with order at least n.

Let H be a component of G with the largest order. Note that $\nu(H) \geqslant n$. If every vertex of H has degree at least $\lfloor n/2 \rfloor$, then by Lemma 2, H contains a P_n, a contradiction. Thus there is a vertex v in H with $d(v) \leqslant \lfloor n/2 \rfloor - 1$. Let $G' = G - v - N(v)$. Then

$$\begin{aligned} \nu(G') &= \nu(G) - 1 - d(v) \\ &\geqslant m + n - s - 1 - \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ &= m + \left\lceil \frac{n}{2} \right\rceil - s \geqslant m. \end{aligned}$$

Since $\nu(H) \geqslant n > 1 + d(v)$, G' is disconnected. Let $\mathcal{H}$ be the union of $\omega(G') - 1$ components of G'. We will prove that $\nu(\mathcal{H}) \geqslant m + \lfloor n/2 \rfloor - \nu(G')$.

Let H' be a component of G other than H. If $H' \not\subset \mathcal{H}$, then $\nu(\mathcal{H}) = \nu(G') - \nu(H') \geqslant \nu(G') - \lfloor \nu(G)/2 \rfloor$, and

$$\nu(\mathcal{H}) + \nu(G') - m - \left\lfloor \frac{n}{2} \right\rfloor \geqslant \nu(G') - \left\lfloor \frac{\nu(G)}{2} \right\rfloor + \nu(G') - m - \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned}
&\geqslant 2\left(m+\left\lceil\frac{n}{2}\right\rceil-s\right)-\left\lfloor\frac{m+n-s}{2}\right\rfloor-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant \left\lceil 2\left(m+\frac{n}{2}-s\right)-\frac{m+n-s}{2}-m-\frac{n}{2}\right\rceil\\
&=\left\lceil\frac{m-3s}{2}\right\rceil \geqslant \left\lceil\frac{2n+1-3s}{2}\right\rceil \geqslant 0.
\end{aligned}$$

If $H' \subset \mathcal{H}$, then $\nu(\mathcal{H}) \geqslant \nu(H') \geqslant \delta(G)+1$, and

$$\begin{aligned}
\nu(\mathcal{H})+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor &\geqslant \delta(G)+1+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant \left\lceil\frac{n}{2}\right\rceil-s+2+m+\left\lceil\frac{n}{2}\right\rceil-s-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&=\left\lceil\frac{n}{2}\right\rceil+\mathrm{par}(n)+2-2s \geqslant 0.
\end{aligned}$$

Now by Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2. G has connectivity 1.

Note that $\delta(G) \geqslant 2$. Every end-block of G is 2-connected.

Case 2.1. Every end-block of G has order at most $\lceil m/2 \rceil$.

Claim 4. *Let G' be a disconnected subgraph of G. If*
(1) *$\nu(G') \geqslant m$; and*
(2) *there are two components of G', each of which is an end-block removing a cut-vertex of G contained in the end-block,*

then the order sum of every $\omega(G')-1$ components in G' is at least $m+\lfloor n/2 \rfloor-\nu(G')$.

Proof. Let $B-x$ and $B'-x'$ be two components of G', where B, B' are two end-blocks of G and x, x' are two cut-vertices of G contained in B and B', respectively.

Let $\mathcal{H}$ be the union of any $\omega(G')-1$ components of G'. We first assume that $\mathcal{H}$ does not contain $B-x$ or $B'-x'$. Without loss of generality, we assume that $\mathcal{H}$ does not contain $B-x$. Then $\nu(\mathcal{H})=\nu(G')-\nu(B-x) \geqslant \nu(G')-\lceil m/2 \rceil+1$, and

$$\begin{aligned}
\nu(\mathcal{H})+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor &\geqslant \nu(G')-\left\lceil\frac{m}{2}\right\rceil+1+\nu(G')-m-\left\lceil\frac{n}{2}\right\rceil\\
&\geqslant 2m-\left\lceil\frac{m}{2}\right\rceil+1-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&=\left\lfloor\frac{m}{2}\right\rfloor-\left\lfloor\frac{n}{2}\right\rfloor+1 \geqslant 0.
\end{aligned}$$

Now we assume that both $B-x$ and $B'-x' \subset \mathcal{H}$. Then $\nu(\mathcal{H}) \geqslant \nu(B-x)+\nu(B'-x') \geqslant 2\delta(G)$, and

$$\nu(\mathcal{H})+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor \geqslant 2\delta(G)+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor$$

$$
\begin{aligned}
&\geqslant 2\left(\left\lceil\frac{n}{2}\right\rceil - s + 1\right) + m - m - \left\lfloor\frac{n}{2}\right\rfloor \\
&= \left\lceil\frac{n}{2}\right\rceil + \mathrm{par}(n) + 2 - 2s \geqslant 0.
\end{aligned}
$$

Thus the claim holds. □

Case 2.1.1. G has only two end-blocks.

Let B and B' be the two end-blocks of G, and let x and x' be the cut-vertices of G contained in B and B', respectively. Note that

$$\nu(G) - \nu(B) - \nu(B') \geqslant m + n - s - 2 \cdot \left\lceil\frac{m}{2}\right\rceil = n - s - \mathrm{par}(m) \geqslant 1.$$

This implies that $V(G)\backslash(V(B) \cup V(B')) \neq \emptyset$.

Note that in this case $G - (B - x) - (B' - x') + xx'$ is 2-connected. If every vertex in $G - B - B'$ has degree at least $2s - \mathrm{par}(n) - 3$, then by Lemma 2, there is a path from x to x' of order at least $2s - \mathrm{par}(n) - 2$. Note that B contains a path from x of order at least $\lceil n/2\rceil - s + 2$, and B' contains a path from x' of order at least $\lceil n/2\rceil - s + 2$. Thus G contains a P_n, a contradiction. This implies that there is a vertex v in $G - B - B'$ with $d(v) \leqslant 2s - \mathrm{par}(n) - 4$.

Let $G' = G - x - x' - v - N(v)$. Then

$$
\begin{aligned}
\nu(G') &\geqslant \nu(G) - 3 - d(v) \\
&\geqslant m + n - s - 3 - 2s + \mathrm{par}(n) + 4 \\
&= m + n + \mathrm{par}(n) + 1 - 3s \geqslant m.
\end{aligned}
$$

By Claim 4, the order sum of every $\omega(G') - 1$ components in G' is at least $m + \lfloor n/2\rfloor - \nu(G')$. By Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2.1.2. G has at least three end-blocks.

Let x and x' be two cut-vertices of G such that the longest path between x and x' in G is as long as possible. Clearly x and x' are both contained in some end-blocks. Let B and B' be two end-blocks of G containing x and x', respectively ($B \neq B'$). Let v be a vertex in $V(B - x) \cup V(B' - x')$ such that $d_{G-x-x'}(v)$ is as small as possible. We assume without loss of generality that $v \in V(B - x)$.

Claim 5.

$$
d_{B-x}(v) \leqslant \begin{cases} \lfloor n/2\rfloor - 2, & \text{if } x = x'; \\ \lceil n/2\rceil - 3, & \text{if } xx' \text{ is a cut-edge of } G; \\ \lfloor n/2\rfloor - 3, & \text{otherwise.} \end{cases}
$$

Proof. We set a parameter a such that $a = 0$ if $x = x'$, 1 if xx' is a cut-edge of G, and 2 otherwise. So there is a path between x and x' of length at least a.

If $\delta(B-x) \geqslant \lfloor(n-a)/2\rfloor - 1$, then $\delta(B'-x') \geqslant \lfloor(n-a)/2\rfloor - 1$. By Lemma 2, B contains a path from x of order at least $\lfloor(n-a)/2\rfloor + 1$ and B' contains a path from x' of order at least $\lfloor(n-a)/2\rfloor + 1$. Thus G contains a path of order at least $n + 1 - \mathrm{par}(n-a) \geqslant n$, a contradiction. Now we obtain that $\delta(B - x) \leqslant \lfloor(n-a)/2\rfloor - 2$. □

Case 2.1.2.1. $x = x'$.

In this case, G has only one cut-vertex x. Let $G' = G - x - v - N(v)$. Then

$$\begin{aligned}\nu(G') &= \nu(G) - 2 - d_{B-x}(v) \\ &\geqslant m + n - s - 2 - \left\lfloor \frac{n}{2} \right\rfloor + 2 \\ &= m + \left\lceil \frac{n}{2} \right\rceil - s \geqslant m.\end{aligned}$$

Note that every end-block of G other than B removing x is a component of G'. By Claim 4 and Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2.1.2.2. xx' is a cut-edge of G.

In this case, G has only two cut-vertices x and x'. Let $G' = G - x - x' - v - N(v)$. Then

$$\begin{aligned}\nu(G') &= \nu(G) - 3 - d_{B-x}(v) \\ &\geqslant m + n - s - 3 - \left\lceil \frac{n}{2} \right\rceil + 3 \\ &= m + \left\lfloor \frac{n}{2} \right\rfloor - s \geqslant m.\end{aligned}$$

Note that every end-block of G other than B removing x or x' is a component of G'. By Claim 4 and Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2.1.2.3. $xx' \notin E(G)$ or xx' is not a cut-edge of G.

Let B'' be an end-block of G other than B and B', and let x'' be the cut-vertex of G contained in B'' (possibly $x'' = x$ or x'). Let $G' = G - x - x' - x'' - v - N(v)$. Then

$$\begin{aligned}\nu(G') &\geqslant \nu(G) - 4 - d_{B-x}(v) \\ &\geqslant m + n - s - 4 - \left\lfloor \frac{n}{2} \right\rfloor + 3 \\ &= m + \left\lceil \frac{n}{2} \right\rceil - s - 1 \geqslant m.\end{aligned}$$

Note that $B' - x'$ and $B'' - x''$ are two components of G'. By Claim 4 and Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2.2. There is an end-block of G with order at least $\lceil m/2 \rceil + 1$.

Let B be an end-block of G with the maximum order, and x be the cut-vertex of G contained in B. Let x' be a cut-vertex of G such that the longest path between x and x' is as long as possible. Clearly x' is contained in some end-blocks. Let B' be an end-block of G containing x' ($B \neq B'$). Let v be a vertex in $B - x$ such that $d_{B-x}(v)$ is as small as possible.

Claim 6.

$$d_{B-x}(v) \leqslant \begin{cases} \lceil (n+2s-\mathrm{par}(n))/4 \rceil - 2, & \text{if } x = x'; \\ \lfloor (n+2s-\mathrm{par}(n))/4 \rfloor - 2, & \text{if } xx' \text{ is a cut-edge of } G; \\ \lceil (n+2s-\mathrm{par}(n))/4 \rceil - 3, & \text{otherwise.} \end{cases}$$

Proof. We set a parameter a such that $a = 0$ if $x = x'$, 1 if xx' is a cut-edge of G, and 2 otherwise. So there is a path between x and x' of length at least a.

By Claim 3 and Lemma 2, B' contains a path from x' of order at least $\lceil n/2 \rceil - s + 2$, and $G - (B - x)$ contains a path from x of order at least $\lceil n/2 \rceil - s + a + 2$.

Note that $\nu(B) \geqslant \lceil m/2 \rceil + 1 \geqslant \lfloor n/2 \rfloor + s - a - 1$. If $\delta(B-x) \geqslant \lfloor (\lfloor n/2 \rfloor + s - a - 1)/2 \rfloor$, then by Lemma 2, B contains a path from x of order at least $\lfloor n/2 \rfloor + s - a - 1$. Thus G contains a P_n, a contradiction. This implies that

$$\delta(B-x) \leqslant \left\lfloor \frac{\lfloor n/2 \rfloor + s - a - 1}{2} \right\rfloor - 1 = \left\lceil \frac{n + 2s - \mathrm{par}(n) - 2a}{4} \right\rceil - 2.$$

Thus the claim holds. □

Note that

$$\begin{aligned} \nu(B - x - v - N(v)) &= \nu(B) - 2 - d_{B-x}(v) \\ &\geqslant \left\lceil \frac{m}{2} \right\rceil + 1 - 2 - \left\lceil \frac{n + 2s - \mathrm{par}(n)}{4} \right\rceil + 2 \\ &\geqslant \left\lceil \frac{m}{2} - \frac{n + 2s + 2}{4} \right\rceil + 1 \\ &\geqslant \left\lceil \frac{3n - 2s}{4} \right\rceil + 1 \geqslant 1. \end{aligned}$$

This implies that $V(B) \backslash (\{x, v\} \cup N(v)) \neq \emptyset$.

Case 2.2.1. $x = x'$.

In this case, G has only one cut-vertex x. Let $G' = G - x - v - N(v)$. Then G' is disconnected and

$$\begin{aligned} \nu(G') &= \nu(G) - 2 - d_{B-x}(v) \\ &\geqslant m + n - s - 2 - \left\lceil \frac{n + 2s - \mathrm{par}(n)}{4} \right\rceil + 2 \\ &= m + \left\lfloor \frac{3n + \mathrm{par}(n) - 6s}{4} \right\rfloor \geqslant m. \end{aligned}$$

Let $\mathcal{H}$ be the union of any $\omega(G') - 1$ components of G'. We will prove that $\nu(\mathcal{H}) \geqslant m + \lfloor n/2 \rfloor - \nu(G')$.

If $B' - x \not\subset \mathcal{H}$, then $\nu(\mathcal{H}) = \nu(G') - \nu(B' - x) \geqslant \nu(G') - \lfloor(\nu(G) - 1)/2\rfloor$, and

$$\begin{aligned}
\nu(\mathcal{H}) + \nu(G') - m - \left\lfloor \frac{n}{2} \right\rfloor &\geqslant \nu(G') - \left\lfloor \frac{\nu(G) - 1}{2} \right\rfloor + \nu(G') - m - \left\lfloor \frac{n}{2} \right\rfloor \\
&\geqslant 2\left(m + \left\lfloor \frac{3n + \mathrm{par}(n) - 6s}{4} \right\rfloor\right) - \left\lfloor \frac{m + n - s - 1}{2} \right\rfloor - m - \left\lfloor \frac{n}{2} \right\rfloor \\
&\geqslant \left\lceil m + 2 \cdot \frac{3n - 6s - 2}{4} - \frac{m + n - s - 1}{2} - \frac{n}{2} \right\rceil \\
&= \left\lceil \frac{m + n - 5s - 1}{2} \right\rceil \geqslant \left\lceil \frac{3n - 5s}{2} \right\rceil \geqslant 0.
\end{aligned}$$

Now we assume that $B' - x \subset \mathcal{H}$. In this case $\nu(\mathcal{H}) \geqslant \nu(B' - x) \geqslant \delta(G)$, and

$$\begin{aligned}
\nu(\mathcal{H}) + \nu(G') - m - \left\lfloor \frac{n}{2} \right\rfloor &\geqslant \delta(G) + \nu(G') - m - \left\lfloor \frac{n}{2} \right\rfloor \\
&\geqslant \left\lceil \frac{n}{2} \right\rceil - s + 1 + m + \left\lfloor \frac{3n + \mathrm{par}(n) - 6s}{4} \right\rfloor - m - \left\lfloor \frac{n}{2} \right\rfloor \\
&\geqslant \left\lfloor \frac{3n + 5\mathrm{par}(n) + 4 - 10s}{4} \right\rfloor.
\end{aligned}$$

Note that $3n + 5\mathrm{par}(n) + 4 - 10s \geqslant 0$ unless $n = 8$ and $s = 3$.
Petty Case. $n = 8$ and $s = 3$.

In this case $\nu(B' - x) \geqslant 2$ and $d_{B-x}(v) \leqslant 2$. If $\nu(\mathcal{H}) \geqslant 3$, or if $d_{B-x}(v) = 1$, then it is easy to see that $\nu(\mathcal{H}) \geqslant m + \lfloor n/2\rfloor - \nu(G')$. Now we assume that $\nu(B' - x) = \nu(\mathcal{H}) = 2$ and $d_{B-x}(v) = 2$. This implies that B' is a triangle, there are only two blocks B, B', and every vertex in $B - x$ has degree at least 2 in $B - x$. If $B - x$ has a cut-vertex, then noting that every end-block of $B - x$ has at least three vertices, B contains a path from x of order at least 6, and G contains a P_8, a contradiction. So we assume that $B - x$ is 2-connected.

Note that $B - x$ contains a cycle of order at least 4. Let C be a longest cycle of $B - x$. If $\nu(C) \geqslant 5$, then there is also an path from x in B of order at least 6, a contradiction. Thus we assume that $\nu(C) = 4$. If there is a component of $B - x - C$ with order at least 2, or if there is a vertex in $B - x - C$ adjacent to two consecutive vertices on C, then it is easy to find a cycle longer than C. Thus $B - x - C$ consists of isolated vertices and every vertex is adjacent to two nonconsecutive vertices on C. If there are two vertices in $B - x - C$ adjacent to different vertices on C, we can also find a longer cycle. Thus all the vertices of $B - x - C$ have the same neighbors on C. This implies that $B - x - v - N(v)$ is disconnected and then $\nu(\mathcal{H}) \geqslant \nu(B' - x) + 1 = 3$. Thus we also have $\nu(\mathcal{H}) \geqslant m + \lfloor n/2\rfloor - \nu(G')$.

By Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2.2.2. xx' is a cut-edge of G and there is only one end-block containing x'.

Let $G' = G - x - v - N(v)$. Then B' is a component of G', and

$$\nu(G') = \nu(G) - 2 - d_{B-x}(v)$$

$$
\begin{aligned}
&\geqslant m+n-s-2-\left\lfloor \frac{n+2s-\mathrm{par}(n)}{4}\right\rfloor+2\\
&= m+\left\lceil \frac{3n+\mathrm{par}(n)-6s}{4}\right\rceil \geqslant m.
\end{aligned}
$$

Now let $\mathcal{H}$ be the union of any $\omega(G')-1$ components of G'. If $B' \not\subset \mathcal{H}$, then $\nu(\mathcal{H}) = \nu(G')-\nu(B') \geqslant \nu(G')-\lfloor \nu(G)/2\rfloor$, and

$$
\begin{aligned}
\nu(\mathcal{H})+\nu(G')-\left\lfloor\frac{n}{2}\right\rfloor-m &\geqslant \nu(G')-\left\lfloor\frac{\nu(G)}{2}\right\rfloor+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant 2\left(m+\left\lceil\frac{3n+\mathrm{par}(n)-6s}{4}\right\rceil\right)-\left\lfloor\frac{m+n-s}{2}\right\rfloor-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant \left\lceil m+2\cdot\frac{3n-6s}{4}-\frac{m+n-s}{2}-\frac{n}{2}\right\rceil\\
&= \left\lceil\frac{m+n-5s}{2}\right\rceil \geqslant \left\lceil\frac{3n+1-5s}{2}\right\rceil \geqslant 0.
\end{aligned}
$$

If $B' \subset \mathcal{H}$, then $\nu(\mathcal{H}) \geqslant \nu(B') \geqslant \delta(G)+1$, and

$$
\begin{aligned}
\nu(\mathcal{H})+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor &\geqslant \delta(G)+1+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant \left\lceil\frac{n}{2}\right\rceil-s+2+m+\left\lceil\frac{3n+\mathrm{par}(n)-6s}{4}\right\rceil-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&= \left\lceil\frac{3n+5\mathrm{par}(n)+8-10s}{4}\right\rceil \geqslant 0.
\end{aligned}
$$

By Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 2.2.3. $xx' \notin E(G)$, or xx' is not a cut-edge of G, or there are at least two end-blocks of G containing x'.

Let $G' = G-x-x'-v-N(v)$. Note that in this case $\omega(G') \geqslant 3$, and we have

$$
\begin{aligned}
\nu(G') &= \nu(G)-3-d_{B-x}(v)\\
&\geqslant m+n-s-3-\left\lfloor\frac{n+2s-\mathrm{par}(n)}{4}\right\rfloor+2\\
&= m+\left\lceil\frac{3n+\mathrm{par}(n)-6s-4}{4}\right\rceil \geqslant m.
\end{aligned}
$$

Now let $\mathcal{H}$ be the union of any $\omega(G')-1$ components of G'. If $B'-x' \not\subset \mathcal{H}$, then $\nu(\mathcal{H}) = \nu(G')-\nu(B'-x') \geqslant \nu(G')-\lfloor\nu(G)/2\rfloor+1$, and

$$
\nu(\mathcal{H})+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor \geqslant \nu(G')-\left\lfloor\frac{\nu(G)}{2}\right\rfloor+1+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor
$$

$$\begin{aligned}
&\geqslant 2\left(m+\left\lceil\frac{3n+\mathrm{par}(n)-6s-4}{4}\right\rceil\right)-\left\lfloor\frac{m+n-s}{2}\right\rfloor+1-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant\left\lceil m+2\cdot\frac{3n-6s-4}{4}-\frac{m+n-s}{2}+1-\frac{n}{2}\right\rceil\\
&=\left\lceil\frac{m+n-5s-2}{2}\right\rceil\geqslant\left\lceil\frac{3n-5s-1}{2}\right\rceil\geqslant 0.
\end{aligned}$$

If $B'-x'\subset\mathcal{H}$, then noting that $\omega(G')\geqslant 3$, $\nu(\mathcal{H})\geqslant\nu(B'-x')+1\geqslant\delta(G)+1$, and

$$\begin{aligned}
\nu(\mathcal{H})+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor&\geqslant\delta(G)+1+\nu(G')-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&\geqslant\left\lceil\frac{n}{2}\right\rceil-s+2+m+\left\lceil\frac{3n+\mathrm{par}(n)-6s-4}{4}\right\rceil-m-\left\lfloor\frac{n}{2}\right\rfloor\\
&=\left\lceil\frac{3n+5\mathrm{par}(n)+4-10s}{4}\right\rceil\geqslant 0.
\end{aligned}$$

By Lemma 6, $\overline{G'}$ contains a C_m, a contradiction.

Case 3. G is 2-connected.

By Claim 3 and Lemma 2, G contains a cycle of order at least $2(\lceil n/2\rceil-s+1)=n-2s+\mathrm{par}(n)+2$. Let C be a longest cycle of G (with a given orientation). Suppose that $\nu(C)=n-r$, where

$$r\leqslant 2s-\mathrm{par}(n)-2.$$

For a vertex x of C, we use x^+ to denote the successor, and x^- the predecessor, of x on C. For a subset X of $V(C)$, we set $X^+=\{x^+:x\in X\}$ and $X^-=\{x^-:x\in X\}$.

Let H be a subgraph of a component of $G-C$, and let $N_C(H)=\{z_1,z_2,\ldots z_k\}$, where $k=d_C(H)$, and z_i, $1\leqslant i\leqslant k$, are in order along C. We call the subpath $\overrightarrow{C}[z_i,z_{i+1}]$ (the indices are taken modulo k) a *good segment* of C (with respect to H); moreover, if z_i and z_{i+1} are joined to two distinct vertices x,y in H, then we call $\overrightarrow{C}[z_i,z_{i+1}]$ a *better segment* of C (with respect to H); moreover, if there is a path from x to y in $G-C$ of order at least 3, then we call $\overrightarrow{C}[z_i,z_{i+1}]$ a *best segment* of C (with respect to H). Since G is 2-connected, we conclude that for any component H of $G-C$, there are at least two good (better, best) segments of C with respect to H if $\nu(H)\geqslant 1$ ($\nu(H)\geqslant 2$, $\nu(H)\geqslant 3$ and H is not a star, respectively). Note that every good (better, best) segment has order at least 3 (4, 5, respectively).

Now we consider a component H of $G-C$. If H is non-separable, then H is a K_1, a K_2 or 2-connected; if H is separable, then H has at least two end-blocks. In the later case, we call an end-block of H removing the cut-vertex contained in the end-block a *branch* of H (also, of $G-C$).

Claim 7. *Let H be a component of $G-C$ and $u\in V(H)$.*

(1) *If H is non-separable, then H contains a path from u of order at least* $\min\{\nu(H),\lceil r/2\rceil\}$.

(2) *If H is separable and D is a branch of H not containing u, then H contains a path from u of order at least* $\min\{\nu(D)+1, \lceil r/2\rceil\}$.

Proof. We first claim that for any two vertices $u, v \in V(H)$, $d_H(u)+d_H(v) \geqslant \lceil r/2\rceil$, unless uv is a cut-edge of H. Assume that uv is not a cut-edge of H. Then H contains a path from u to v of order at least 3. Let $N_C(\{u,v\}) = \{z_1, z_2, \dots, z_k\}$, where z_i, $1 \leqslant i \leqslant k$, are in order along C. If z_i is joined to exactly one vertex of u, v, then $\overrightarrow{C}[z_i, z_{i+1}]$ is a good segment of C with respect to $\{u,v\}$; if z_i is adjacent to both u and v, then $\overrightarrow{C}[z_i, z_{i+1}]$ is a best segment with respect to $\{u,v\}$. This implies that $d_C(u)+d_C(v) \leqslant \lfloor (n-r)/2\rfloor$ and

$$\begin{aligned} d_H(u)+d_H(v) &= d(u)+d(v)-d_C(u)-d_C(v) \\ &\geqslant 2\cdot\left(\left\lceil\frac{n}{2}\right\rceil - s + 1\right) - \left\lfloor\frac{n-r}{2}\right\rfloor \\ &= \left\lceil\frac{n+r}{2}\right\rceil + \mathrm{par}(n) + 2 - 2s \\ &\geqslant \left\lceil\frac{n}{2}\right\rceil + \left\lceil\frac{r}{2}\right\rceil + 2 - 2s \geqslant \left\lceil\frac{r}{2}\right\rceil. \end{aligned}$$

Now we prove the claim.

(1) If H contains only one or two vertices, then the assertion is trivially true. So we assume that $\nu(H) \geqslant 3$. Let u' be a vertex in H such that $d_H(u')$ is as small as possible. Thus $d_H(v) \geqslant \lceil\lceil r/2\rceil/2\rceil$ for any vertex $v \in V(H)\backslash\{u, u'\}$. By Lemma 2, H contains a path from u of order at least $\min\{\nu(H), \lceil r/2\rceil\}$.

(2) Let B be the end-block of H containing D and b be the cut-vertex of H contained in B. If D contains only two vertices, then the assertion is trivially true. So we assume that $\nu(D) \geqslant 3$, from which we can see that B is 2-connected. Let u' be a vertex in $B - b$ such that $d_H(u')$ is as small as possible. Thus every vertex in $V(B)\backslash\{b, u'\}$ has degree at least $\lceil\lceil r/2\rceil/2\rceil$ in B. By Lemma 2, B contains a path from b of order at least $\min\{\nu(B), \lceil r/2\rceil\}$, and H contains a path from u of order at least $\min\{\nu(B), \lceil r/2\rceil\} = \min\{\nu(D)+1, \lceil r/2\rceil\}$. □

Now we choose D among all the non-separable components and branches of $G - C$ such that the order of D is as small as possible. We set a parameter a such that $a = 0$ if D is a non-separable component, and $a = 1$ if D is a branch of $G - C$.

If D is a branch of $G-C$, then let H be the component of $G-C$, and B the end-block of $G - C$, containing D; if D is a component of $G - C$, then let $H = B = D$.

Case 3.1. $\nu(D) = 1$.

Let v be the vertex in D. If $D = H$, then let $R = G-C-H$, $X = N_C^+(H)$. If $D \neq H$, then let y be a vertex in $H - B$, $R = G - C - B - y$ and $X = N_C^+(H) \cup \{y\}$. Thus every component of R is joined to at most one vertex in X. Moreover, we have

$$\begin{aligned} \nu(R) &= \nu(G) - \nu(C) - 1 - 2a \\ &= m + n - s - n + r - 1 - 2a \end{aligned}$$

$$= m + r - s - 2a - 1,$$

and

$$|X| = d_C(H) + a \geqslant d_C(v) + a = d(v) \geqslant \left\lceil \frac{n}{2} \right\rceil - s + 1.$$

Let $G' = G[V(R) \cup X]$. Note that there is a path of order at least $2+2a$ with an end-vertex in C and all other vertices in H. We have $r \geqslant 2 + 2a$, and

$$\begin{aligned}
\nu(G') &= \nu(R) + |X| \\
&\geqslant m + r - s - 2a - 1 + \left\lceil \frac{n}{2} \right\rceil - s + 1 \\
&= m + \left\lceil \frac{n}{2} \right\rceil + r - 2s - 2a \\
&\geqslant m + \left\lceil \frac{n}{2} \right\rceil + 2 - 2s \geqslant m.
\end{aligned}$$

Claim 8. *$D \neq H$ or $d_C(H) \geqslant 3$.*

Proof. Assume that $D = H$ and $d_C(H) = 2$. Since $d_C(H) = d(v) \geqslant \lceil n/2 \rceil - s + 1$, we have $n \leqslant 8$. We claim that every component of $G - C$ is an isolated vertex. Suppose on the contrary that there is a component H' of $G - C$ with order at least 2. Note that there are at least two better segments of C with respect to H'. We have $\nu(C) \geqslant 6$, and $G[V(C) \cup V(H')]$ contains a P_8, a contradiction. Thus as we claimed, every component of $G - C$ is an isolated vertex.

Note that $\nu(R) = m + r - s - 1$. Since $s \leqslant 3$ (when $n \leqslant 8$) and $r \geqslant 2$, we have $\nu(R) \geqslant m - 2$. If $\nu(R) \geqslant m$, then there is a C_m in $\overline{G'}$; if $\nu(R) = m - 1$, then $r = s \leqslant 3$, and one of the two vertices in $N_C^+(H)$ is nonadjacent to every vertex in R, and there is a C_m in $\overline{G'}$; if $\nu(R) = m - 2$, then $r = s - 1 \leqslant 2$, and both of the two vertices in $N_C^+(H)$ are nonadjacent to every vertex in R, and there is a C_m in $\overline{G'}$. In any case we get a contradiction. So we conclude that $D \neq H$ or $d_C(H) \geqslant 3$. □

By Claim 8, we can see that $|X| \geqslant 3$.

If there is a cycle C' in R with order $r + \mathrm{par}(r)$, then let P be a path between C and C', and $C \cup P \cup C'$ will contain a P_n, a contradiction. Thus we assume that R contains no cycle of order $r + \mathrm{par}(r)$. Since

$$\begin{aligned}
\nu(R) + 1 - \frac{3}{2}(r + \mathrm{par}(r)) &= m + r - s - 2a - 1 + 1 - \frac{3}{2}(r + \mathrm{par}(r)) \\
&\geqslant m - s - 2a - \left\lceil \frac{r}{2} \right\rceil - \mathrm{par}(r) \\
&\geqslant 2n - 2s - 1 \geqslant 0,
\end{aligned}$$

by Lemma 4, there is a path in $\overline{R}$ of order at least

$$p = \nu(R) + 1 - \frac{r + \mathrm{par}(r)}{2}$$

$$= m + r - s - 2a - 1 + 1 - \left\lceil \frac{r}{2} \right\rceil$$
$$= m + \left\lfloor \frac{r}{2} \right\rfloor - s - 2a.$$

Note that

$$\begin{aligned} p + 2|X| - 3 &\geqslant m + \left\lfloor \frac{r}{2} \right\rfloor - s - 2a + 2\left(\left\lceil \frac{n}{2} \right\rceil - s + 1\right) - 3 \\ &= m + n + \mathrm{par}(n) + \left\lfloor \frac{r}{2} \right\rfloor - 3s - 2a - 1 \\ &\geqslant m + n + \mathrm{par}(n) - 3s - a. \end{aligned}$$

We can see that $p + 2|X| - 3 \geqslant m$, when $n \geqslant 9$, unless $n = 11$ or 12 and $a = 1$. If $n \leqslant 8$, then noting that $|X| \geqslant 3$, we also have

$$p + 2|X| - 3 \geqslant m + \left\lfloor \frac{r}{2} \right\rfloor - s - 2a + 3 \geqslant m - s + 3 \geqslant m.$$

By Lemma 7, $\overline{G'}$ contains a C_m, a contradiction.
Petty Case. $n = 11$ or 12 and $a = 1$.

We claim that every component of $G - C$ is a K_1, K_2, K_3 or a star $K_{1,k}$. Suppose the contrary that there is a component H' of order at least 4 which is not a star. Since there are at least two best segments of C with respect to H', we can see that $\nu(C) \geqslant 8$. Note that there is a path of order at least 5 with one end-vertex in C and all other vertices in H'. This implies that $G[V(C) \cup V(H')]$ contains a P_{12}, a contradiction. Thus as we claimed, every component of $G - C$ is a K_1, K_2, K_3 or a star $K_{1,k}$.

Since H is not a K_1, K_2 or K_3, we conclude that H is a star. Now we choose a component H' of $G - C$ that is a maximum star of $G - C$, and let u' be the center of H', v' and y' be two end-vertices of H'. Let $R' = G - C - \{u', v', y'\}$, $X' = N_C^+(H') \cup \{y'\}$ and $G'' = G[V(R') \cup X']$. By the analysis above, we have

$$\nu(R') \geqslant m + r - s - 3 \text{ and } |X'| \geqslant \left\lceil \frac{n}{2} \right\rceil - s + 1.$$

Since $\nu(R') \geqslant m + r - s - 3 \geqslant 2n + 2 - s \geqslant 20$. If $G - C$ has at least three components, then R' is disconnected; if $G - C$ has exactly two components, then H' is a star with at least 4 vertices, and R' is disconnected; if $G - C$ consists of only one component H', then $R' = H' - \{u', v', y'\}$ is empty, and thus disconnected. Thus in any case, $\overline{R'}$ is connected.

Let H'' be a component of R' with the maximum order. If $\nu(H'') \leqslant \lceil \nu(R')/2 \rceil$, then every vertex of R' has degree at least $\lfloor \nu(R')/2 \rfloor$ in $\overline{R'}$. By Lemma 2, R' contains a Hamilton path. If $\nu(H'') \geqslant \lceil \nu(R')/2 \rceil + 1$, then H'' is a star with at least 4 vertices. Let u'' be the center of the star. Then every vertex in $V(R') \backslash \{u''\}$ has degree at least $\lceil \nu(R')/2 \rceil$ in $\overline{R' - u''}$. By Lemma 2, $\overline{R' - u''}$ contains a Hamilton cycle and $\overline{R'}$ contains a Hamilton path. In any case R' contains a path of order at least $p' = \nu(R')$. Thus we have

$$p' + 2|X'| - 3 \geqslant \nu(R') + |X'| \geqslant m.$$

By Lemma 7, $\overline{G''}$ contains a C_m, a contradiction.

Case 3.2. $\nu(D) = 2$.

Let v, v' be the two vertices in D. If $D = H$, then let $R = G - C - H$, $X_1 = N_C^+(H)$, $X_2 = N_C^-(H)$. If $D \neq H$, then let y be a vertex in $H - B$, let $R = G - C - B - y$, $X_1 = N_C^+(H) \cup \{y\}$, $X_2 = N_C^-(H) \cup \{y\}$. Thus every component of R is joined to at most one vertex in X_i, $i = 1, 2$, and

$$\begin{aligned}\nu(R) &= \nu(G) - \nu(C) - 2 - 2a \\ &= m + n - s - n + r - 2 - 2a \\ &= m + r - s - 2a - 2.\end{aligned}$$

Let $X = X_1 \cup X_2$ and $G' = G[V(R) \cup X]$. Note that there is a path of order at least $3 + 3a$ with an end-vertex in C and all other vertices in H. We have that $r \geqslant 3 + 3a$.

Let $N_C(H) = \{z_1, z_2, \ldots, z_k\}$, where z_i, $1 \leqslant i \leqslant k$, are in order along C. Since there are at least two better segments, we have $|X_1 \backslash X_2| = |X_2 \backslash X_1| \geqslant 2$. For any vertex $z_i \in N_C(H)$: if z_i is adjacent to exactly one vertex in $\{v, v'\}$, then $\overrightarrow{C}[z_i, z_{i+1}]$ is a good segment; if z_i is adjacent to both v and v', then $\overrightarrow{C}[z_i, z_{i+1}]$ is a better segment. This implies that

$$\begin{aligned}|X| &\geqslant d_C(v) + d_C(v') + a \\ &\geqslant 2\left(\left\lceil \frac{n}{2} \right\rceil - s + 1 - 1 - a\right) + a \\ &= n + \mathrm{par}(n) - 2s - a,\end{aligned}$$

and

$$\begin{aligned}\nu(G') &= \nu(R) + |X| \\ &\geqslant m + r - s - 2a - 2 + n + \mathrm{par}(n) - 2s - a \\ &\geqslant m + 3 + 3a - s - 2a - 2 + n + \mathrm{par}(n) - 2s - a \\ &\geqslant m + n + \mathrm{par}(n) + 1 - 3s \geqslant m.\end{aligned}$$

Since there are at least two better segments of C with respect to H, $\nu(C) \geqslant 6$. Thus there is a path in $G[V(C) \cup V(H)]$ of order at least 8, which implies that $n \geqslant 9$.

Claim 9. *$D \neq H$ or $d_C(H) \geqslant 3$.*

Proof. Assume that $D = H$ and $d_C(H) = 2$. Note that the two segments of C with respect to H are both better. Since $d_C(H) \geqslant d(v) - 1 \geqslant \lceil n/2 \rceil - s$, we have $n \leqslant 12$. We claim that every component of R has order at most 3. Suppose on the contrary that there is a component H' of $G - C$ that has order at least 4. Note that H' is not a star. There are at least two best segments of C with respect to H', which implies that $\nu(C) \geqslant 8$. Recall that H' is not a star and has order at least 4. We can see that there is a path of order at least 5 with one end-vertex in C and all other vertices in H'. Thus $G[V(C) \cup V(H')]$ contains a P_{12}, a contradiction. Thus as we claimed, every component of R has order 2 or 3.

Note that $\nu(R) = m + r - s - 2$. Since $s \leqslant 4$ (when $n \leqslant 12$) and $r \geqslant 3$, we have $\nu(R) \geqslant m - 3$. If $\nu(R) \geqslant m$, then by Lemma 3 there is a C_m in $\overline{G'}$; if $\nu(R) = m - 1$ or $m - 2$, then we have $r \leqslant s + 1 \leqslant 5$, and one of the two vertices in $N_C^+(H)$ ($N_C^-(H)$) is nonadjacent to every vertex in R, and there is a C_m in $\overline{G'}$; if $\nu(R) = m - 3$, then $r = s - 1 \leqslant 3$, and every vertex in $N_C^+(H)$ and $N_C^-(H)$ is nonadjacent to every vertex in R, and there is a C_m in $\overline{G'}$. In any case, we get a contradiction. So we conclude that $D \neq H$ or $d_C(H) \geqslant 3$. □

By Claim 9, we have $|X_1| = |X_2| \geqslant 3$.

If there is a cycle in R of order $r + \mathrm{par}(r)$, then there will be a path of order at least n in G. Thus we assume that R contains no cycle of order $r + \mathrm{par}(r)$. Since

$$\begin{aligned}\nu(R) + 1 - \frac{3}{2}(r + \mathrm{par}(r)) &= m + r - s - 2 - 2a + 1 - \frac{3}{2}(r + \mathrm{par}(r)) \\ &\geqslant m - s - 2a - \left\lceil \frac{r}{2} \right\rceil - \mathrm{par}(r) - 1 \\ &\geqslant 2n - 2s - 2 \geqslant 0,\end{aligned}$$

by Lemma 4, there is a path in $\overline{R}$ of order at least

$$\begin{aligned}p &= \nu(R) + 1 - \frac{r + \mathrm{par}(r)}{2} \\ &= m + r - s - 2 - 2a + 1 - \left\lceil \frac{r}{2} \right\rceil \\ &= m + \left\lfloor \frac{r}{2} \right\rfloor - s - 2a - 1.\end{aligned}$$

Note that

$$\begin{aligned}p + 2|X| - 5 &\geqslant m + \left\lfloor \frac{r}{2} \right\rfloor - s - 2a - 1 + 2(n + \mathrm{par}(n) - 2s - a) - 5 \\ &= m + 2n + 2\mathrm{par}(n) + \left\lfloor \frac{r}{2} \right\rfloor - 5s - 4a - 6 \\ &\geqslant m + 2n + 2\mathrm{par}(n) - 5s - 7.\end{aligned}$$

We can see that $p+2|X|-5 \geqslant m$, when $n \geqslant 13$. If $n \leqslant 12$, then noting that $d_C(H)+a \geqslant 3$ and $|X| \geqslant 5$, we also have

$$p + 2|X| - 5 \geqslant m + \left\lfloor \frac{r}{2} \right\rfloor - s - 2a - 1 + 5 \geqslant m - s + 5 \geqslant m.$$

By Lemma 8, $\overline{G'}$ contains a C_m, a contradiction.

Case 3.3. $3 \leqslant \nu(D) \leqslant \lceil r/2 \rceil - 1$.

In this case, $r \geqslant 7$. If $D = H$, then let $R = G - C - H$, $X_1 = N_C^+(H)$ and $X_2 = N_C^-(H)$. If $D \neq H$, then let y be a vertex in $H - B$ which is not a cut-vertex of

$H-B$, let $R = G-C-B-y$, $X_1 = N_C^+(H)\cup\{y\}$ and $X_2 = N_C^-(H)\cup\{y\}$. Thus every component of R is joined to at most one vertex in X_i, $i=1,2$, and

$$\begin{aligned}\nu(R) &= \nu(G)-\nu(C)-\nu(D)-2a\\ &\geqslant m+n-s-n+r-\left\lceil\frac{r}{2}\right\rceil+1-2a\\ &= m+\left\lfloor\frac{r}{2}\right\rfloor+1-s-2a.\end{aligned}$$

Clearly, every component of R has order at least 2, and

$$\begin{aligned}\left\lceil\frac{3\nu(R)}{2}\right\rceil+4 &\geqslant \left\lceil\frac{3(m+\lfloor r/2\rfloor+1-s-2a)}{2}\right\rceil+4\\ &\geqslant m+\left\lceil\frac{m+3(3+1-s-2a)}{2}\right\rceil+4\\ &\geqslant m+\left\lceil\frac{2n+15-3s}{2}\right\rceil\geqslant m.\end{aligned}$$

Let $X = X_1\cup X_2$ and $G' = G[V(G-B)\backslash N_C(H)]$. Since there are at least two best segments with respect to H, we have $|X_1\backslash X_2| = |X_2\backslash X_1|\geqslant 2$. Let v be a vertex in D.

Since R contains no cycle of length $r+\mathrm{par}(r)$ and

$$\begin{aligned}\nu(R)+1-\frac{3}{2}(r+\mathrm{par}(r)) &= m+\left\lfloor\frac{r}{2}\right\rfloor+1-s-2a+1-3\cdot\left\lceil\frac{r}{2}\right\rceil\\ &\geqslant m+2-r-2\mathrm{par}(r)-s-2a\\ &\geqslant 2n+3-2s+\mathrm{par}(n)+2-2\mathrm{par}(r)-s-2a\\ &\geqslant 2n+\mathrm{par}(n)+1-3s\geqslant 0,\end{aligned}$$

$\overline{R}$ contains a path of order at least

$$\begin{aligned}p &= \nu(R)+1-\frac{r+\mathrm{par}(r)}{2}\\ &\geqslant m+\left\lfloor\frac{r}{2}\right\rfloor+1-s-2a+1-\left\lceil\frac{r}{2}\right\rceil\\ &= m+2-s-\mathrm{par}(r)-2a.\end{aligned}$$

Claim 10. *$D\neq H$ or $d_C(H)\geqslant 3$.*

Proof. Assume that $D=H$ and $d_C(H)=2$. Thus

$$d_D(v)\geqslant\left\lceil\frac{n}{2}\right\rceil-s+1-2=\left\lceil\frac{n}{2}\right\rceil-s-1,$$

and

$$\nu(D)\geqslant 1+d_D(v)\geqslant\left\lceil\frac{n}{2}\right\rceil-s\geqslant s-2\geqslant\left\lceil\frac{r}{2}\right\rceil-1.$$

This implies that $\lceil r/2 \rceil = s-1$, $\nu(D) = \lceil r/2 \rceil - 1$, and $d_D(v) = \nu(D) - 1$. Note that in this case every vertex in D has degree $\nu(D)-1$, and thus D is a clique.

If every vertex in $N_C^+(H)$ is joined to some component of $G-C$, then by Claim 7, we can find a path from the cycle C, component H and the two components joined to the two vertices in $N_C^+(H)$, of order at least

$$\begin{aligned}\nu(C) + 3\nu(D) &= \nu(C) + 3 \cdot \left(\left\lceil \frac{r}{2} \right\rceil - 1 \right) \\ &= n - r + r + \mathrm{par}(r) + \left\lceil \frac{r}{2} \right\rceil - 3 \\ &\geqslant n,\end{aligned}$$

a contradiction. Thus there is a vertex v' in $N_C^+(H)$ that is not joined to every component of $G-C$. Let $G'' = G - C$.

Since $\lceil r/2 \rceil = s-1$ and $r \geqslant 7$, we can see that $r \geqslant s+1$. Thus

$$\nu(G'') = \nu(G) - \nu(C) \geqslant m + n - s - n + r = m + r - s \geqslant m.$$

Note that in this case, $\overline{G'' - H} = \overline{R}$ contains a path of order at least $p \geqslant m + 2 - s - \mathrm{par}(r) \geqslant m + 3 - r - \mathrm{par}(r)$ and

$$\begin{aligned}p + 2\nu(H) - 1 &\geqslant m + 3 - r - \mathrm{par}(r) + 2 \cdot \left(\left\lceil \frac{r}{2} \right\rceil - 1 \right) - 1 \\ &= m + 3 - r - \mathrm{par}(r) + r + \mathrm{par}(r) - 2 - 1 \\ &= m.\end{aligned}$$

Since

$$\nu(R) \geqslant m + \left\lfloor \frac{r}{2} \right\rfloor + 1 - s = m - \mathrm{par}(r) \geqslant \left\lceil \frac{m}{2} \right\rceil,$$

by Lemma 5, $\overline{G''}$ contains a C_m, and $\overline{G}$ contains a W_m with the hub v', a contradiction. □

By Claim 10, $|X_1| = |X_2| \geqslant 3$. If $D \neq H$, then since there are at least two best segments with respect to H, we can see that $\nu(C) \geqslant 8$; if $D = H$ and $d_C(H) \geqslant 3$, noting that at least two segments of C with respect to H are best, we have $\nu(C) \geqslant 10$. Since $\nu(C) = n - r$ and $r \geqslant 7$, we conclude that $n \geqslant 15$.

Let H' be a component of R, and let W be the union of X and the set of vertices in $V(C) \backslash N_C(H)$ not joined to H'. For any two vertices x, y with $xy \in E(C)$: if one of x, y is in $N_C(H)$, then the other one will be in $X \subset W$; if none of them is in $N_C(H)$, then at least one of them will not be joined to H', otherwise there will be a cycle longer than C. This implies that $|W| \geqslant \lceil (n-r)/2 \rceil + a = q$.

Since

$$\begin{aligned}\nu(R) + q - 1 &\geqslant m + \left\lfloor \frac{r}{2} \right\rfloor + 1 - s - 2a + \left\lceil \frac{n-r}{2} \right\rceil + a - 1 \\ &\geqslant m + \left\lfloor \frac{n}{2} \right\rfloor - s - a \geqslant m,\end{aligned}$$

($n \geqslant 14$) and

$$\begin{aligned} p + 2q - 5 &\geqslant m + 2 - s - \mathrm{par}(r) - 2a + 2 \cdot \left(\left\lceil \frac{n-r}{2} \right\rceil + a \right) - 5 \\ &= m + n - r - s - 3 + \mathrm{par}(n-r) - \mathrm{par}(r) \\ &= m + n - 3s - 1 + \mathrm{par}(n) + \mathrm{par}(n-r) - \mathrm{par}(r) \\ &\geqslant m + n - 3s - 1, \end{aligned}$$

we can see that $p + 2q - 5 \geqslant m$, unless $n = 15$, $s = 5$ and $r = 7$.
Petty Case. $n = 15$, $s = 5$ and $r = 7$.

In this case, $\nu(C) = 8$ which implies that $D \neq H$. It is easy to find a path with two end-vertices in C and all internal vertices in H of order at least 7. Thus $\nu(C) \geqslant 12$, a contradiction.

By Lemma 9, $\overline{G'}$ contains a C_m, a contradiction.

Case 3.4. $\nu(D) \geqslant \max\{\lceil r/2 \rceil, 3\}$.

In this case, there is a path of order at least 4 with an end-vertex in C and all other vertices in H. Thus we have $r \geqslant 4$. Let H' be an arbitrary component of $G - C$ and $u \in V(H')$. By Claim 7, H' contains a path from u of order at least $\lceil r/2 \rceil$. Thus for any edge $xy \in E(C)$, either x or y is not joined to any components of $G - C$, otherwise there will be a P_n in G. Moreover, if r is odd and x is joined to some component, say H', of $G - C$, then x^{++} will not be joined to any component of $G - C$ other than H' as well.

Case 3.4.1. Every component of $G - C$ has order less than r.

Let v be a vertex in $N_C^+(H)$, and let $G' = G[V(G - C) \cup N_C^+(H) \backslash \{v\}]$. Note that v is nonadjacent to every vertex in G', and every component of G' has order at most

$$r - 1 \leqslant 2s - \mathrm{par}(n) - 2 - 1 \leqslant \left\lceil \frac{n}{2} \right\rceil \leqslant \left\lfloor \frac{m}{2} \right\rfloor .$$

Let u be a vertex in H. Since

$$d_C(H) \geqslant d_C(u) \geqslant d(u) - \nu(H) + 1 \geqslant d(u) + 2 - r,$$

and

$$\begin{aligned} \nu(G') &= \nu(G) - \nu(C) + d_C(H) - 1 \\ &\geqslant m + n - s - n + r + d(v) + 1 - r \\ &\geqslant m - s + \left\lceil \frac{n}{2} \right\rceil - s + 1 + 1 \\ &= m + \left\lceil \frac{n}{2} \right\rceil + 2 - 2s \geqslant m, \end{aligned}$$

by Lemma 3, there is a C_m in $\overline{G'}$, a contradiction.

Case 3.4.2. There is a component of $G - C$ of order at least r.

Let H' be a component of $G-C$ with order at least r. We claim that there is a vertex u in H' with $d_{H'}(u) \leqslant \lceil r/2 \rceil - 1$. Suppose the contrary that every vertex of H' has degree at least $\lceil r/2 \rceil$ in H'. If H' is 2-connected, then by Lemma 2, there is a cycle of order at least r in H', and G will contain a P_n; if G is separable, letting B' be any end-block of H', b' be the cut-vertex of H' contained in B', and u' be any vertex in $V(B')\backslash\{b'\}$, then there is a path from b' to u' of order at least $\lceil r/2 \rceil + 1$. Thus G will contain a P_n as well. So we assume that there is a vertex u in H' with $d_{H'}(u) \leqslant \lceil r/2 \rceil - 1$.

Let v be a vertex in $N_C^+(H')$, $X = N_C^+(H')\backslash\{v\}$. If r is odd, then let $\overrightarrow{C}[z, z']$ be a better segment of C with respect to H' not containing v, and we add z^{++} to X. Let $G' = G[V(G - C) \cup X]$. Note that v is nonadjacent to every vertex in G', and there are no edges between $G - C$ and X.

Since

$$\begin{aligned} d_C(H') \geqslant d_C(u) &= d(u) - d_{H'}(u) \\ &\geqslant \left\lceil \frac{n}{2} \right\rceil - s + 1 - \left\lceil \frac{r}{2} \right\rceil + 1 \\ &= \left\lceil \frac{n}{2} \right\rceil + 2 - \left\lceil \frac{r}{2} \right\rceil - s, \end{aligned}$$

we have

$$|X| = d_C(H) - 1 + \mathrm{par}(r) \geqslant \left\lceil \frac{n}{2} \right\rceil + 1 - \left\lfloor \frac{r}{2} \right\rfloor - s$$

and

$$\begin{aligned} \nu(G') &= \nu(G) - \nu(C) + |X| \\ &\geqslant m + n - s - n + r + \left\lceil \frac{n}{2} \right\rceil + 1 - \left\lfloor \frac{r}{2} \right\rfloor - s \\ &\geqslant m - s + \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - s + 1 \\ &\geqslant m + \left\lceil \frac{n}{2} \right\rceil + 3 - 2s \geqslant m. \end{aligned}$$

Since $G - C$ contains no cycle of length $r + \mathrm{par}(r)$ and

$$\begin{aligned} \nu(G - C) + 1 - \frac{3}{2}(r + \mathrm{par}(r)) &= m + r - s + 1 - 3 \cdot \left\lceil \frac{r}{2} \right\rceil \\ &\geqslant m - \left\lceil \frac{r}{2} \right\rceil - \mathrm{par}(r) - s \\ &\geqslant 2n - 2s \geqslant 0, \end{aligned}$$

$\overline{G - C}$ contains a path of order at least

$$p = \nu(G - C) + 1 - \frac{r + \mathrm{par}(r)}{2}$$

$$
\begin{aligned}
&= m + r - s + 1 - \left\lceil \frac{r}{2} \right\rceil \\
&= m + \left\lfloor \frac{r}{2} \right\rfloor + 1 - s.
\end{aligned}
$$

Clearly $|X| \geqslant 1$. If $\lfloor r/2 \rfloor \geqslant s - 2$, then

$$
\begin{aligned}
p + 2|X| - 1 &\geqslant m + \left\lfloor \frac{r}{2} \right\rfloor + 1 - s + 2 - 1 \\
&\geqslant m + s - 2 + 1 - s + 2 - 1 \\
&= m.
\end{aligned}
$$

If $\lfloor r/2 \rfloor \leqslant s - 3$, then

$$
\begin{aligned}
p + 2|X| - 1 &\geqslant m + \left\lfloor \frac{r}{2} \right\rfloor + 1 - s + 2 \cdot \left(\left\lceil \frac{n}{2} \right\rceil + 1 - \left\lfloor \frac{r}{2} \right\rfloor - s \right) - 1 \\
&= m + n + \mathrm{par}(n) + 2 - \left\lfloor \frac{r}{2} \right\rfloor - 3s \\
&\geqslant m + n + \mathrm{par}(n) + 5 - 4s \geqslant m.
\end{aligned}
$$

Since

$$
\nu(G - C) = m + r - s \geqslant \left\lceil \frac{m}{2} \right\rceil,
$$

by Lemma 5, there is a C_m in $\overline{G'}$, a contradiction.

The proof is complete. □

4 Remarks

A *linear forest* is a forest such that every component of it is a path. From our main result of the paper, we can conclude the following result.

Corollary 1. *Let $n \geqslant 2$, $m \geqslant 2n + 1$ and F be a linear forest on m vertices. Then*

$$
R(P_n, K_1 \vee F) = t(n, m).
$$

Proof. Note that the graph constructed at the beginning of Section 3 contains no P_n and its complement contains no $K_1 \vee F$. We conclude that $R(P_n, K_1 \vee F) \geqslant t(n, m)$. On the other hand, since $K_1 \vee F$ is a subgraph of W_m, we have $R(P_n, K_1 \vee F) \leqslant R(P_n, W_m) \leqslant t(n, m)$. □

For the case F is an empty graph, the above formula gives the Ramsey numbers of paths versus stars when $m \geqslant 2n + 1$. In fact, Parsons [10] gave all the values of the path-star Ramsey numbers by a recursive formula.

Acknowledgements

The authors are very grateful to Professor Yunqing Zhang for providing them the paper [12]. The research is supported by NSFC (No. 11271300), the Doctorate Foundation of Northwestern Polytechnical University (No. cx201202 and No. cx201326) and the project NEXLIZ - CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

References

[1] E.T. Baskoro and Surahmat. The Ramsey number of paths with respect to wheels. *Discrete Math.*, 294:275–277, 2005.

[2] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications. Macmillan, London and Elsevier, New York, 1976.

[3] Y. Chen, Y. Zhang and K. Zhang. The Ramsey numbers of paths versus wheels. *Discrete Math.*, 290:85–87, 2005.

[4] G.A. Dirac. Some theorems on abstract graphs. *Proc. London. Math. Soc.*, 2:69–81, 1952.

[5] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[6] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, 10:337–356, 1959.

[7] G. Fan. New sufficient conditions for cycles in graphs. *J. Combin. Theory, Ser. B*, 37:221–227, 1984.

[8] R.J. Faudree, S.L. Lawrence, T.D. Parsons and R.H. Schelp. Path-cycle Ramsey numbers. *Discrete Math.*, 10:269–277, 1974.

[9] L. Gerencsér and A. Gyárfás. On Ramsey-type problems. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 10:167–170, 1967.

[10] T.D. Parsons. Path-star Ramsey numbers. *J. Combin. Theory, Ser. B*, 17:51–58, 1974.

[11] A.N.M. Salman and H.J. Broersma. On Ramsey numbers for paths versus wheels. *Discrete Math.*, 307:975–982, 2007.

[12] Y. Zhang. On Ramsey numbers of short paths versus large wheels. *Ars Combin.*, 89:11–20, 2008.