On bipartite Q-polynomial distance-regular graphs with $c_2 \leqslant 2$

Štefko Miklavič*

Andrej Marušič Institute
University of Primorska
Muzejski trg 2, 6000 Koper, Slovenia

stefko.miklavic@upr.si

Safet Penjić

Department of Mathematics and Informatics
Faculty of Philosophy
University of Zenica
Zmaja od Bosne 56, 72000 Zenica, Bosnia and Herzegovina

safet.penjic@ff.unze.ba

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Abstract

Let Γ denote a bipartite Q-polynomial distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$ and intersection number $c_2 \leqslant 2$. We show that Γ is either the D-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube, or $D = 5$.

Keywords: bipartite distance-regular graph; Q-polynomial property; equitable partition

1 Introduction

Let Γ denote a bipartite Q-polynomial distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$ and intersection numbers b_i, c_i (see Sections 2 and 3 for formal definitions). The present paper is a part of an effort to classify the examples with $c_2 \leqslant 2$. In order to motivate our results we give some comments on this case.

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Caughman proved in [3] that if $D \geqslant 12$ then Γ is either the D-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube, or the intersection numbers of Γ satisfy $c_i = (q^i - 1)/(q - 1)$ $(0 \leqslant i \leqslant D)$ for some integer q at least 2. Note that if $c_2 \leqslant 2$, then the last of the above possibilities cannot occur. It is the aim of the present paper to further investigate these graphs. Our main result is the following theorem.

Theorem 1. *Let Γ denote a bipartite Q-polynomial distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$, and intersection number $c_2 \leqslant 2$. Then one of the following holds:*

(i) Γ *is the D-dimensional hypercube;*

(ii) Γ *is the antipodal quotient of the $2D$-dimensional hypercube;*

(iii) Γ *is a graph with $D = 5$ not listed above.*

To prove the above theorem we use the results of Caughman [3] and, in case when $c_2 = 2$, a certain equitable partition of the vertex set of Γ which involves $4(D-1) + 2\ell$ cells for some integer ℓ with $0 \leqslant \ell \leqslant D - 2$.

Our paper is organized as follows. In Sections 2 and 3 we review some definitions and basic concepts and set up some necessary tools for the proof of our main results. We consider the case $D \geqslant 6$ in Section 4. In Section 5 we describe a partition of the vertex set of Γ and in Section 6 we show that this partition is equitable. In Section 7 we consider the case $D = 4$.

For the current status of the classification of the Q-polynomial distance-regular graphs see a recent survey by E. R. Van Dam, J. H. Koolen and H. Tanaka [5].

2 Preliminaries

In this section, we review some definitions and basic concepts. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by X. Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with entries in $\mathbb{R}$ and rows indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication. We refer to V as the *standard module* of $\mathrm{Mat}_X(\mathbb{R})$. For $x \in X$ let $\hat{x}$ denote the vector in V that has x-coordinate 1 and all other coordinates 0. We endow V with the standard dot product $\langle\, ,\, \rangle$, where $\langle u, v\rangle = u^t v \quad (u, v \in V)$.

Throughout the paper let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $D := \max\{\partial(x, y) |\; x, y \in X\}$. For $x \in X$ and an integer i let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geqslant 0$ we say Γ is *regular with valency k* whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers $0 \leqslant h, i, j \leqslant D$ and all $x, y \in X$ with $\partial(x, y) = h$ the number $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. Observe that for h, i, j $(0 \leqslant h, i, j \leqslant D)$, $p^h_{ij} = 0$ (resp. $p^h_{ij} \neq 0$) if one of h, i, j is greater than (resp. equal to) the sum of the other two. Note that $p^h_{ij} = p^h_{ji}$ for $0 \leqslant h, i, j \leqslant D$. For the rest of this paper we assume Γ is distance-regular with diameter $D \geqslant 3$ and valency $k \geqslant 3$. For convenience set $c_i := p^i_{1,i-1}$ $(1 \leqslant i \leqslant D)$, $a_i := p^i_{1i}$ $(0 \leqslant i \leqslant D)$, $b_i := p^i_{1,i+1}$ $(0 \leqslant i \leqslant D-1)$, $k_i := p^0_{ii}$ $(0 \leqslant i \leqslant D)$, and $c_0 = 0$, $b_D = 0$. Observe that Γ is regular with valency $k = b_0 = k_1 = p^0_{11}$. Moreover, for $0 \leqslant i \leqslant D$

$$c_i + a_i + b_i = k, \tag{1}$$

where $k := k_1$. Observe also that Γ is bipartite if and only if $a_i = 0$ for $0 \leqslant i \leqslant D$. In this case $b_i + c_i = k$ for $0 \leqslant i \leqslant D$ and $p^h_{ij} = 0$ unless $i + j + h$ is even.

By [2, page 127] we have

$$k_0 = 1 \text{ and } k_i = (b_0 b_1 \cdots b_{i-1})/(c_1 c_2 \cdots c_i) \quad (1 \leqslant i \leqslant D). \tag{2}$$

The following formulae will be useful.

Lemma 2. ([2, Lemma 4.1.7]) *Let Γ denote a distance-regular graph with diameter $D \geqslant 3$. Then the following* (i), (ii) *hold.*

(i) $p^2_{02} = 1$ *and* $p^2_{i-1,i+1} = (b_2 b_3 \cdots b_i)/(c_1 c_2 \cdots c_{i-1})$ $(2 \leqslant i \leqslant D-1)$*;*

(ii) $p^2_{22} = (c_2 b_1 + a_2^2 + c_3 b_2 - k - a_1 a_2)/c_2$ *and*
$p^2_{ii} = (b_2 b_3 \cdots b_{i-1})(c_i b_{i-1} + a_i^2 + c_{i+1} b_i - k - a_1 a_i)/(c_1 c_2 \cdots c_i)$ $(3 \leqslant i \leqslant D-1)$.

We recall the Bose-Mesner algebra of Γ. For $0 \leqslant i \leqslant D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, z)-entry

$$(A_i)_{yz} = \begin{cases} 1 & \text{if } \partial(y, z) = i, \\ 0 & \text{if } \partial(y, z) \neq i \end{cases} \qquad (y, z \in X). \tag{3}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (ai) $A_0 = I$; (aii) $J = \sum_{i=0}^{D} A_i$; (aiii) $A_i^t = A_i$ $(0 \leqslant i \leqslant D)$; (aiv) $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h$ $(0 \leqslant i, j \leqslant D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{R})$. Using these facts we find $\{A_i\}_{i=0}^{D}$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. We call M the *Bose-Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [2, p. 45], M has a second basis $\{E_i\}_{i=0}^{D}$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $I = \sum_{i=0}^{D} E_i$; (eiii) $E_i^t = E_i$ $(0 \leqslant i \leqslant D)$; (eiv) $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant D)$. We call $\{E_i\}_{i=0}^{D}$ the *primitive idempotents* of Γ. We call E_0 the *trivial primitive idempotent* of Γ. By (eii)–(eiv) above,

$$V = E_0 V + E_1 V + \cdots + E_D V \qquad \text{(orthogonal direct sum)}. \tag{4}$$

We recall the eigenvalues of Γ. Since $\{E_i\}_{i=0}^{D}$ form a basis for M, there exist real scalars $\{\theta_i\}_{i=0}^{D}$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Combining this with (eiv) we find

$$A E_i = E_i A = \theta_i E_i \qquad (0 \leqslant i \leqslant D).$$

We call θ_i the *eigenvalue* of Γ associated with E_i. For $0 \leqslant i \leqslant D$ the space E_iV is the eigenspace of A associated with θ_i. Let m_i denote the rank of E_i $(0 \leqslant i \leqslant D)$. Observe that m_i is the dimension of the eigenspace E_iV $(0 \leqslant i \leqslant D)$. We call m_i the *multiplicity* of θ_i. Observe that $\{\theta_i\}_{i=0}^D$ are mutually distinct since A generates M. By (ei) we have $\theta_0 = k$.

Let θ denote an eigenvalue of Γ, and let E denote the associated primitive idempotent. For $0 \leqslant i \leqslant D$ define a real number θ_i^* by

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i.$$

We call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ the *dual eigenvalue sequence* associated with θ, E. We say the sequence is *trivial* whenever $E = E_0$ (in which case $\theta_0^* = \theta_1^* = \cdots = \theta_D^* = 1$). In the following lemma, we cite a well known result about primitive idempotents.

Lemma 3. ([7, Lemma 1.1]) *Let Γ denote a distance-regular graph with diameter $D \geqslant 3$, let E denote a primitive idempotent of Γ, and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the corresponding dual eigenvalue sequence. Then for $0 \leqslant i \leqslant D$ and for all $x, y \in X$ with $\partial(x, y) = i$ we have $\langle E\hat{x}, E\hat{y}\rangle = |X|^{-1}\theta_i^*$.*

An *equitable partition* of a graph is a partition $\pi = \{C_1, C_2, \ldots, C_s\}$ of its vertex set into nonempty cells such that for all integers i, j $(1 \leqslant i, j \leqslant s)$ the number c_{ij} of neighbours, which a vertex in the cell C_i has in the cell C_j, is independent of the choice of the vertex in C_i. We call the c_{ij} the *corresponding parameters*.

3 The Q-polynomial property

We continue to discuss the distance-regular graph $\Gamma = (X, R)$ from Section 2. In this section we recall the Q-polynomial property of Γ. We first recall the Krein parameters of Γ. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{R})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leqslant i, j \leqslant D$, so M is closed under $\circ$. Thus there exist $q_{ij}^h \in \mathbb{R}$ $(0 \leqslant h, i, j \leqslant D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad (0 \leqslant i, j \leqslant D).$$

The parameters q_{ij}^h are called the *Krein parameters of* Γ. By [2, Proposition 4.1.5] the Krein parameters of Γ are nonnegative.

We recall the Q-polynomial property of Γ. Let $\{E_i\}_{i=0}^D$ denote an ordering of the primitive idempotents of Γ. This ordering is said to be *Q-polynomial* whenever for $0 \leqslant h, i, j \leqslant D$ the Krein parameter $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote a nontrivial primitive idempotent of Γ and let θ denote the corresponding eigenvalue. We say Γ is *Q-polynomial with respect to E* (or *Q-polynomial with respect to θ*) whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents such that $E_1 = E$. We have the following useful lemmas about the Q-polynomial property.

Lemma 4. ([2, Thm. 8.1.1]) *Let Γ denote a distance-regular graph with diameter $D \geqslant 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Suppose Γ is Q-polynomial with respect to E. Then $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are mutually distinct.*

Lemma 5. ([7, Thm. 3.3]) *Let Γ denote a distance-regular graph with diameter $D \geqslant 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Then the following* (i), (ii) *are equivalent.*

(i) *Γ is Q-polynomial with respect to E.*

(ii) *$\theta^*_0 \neq \theta^*_i$ for $1 \leqslant i \leqslant D$; for all integers h, i, j $(1 \leqslant h \leqslant D)$, $(0 \leqslant i, j \leqslant D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$ the following hold:*

$$\sum_{\substack{z\in X\\ \partial(x,z)=i\\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z\in X\\ \partial(x,z)=j\\ \partial(y,z)=i}} E\hat{z} \in span\{E\hat{x} - E\hat{y}\}.$$

Suppose (i), (ii) *hold. Then for all integers h, i, j $(1 \leqslant h \leqslant D)$, $(0 \leqslant i, j \leqslant D)$ and for all $x, y \in X$ such that $\partial(x, y) = h$,*

$$\sum_{\substack{z\in X\\ \partial(x,z)=i\\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z\in X\\ \partial(x,z)=j\\ \partial(y,z)=i}} E\hat{z} = p^h_{ij}\frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h}(E\hat{x} - E\hat{y}). \tag{5}$$

We have the following important result about bipartite Q-polynomial distance-regular graphs, see [3, Lemma 3.2, Lemma 3.3].

Lemma 6. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$, and intersection numbers b_i, c_i. Let $\{E_i\}_{i=0}^D$ be a Q-polynomial ordering of primitive idempotents of Γ, and let $\{\theta^*_i\}_{i=0}^D$ denote the dual eigenvalue sequence associated with E_1. For $0 \leqslant i \leqslant D$ let θ_i denote the eigenvalue associated with E_i. Assume Γ is not the D-cube or the antipodal quotient of the $2D$-cube. Then there exist scalars $q, s^* \in \mathbb{R}$ such that* (i)–(iii) *hold below.*

(i) *$|q| > 1$, $s^* q^i \neq 1$ $\quad (2 \leqslant i \leqslant 2D+1)$;*

(ii) *$\theta_i = h(q^{D-i} - q^i)$, $\theta^*_i = \theta^*_0 + h^*(1 - q^i)(1 - s^* q^{i+1})q^{-i}$ for $0 \leqslant i \leqslant D$, where*

$$h = \frac{1 - s^* q^3}{(q-1)(1 - s^* q^{D+2})}, \quad h^* = \frac{(q^D + q^2)(q^D + q)}{q(q^2 - 1)(1 - s^* q^{2D})}, \quad \theta^*_0 = \frac{h^*(q^D - 1)(1 - s^* q^2)}{q(q^{D-1} + 1)};$$

(iii) *$k = c_D = h(q^D - 1)$, and*

$$c_i = \frac{h(q^i - 1)(1 - s^* q^{D+i+1})}{1 - s^* q^{2i+1}}, \qquad b_i = \frac{h(q^D - q^i)(1 - s^* q^{i+1})}{1 - s^* q^{2i+1}} \qquad (1 \leqslant i \leqslant D-1).$$

4 Case $D \geqslant 6$

Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D \geqslant 6$, valency $k \geqslant 3$, and intersection numbers b_i, c_i. In this section we show that if $c_2 \leqslant 2$, then Γ is either the D-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube.

Theorem 7. *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D \geqslant 6$ and valency $k \geqslant 3$. If $c_2 \leqslant 2$, then Γ is either the D-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube.*

Proof. Assume that Γ is not the D-dimensional hypercube or the antipodal quotient of the $2D$-dimensional hypercube. Let scalars s^*, q be as in Lemma 6.

By [3, Lemma 4.1 and Lemma 5.1], scalars s^* and q satisfy

$$q > 1, \quad \text{and} \quad -q^{-D-1} \leqslant s^* < q^{-2D-1}. \tag{6}$$

Assume first $c_2 = 1$. Abbreviate $\alpha = 1 + q - q^2 - q^{D-1} + q^D + q^{D+1}$ and observe $\alpha > 2$. By Lemma 6(iii) we find

$$s^* = \frac{\alpha \pm \sqrt{\alpha^2 - 4q^{D+1}}}{2q^{D+3}}.$$

Note that $\alpha^2 - 4q^{D+1} \geqslant 0$, and so we have

$$s^* \geqslant \frac{\alpha - \sqrt{\alpha^2 - 4q^{D+1}}}{2q^{D+3}}.$$

We claim

$$\frac{\alpha - \sqrt{\alpha^2 - 4q^{D+1}}}{2q^{D+3}} > q^{-2D-1}.$$

First observe that $(\alpha q^{D-2} - 2)^2 - q^{2D-4}(\alpha^2 - 4q^{D+1}) = 4(q^D + 1)(q^{D-1} - 1)(q^{D-2} - 1) > 0$. Therefore,

$$(\alpha q^{D-2} - 2)^2 > q^{2D-4}(\alpha^2 - 4q^{D+1}).$$

Furthermore, $\alpha q^{D-2} - 2 > 0$ implies

$$\alpha q^{D-2} - 2 > q^{D-2}\sqrt{\alpha^2 - 4q^{D+1}},$$

and the claim follows. Therefore,

$$s^* \geqslant \frac{\alpha - \sqrt{\alpha^2 - 4q^{D+1}}}{2q^{D+3}} > q^{-2D-1},$$

contradicting (6).

Next assume $c_2 = 2$. Abbreviate $\beta = 1 + 2q - 2q^{D-1} - q^D$ and observe $\beta < 0$. By Lemma 6(iii) we find

$$s^* = \frac{\beta \pm \sqrt{\beta^2 + 4q^D}}{2q^{D+3}}.$$

Assume first $s^* = (\beta - \sqrt{\beta^2 + 4q^D})/(2q^{D+3})$. If $\beta + 2q^2 < 0$, then clearly $\beta + 2q^2 < \sqrt{\beta^2 + 4q^D}$. On the other hand, if $\beta + 2q^2 > 0$, then $(\beta + 2q^2)^2 < \beta^2 + 4q^D$ again implies $\beta + 2q^2 < \sqrt{\beta^2 + 4q^D}$. Therefore, in both cases we find $\beta + 2q^2 < \sqrt{\beta^2 + 4q^D}$. But now

$$-\frac{1}{q^{D+1}} = \frac{\beta - (\beta + 2q^2)}{2q^{D+3}} > \frac{\beta - \sqrt{\beta^2 + 4q^D}}{2q^{D+3}} = s^*,$$

contradicting (6).
Finally, assume $s^* = (\beta + \sqrt{\beta^2 + 4q^D})/(2q^{D+3})$. We observe that $q^{3D-4} + \beta q^{D-2} - 1 = (q^{D-1} - 1)^2(q^{D-2} - 1) > 0$. Therefore $q^{3D-4} > 1 - \beta q^{D-2}$, implying

$$\beta^2 q^{2D-4} + 4q^{3D-4} > 4 - 4\beta q^{D-2} + \beta^2 q^{2D-4} = (2 - \beta q^{D-2})^2.$$

Taking the square root of the above inequality and dividing by q^{D-2} we obtain

$$\sqrt{\beta^2 + 4q^D} > \frac{2}{q^{D-2}} - \beta.$$

But now we have

$$s^* = \frac{\beta + \sqrt{\beta^2 + 4q^D}}{2q^{D+3}} > \frac{1}{q^{2D+1}},$$

contradicting (6). This finishes the proof. □

5 The partition - part I

We continue to discuss the distance-regular graph $\Gamma = (X, R)$ from Section 2. Up to Section 7 we will assume that Γ is bipartite with diameter $D \geqslant 4$, valency $k \geqslant 3$ and intersection number $c_2 = 2$. In this section we describe certain partition of the vertex set X.

Definition 8. Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$ and intersection number $c_2 = 2$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. For all integers i, j we define $D^i_j = D^i_j(x, y)$ by

$$D^i_j = \{w \in X \mid \partial(x, w) = i \text{ and } \partial(y, w) = j\}.$$

We observe $D^i_j = \emptyset$ unless $0 \leqslant i, j \leqslant D$. Moreover $|D^i_j| = p^2_{ij}$ for $0 \leqslant i, j \leqslant D$.

Lemma 9. ([6, Lemma 3.2]) *With reference to Definition 8, the following* (i), (ii) *hold for* $0 \leqslant i, j \leqslant D$.

(i) *If* $|i - j| > 2$ *then* $D^i_j = \emptyset$.

(ii) *If* $i + j$ *is odd then* $D^i_j = \emptyset$.

Lemma 10. ([6, Lemma 3.3]) *With reference to Definition 8, the following* (i), (ii) *hold.*

(i) $|D_0^2| = |D_2^0| = 1$ *and* $|D_{i-1}^{i+1}| = |D_{i+1}^{i-1}| = (b_2b_3 \cdots b_i)/(c_1c_2 \cdots c_{i-1})$ $(2 \leqslant i \leqslant D-1)$*;*

(ii) $D_{i-1}^{i+1} \neq \emptyset$, $D_{i+1}^{i-1} \neq \emptyset$ $(1 \leqslant i \leqslant D-1)$.

Lemma 11. ([6, Lemma 3.4]) *With reference to Definition 8, there are no edges inside the set D_j^i for $0 \leqslant i, j \leqslant D$.*

Lemma 12. *With reference to Definition 8, let z, v denote the common neighbours of x and y. For $1 \leqslant i \leqslant D$ and for $w \in D_i^i$ we have $\partial(w, z) \in \{i-1, i+1\}$ and $\partial(w, v) \in \{i-1, i+1\}$.*

Proof. Let $u \in \{z, v\}$. From the triangle inequality we find $i - 1 \leqslant \partial(w, u) \leqslant i + 1$. Now if $\partial(w, u) = i$, then we have a cycle of an odd length in Γ, a contradiction. $\square$

Definition 13. Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$ and intersection number $c_2 = 2$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$ and let z, v denote the common neighbours of x, y. For $0 \leqslant i, j \leqslant D$ let the sets D_j^i be as defined in Definition 8. For $1 \leqslant i \leqslant D$ we define $\mathcal{A}_i = \mathcal{A}_i(x, y)$, $\mathcal{C}_i = \mathcal{C}_i(x, y)$, $\mathcal{B}_i(z) = \mathcal{B}_i(z)(x, y)$, $\mathcal{B}_i(v) = \mathcal{B}_i(v)(x, y)$ by

$$\mathcal{A}_i = \{w \in D_i^i \mid \partial(w, z) = i + 1 \text{ and } \partial(w, v) = i + 1\},$$

$$\mathcal{C}_i = \{w \in D_i^i \mid \partial(w, z) = i - 1 \text{ and } \partial(w, v) = i - 1\},$$

$$\mathcal{B}_i(z) = \{w \in D_i^i \mid \partial(w, z) = i - 1 \text{ and } \partial(w, v) = i + 1\},$$

$$\mathcal{B}_i(v) = \{w \in D_i^i \mid \partial(w, z) = i + 1 \text{ and } \partial(w, v) = i - 1\}.$$

We observe D_i^i is a disjoint union of $\mathcal{A}_i, \mathcal{B}_i(z), \mathcal{B}_i(v), \mathcal{C}_i$.

Remark 14. With reference to Definition 13, note that $\partial(z, v) = 2$ and that x, y are the common neighbours of z, v. Consequently, if we have a result that holds for x, y (and z, v as their common neighbours), then an analogous result for z, v (and x, y as their common neighbours) also holds. We will be using this fact extensively throughout the paper.

Lemma 15. *With reference to Definition 13, the following* (i)–(iii) *hold.*

(i) $\mathcal{A}_1 = \emptyset$, $\mathcal{C}_1 = \emptyset$, $\mathcal{B}_1(z) = \{z\}$, $\mathcal{B}_1(v) = \{v\}$.

(ii) $\mathcal{C}_2 = \emptyset$.

(iii) $\mathcal{C}_D = D_D^D$ *and* $\mathcal{A}_D = \mathcal{B}_D(z) = \mathcal{B}_D(v) = \emptyset$.

Proof. (i) and (iii) follows immediately from Definition 13. (ii) follows from the fact that $c_2 = 2$. $\square$

Lemma 16. *With reference to Definition 13, the following* (i)–(vi) *hold.*

(i) $D_{i+1}^{i-1}(x, y) = \mathcal{B}_i(x)(z, v)$ *for* $1 \leqslant i \leqslant D - 1$.

(ii) $D^{i+1}_{i-1}(x,y) = \mathcal{B}_i(y)(z,v)$ *for* $1 \leqslant i \leqslant D-1$.

(iii) $\mathcal{A}_i(x,y) = \mathcal{C}_{i+1}(z,v)$ *for* $1 \leqslant i \leqslant D-1$.

(iv) $\mathcal{C}_i(x,y) = \mathcal{A}_{i-1}(z,v)$ *for* $2 \leqslant i \leqslant D$.

(v) $\mathcal{B}_i(z)(x,y) = D^{i-1}_{i+1}(z,v)$ *for* $1 \leqslant i \leqslant D-1$.

(vi) $\mathcal{B}_i(v)(x,y) = D^{i+1}_{i-1}(z,v)$ *for* $1 \leqslant i \leqslant D-1$.

Proof. (i) Pick $w \in D^{i-1}_{i+1}(x,y)$ and note that $\partial(w,x) = i-1$, $\partial(w,y) = i+1$ and $\partial(w,z) = \partial(w,v) = i$. Therefore $w \in \mathcal{B}_i(x)(z,v)$, implying $D^{i-1}_{i+1}(x,y) \subseteq \mathcal{B}_i(x)(z,v)$. Similarly, if $w \in \mathcal{B}_i(x)(z,v)$, then $\partial(w,z) = \partial(w,v) = i$, $\partial(w,x) = i-1$ and $\partial(w,y) = i+1$. Therefore $w \in D^{i-1}_{i+1}(x,y)$, implying $\mathcal{B}_i(x)(z,v) \subseteq D^{i-1}_{i+1}(x,y)$. The result follows.
(ii)-(vi) Similarly as the proof of (i) above. □

To compute the cardinalities of the sets $\mathcal{A}_i, \mathcal{B}_i(z), \mathcal{B}_i(v)$ and $\mathcal{C}_i$ we make the following definition. For $2 \leqslant i \leqslant D-1$ define

$$M_i = p^2_{ii} - p^2_{i-1,i-1} + p^2_{i-2,i-2} - \cdots \pm p^2_{22}$$

and

$$N_i = p^2_{i-1,i+1} - p^2_{i-2,i} + p^2_{i-3,i-1} - \cdots \pm p^2_{13}.$$

Lemma 17. *With reference to Definition 13, the following* (i)–(iv) *hold.*

(i) $|\mathcal{B}_i(z)| = p^2_{i-1,i+1}$ $(1 \leqslant i \leqslant D-1)$;

(ii) $|\mathcal{B}_i(v)| = p^2_{i-1,i+1}$ $(1 \leqslant i \leqslant D-1)$;

(iii) $|\mathcal{A}_i| = M_i - 2N_i$ $(2 \leqslant i \leqslant D-1)$;

(iv) $|\mathcal{C}_i| = M_{i-1} - 2N_{i-1}$ $(3 \leqslant i \leqslant D)$;

Proof. (i), (ii) This follows from Lemma 16(v),(vi) and Lemma 10.
(iii) As $|\mathcal{B}_2(z) \cup \mathcal{B}_2(v) \cup \mathcal{A}_2| = p^2_{22}$, the result is true for $i = 2$. Now assume that the result is true for some i $(2 \leqslant i \leqslant D-2)$. We will show that it is true also for $i+1$. Note that D^{i+1}_{i+1} is a disjoint union of $\mathcal{A}_{i+1}$, $\mathcal{B}_{i+1}(z)$, $\mathcal{B}_{i+1}(v)$ and $\mathcal{C}_{i+1}$. It follows from (i), (ii) above, Lemma 16(iv) and the induction hypothesis that $|\mathcal{A}_{i+1}| = p^2_{i+1,i+1} - 2p^2_{i,i+2} - M_i + 2N_i$. The result follows.
(iv) The result follows from (iii) above and Lemma 16(iv). □

Corollary 18. *With reference to Definition 13, the following* (i), (ii) *hold.*

(i) $\mathcal{B}_i(z) \neq \emptyset$ $(1 \leqslant i \leqslant D-1)$;

(ii) $\mathcal{B}_i(v) \neq \emptyset$ $(1 \leqslant i \leqslant D-1)$;

Proof. Immediate from Lemma 17(i),(ii). □

Lemma 19. *With reference to Definition 13, the following* (i)–(iv) *hold.*

(i) *For* $1 \leqslant i \leqslant D-1$*, there is no edge between any of the sets* $\mathcal{A}_i, \mathcal{B}_i(z), \mathcal{B}_i(v), \mathcal{C}_i$.

(ii) *For* $2 \leqslant i \leqslant D-1$*, there is no edge between* $\mathcal{A}_i$ *and* $\mathcal{B}_{i-1}(z) \cup \mathcal{B}_{i-1}(v) \cup \mathcal{C}_{i-1}$.

(iii) *For* $2 \leqslant i \leqslant D-1$*, there is no edge between* $\mathcal{B}_i(z)$ *and* $\mathcal{B}_{i-1}(v) \cup \mathcal{C}_{i-1}$.

(iv) *For* $2 \leqslant i \leqslant D-1$*, there is no edge between* $\mathcal{B}_i(v)$ *and* $\mathcal{B}_{i-1}(z) \cup \mathcal{C}_{i-1}$.

Proof. (i) Immediate from Lemma 11.
(ii), (iii), (iv) By the definition of the sets $\mathcal{A}_i, \mathcal{B}_i(z), \mathcal{B}_i(v), \mathcal{C}_i$. □

With reference to Definition 13, we visualize $D^{i-1}_{i+1}, D^{i+1}_{i-1}, \mathcal{A}_i, \mathcal{B}_i(z), \mathcal{B}_i(v), \mathcal{C}_i$ and edges between these sets in Figure 1.

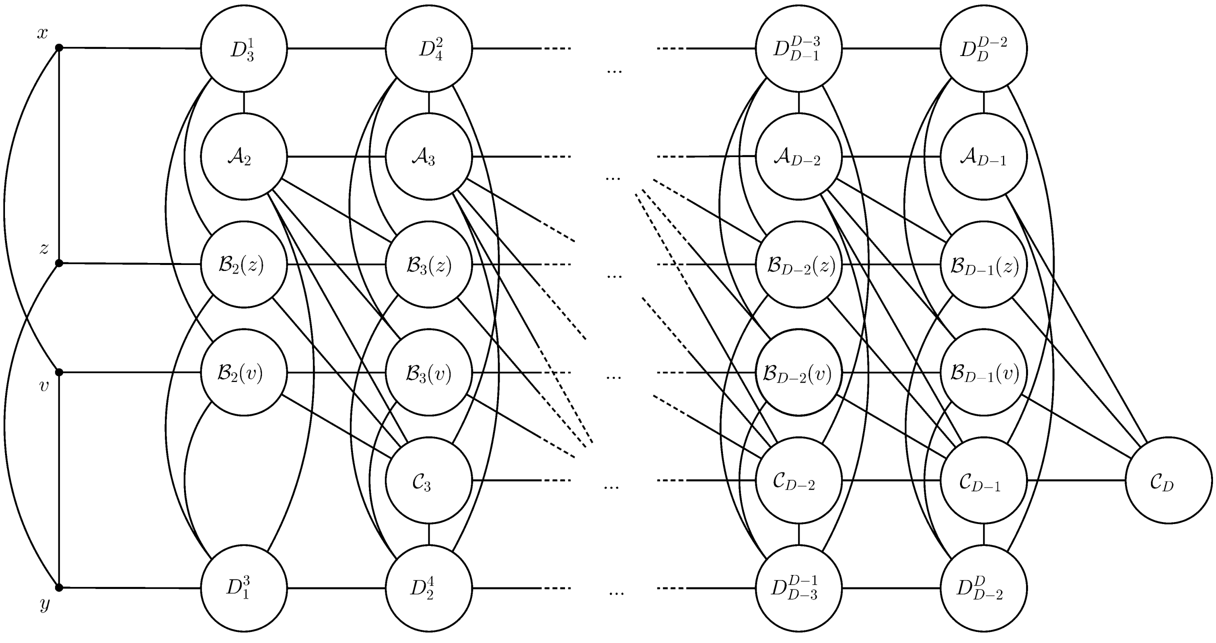

Figure 1: The partition of graph Γ.

Lemma 20. *With reference to Definition 13, the following holds. For each integer* i $(1 \leqslant i \leqslant D-1)$*, each* $w \in D^{i+1}_{i-1}$ *(resp.* D^{i-1}_{i+1}*) is adjacent to*

(a) *precisely* c_{i-1} *vertices in* D^i_{i-2} *(resp.* D^{i-2}_i*),*
(b) *precisely* b_{i+1} *vertices in* D^{i+2}_i *(resp.* D^i_{i+2}*),*
(c) *precisely* $c_i - c_{i-1} - |\Gamma(w) \cap \mathcal{C}_i|$ *vertices in* $\mathcal{B}_i(z)$*,*
(d) *precisely* $c_i - c_{i-1} - |\Gamma(w) \cap \mathcal{C}_i|$ *vertices in* $\mathcal{B}_i(v)$*,*
(e) *precisely* $b_i - b_{i+1} - c_i + c_{i-1} + |\Gamma(w) \cap \mathcal{C}_i|$ *vertices in* $\mathcal{A}_i$*,*
(f) *precisely* $|\Gamma(w) \cap \mathcal{C}_i|$ *vertices in* $\mathcal{C}_i$*,*

and no other vertices in X*.*

Proof. The proof of (a), (b) and (f) is a routine. We now prove (c). We prove (c) for the case $w \in D_{i-1}^{i+1}$. The case $w \in D_{i+1}^{i-1}$ is treated similarly. First note that w is at distance i from z, and so w must have c_i neighbours in $\Gamma_{i-1}(z)$. Observe also that $\Gamma_{i-1}(z) = D_{i-2}^{i} \cup D_{i}^{i-2} \cup \mathcal{B}_i(z) \cup \mathcal{C}_i \cup \mathcal{A}_{i-2} \cup \mathcal{B}_{i-2}(v)$. As w only can have neighbours in $D_{i-2}^{i} \cup \mathcal{B}_i(z) \cup \mathcal{C}_i$, the result follows from (a) above. The proof of (d) is similar, and the proof of (e) is clear as w must have k neighbours. □

Lemma 21. *With reference to Definition 13, the following* (i), (ii) *hold.*

(i) *Vertex v (resp. z) is adjacent to precisely one neighbour in D_2^0, precisely one neighbour in D_0^2, precisely $b_2 = k - 2$ neighbours in $\mathcal{B}_2(v)$ (resp. $\mathcal{B}_2(z)$), and no other vertices in X.*

(ii) *For each integer i ($2 \leqslant i \leqslant D-1$), each $w \in \mathcal{B}_i(v)$ (resp. $\mathcal{B}_i(z)$) is adjacent to*

(a) *precisely*	c_{i-1}	*vertices in $\mathcal{B}_{i-1}(v)$ (resp. $\mathcal{B}_{i-1}(z)$),*
(b) *precisely*	b_{i+1}	*vertices in $\mathcal{B}_{i+1}(v)$ (resp. $\mathcal{B}_{i+1}(z)$),*
(c) *precisely*	$c_i - c_{i-1} - \lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in D_{i+1}^{i-1},*
(d) *precisely*	$c_i - c_{i-1} - \lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in D_{i-1}^{i+1},*
(e) *precisely*	$b_i - b_{i+1} - c_i + c_{i-1} + \lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in $\mathcal{C}_{i+1}$,*
(f) *precisely*	$\lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in $\mathcal{A}_{i-1}$,*

and no other vertices in X.

Proof. (i) This is clear.
(ii) This follows from Lemma 16 and Lemma 20. □

Lemma 22. *With reference to Definition 13, the following holds. For each integer i ($2 \leqslant i \leqslant D-1$), each $w \in \mathcal{A}_i$ is adjacent to*

(a) *precisely*	$\lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in $\mathcal{A}_{i-1}$,*
(b) *precisely*	$c_i - \lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in D_{i-1}^{i+1},*
(c) *precisely*	$c_i - \lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert$	*vertices in D_{i+1}^{i-1},*
(d) *precisely*	$\lvert\Gamma(w) \cap \mathcal{A}_{i+1}\rvert$	*vertices in $\mathcal{A}_{i+1}$,*
(e) *precisely*	$b_{i+1} - \lvert\Gamma(w) \cap \mathcal{A}_{i+1}\rvert$	*vertices in $\mathcal{B}_{i+1}(v)$,*
(f) *precisely*	$b_{i+1} - \lvert\Gamma(w) \cap \mathcal{A}_{i+1}\rvert$	*vertices in $\mathcal{B}_{i+1}(z)$,*
(g) *precisely*	$k - 2c_i - 2b_{i+1} + \lvert\Gamma(w) \cap \mathcal{A}_{i-1}\rvert + \lvert\Gamma(w) \cap \mathcal{A}_{i+1}\rvert$	*vertices in $\mathcal{C}_{i+1}$,*

and no other vertices in X.

Proof. The proof of (a) and (d) is a routine. The proof of (b) (resp. (c)) follows from the fact that $\partial(w, x) = \partial(w, y) = i$, and so w must have c_i neighbours in $\Gamma_{i-1}(x)$ (resp. $\Gamma_{i-1}(y)$). We now prove (e). First note that w is at distance $i+1$ from v, and so w must have b_{i+1} neighbours in $\Gamma_{i+2}(v)$. As $\Gamma_{i+2}(v) \cap \Gamma(w) \subseteq \mathcal{A}_{i+1} \cup \mathcal{B}_{i+1}(z)$, the result follows from (d) above. The proof of (f) is similar, and the proof of (g) is clear as w must have k neighbours. □

Lemma 23. *With reference to Definition 13, the following holds. For each integer i ($3 \leqslant i \leqslant D$), each $w \in \mathcal{C}_i$ is adjacent to*

(a) *precisely*	$\lvert\Gamma(w) \cap \mathcal{C}_{i-1}\rvert$	*vertices in* $\mathcal{C}_{i-1}$,
(b) *precisely*	$c_{i-1} - \lvert\Gamma(w) \cap \mathcal{C}_{i-1}\rvert$	*vertices in* $\mathcal{B}_{i-1}(v)$,
(c) *precisely*	$c_{i-1} - \lvert\Gamma(w) \cap \mathcal{C}_{i-1}\rvert$	*vertices in* $\mathcal{B}_{i-1}(z)$,
(d) *precisely*	$\lvert\Gamma(w) \cap \mathcal{C}_{i+1}\rvert$	*vertices in* $\mathcal{C}_{i+1}$,
(e) *precisely*	$b_i - \lvert\Gamma(w) \cap \mathcal{C}_{i+1}\rvert$	*vertices in* D^{i+1}_{i-1},
(f) *precisely*	$b_i - \lvert\Gamma(w) \cap \mathcal{C}_{i+1}\rvert$	*vertices in* D^{i-1}_{i+1},
(g) *precisely*	$k - 2b_i - 2c_{i-1} + \lvert\Gamma(w) \cap \mathcal{C}_{i+1}\rvert + \lvert\Gamma(w) \cap \mathcal{C}_{i-1}\rvert$	*vertices in* $\mathcal{A}_{i-1}$,

and no other vertices in X.

Proof. This follows from Lemma 16 and Lemma 22. □

6 The partition - part II

We continue to discuss the distance-regular graph $\Gamma = (X, R)$ from Section 5. In this section we further assume Γ is Q-polynomial. We show the partition from Section 5 is equitable, and that the corresponding parameters are independent of x, y.

Lemma 24. *With reference to Definition 13, let E denote a nontrivial primitive idempotent of Γ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then for $1 \leqslant i \leqslant D-1$ and for $w \in D^{i+1}_{i-1} \cup D^{i-1}_{i+1}$,*

$$|\Gamma(w) \cap \mathcal{C}_i| = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})} - c_{i-1} + \frac{\theta^*_1 - \theta^*_3}{\theta^*_{i-1} - \theta^*_{i+1}}.$$

Proof. Assume $w \in D^{i+1}_{i-1}$. If $w \in D^{i-1}_{i+1}$, then the proof is similar. We abbreviate $\tau = |\Gamma(w) \cap \mathcal{C}_i|$. By Lemma 5 we find

$$\sum_{\substack{u \in X \\ \partial(u,v)=i-1 \\ \partial(u,w)=1}} E\hat{u} - \sum_{\substack{u \in X \\ \partial(u,v)=1 \\ \partial(u,w)=i-1}} E\hat{u} = c_i \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(E\hat{v} - E\hat{w}). \tag{7}$$

Observe that beside y, all vertices of the set $\{u \in X \mid \partial(u,v) = 1,\ \partial(u,w) = i-1\}$ are contained in $\mathcal{B}_2(v)$. On the other hand, vertices of the set $\{u \in X \mid \partial(u,v) = i-1,\ \partial(u,w) = 1\}$ are contained in D^i_{i-2} (there is c_{i-1} of these vertices and all are at distance $i-1$ from z), in $\mathcal{C}_i$ (there is τ of these vertices and all are at distance $i-1$ from z), and in $\mathcal{B}_i(v)$ (there is $c_i - c_{i-1} - \tau$ of these vertices and all are at distance $i+1$ from z). Taking the inner product of (7) with $E\hat{z}$, using Lemma 3 and the above comments, we get (after multiplying by $|V\Gamma|$)

$$c_{i-1}\theta^*_{i-1} + \tau\theta^*_{i-1} + (c_i - c_{i-1} - \tau)\theta^*_{i+1} - \theta^*_1 - (c_i - 1)\theta^*_3 = c_i \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(\theta^*_2 - \theta^*_i).$$

Evaluating the above line using $\theta^*_{i-1} \neq \theta^*_{i+1}$ we obtain

$$\tau = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})} - c_{i-1} + \frac{\theta^*_1 - \theta^*_3}{\theta^*_{i-1} - \theta^*_{i+1}}.$$

The assertion now follows. □

Lemma 25. *With reference to Definition 13, let E denote a nontrivial primitive idempotent of Γ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then for $2 \leqslant i \leqslant D-1$ and for $w \in \mathcal{B}_i(z) \cup \mathcal{B}_i(v)$,*

$$|\Gamma(w) \cap \mathcal{A}_{i-1}| = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})} - c_{i-1} + \frac{\theta^*_1 - \theta^*_3}{\theta^*_{i-1} - \theta^*_{i+1}}.$$

Proof. This follows from Lemma 16 and Lemma 24. □

Lemma 26. *With reference to Definition 13, let E denote a nontrivial primitive idempotent of Γ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then for $2 \leqslant i \leqslant D-1$ and for $w \in \mathcal{A}_i$ the following* (i), (ii) *hold.*

(i)

$$|\Gamma(w) \cap \mathcal{A}_{i-1}| = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})}.$$

(ii)

$$|\Gamma(w) \cap \mathcal{A}_{i+1}| = b_{i+1} \frac{(\theta^*_0 - \theta^*_{i+1})(\theta^*_3 - \theta^*_i) - (\theta^*_1 - \theta^*_{i+2})(\theta^*_2 - \theta^*_{i+1})}{(\theta^*_0 - \theta^*_{i+1})(\theta^*_{i+2} - \theta^*_i)}.$$

Proof. (i) We abbreviate $\tau = |\Gamma(w) \cap \mathcal{A}_{i-1}|$. By Lemma 5 we find

$$\sum_{\substack{u \in X \\ \partial(u,x)=i-1 \\ \partial(u,w)=1}} E\hat{u} - \sum_{\substack{u \in X \\ \partial(u,x)=1 \\ \partial(u,w)=i-1}} E\hat{u} = c_i \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(E\hat{x} - E\hat{w}). \tag{8}$$

Observe that all vertices of the set $\{u \in X \mid \partial(u,x) = 1,\ \partial(u,w) = i-1\}$ are contained in D^1_3. On the other hand, vertices of the set $\{u \in X \mid \partial(u,x) = i-1,\ \partial(u,w) = 1\}$ are contained in $\mathcal{A}_{i-1}$ (there is τ of these vertices and all are at distance $i-1$ from y), and in D^{i-1}_{i+1} (there is $c_i - \tau$ of these vertices and all are at distance $i+1$ from y). Taking the inner product of (8) with $E\hat{y}$, using Lemma 3 and the above comments, we get (after multiplying by $|V\Gamma|$)

$$\tau\theta^*_{i-1} + (c_i - \tau)\theta^*_{i+1} - c_i\theta^*_3 = c_i \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(\theta^*_2 - \theta^*_i).$$

Evaluating the above line using $\theta^*_{i-1} \neq \theta^*_{i+1}$ we obtain

$$\tau = c_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i+1}) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_{i+1})}.$$

The assertion now follows.

(ii) We abbreviate $\tau = |\Gamma(w) \cap \mathcal{A}_{i+1}|$. By Lemma 5 we find

$$\sum_{\substack{u \in X \\ \partial(u,v)=i+2 \\ \partial(u,w)=1}} E\hat{u} - \sum_{\substack{u \in X \\ \partial(u,v)=1 \\ \partial(u,w)=i+2}} E\hat{u} = b_{i+1} \frac{\theta^*_{i+2} - \theta^*_1}{\theta^*_0 - \theta^*_{i+1}} (E\hat{v} - E\hat{w}). \tag{9}$$

Observe that all vertices of the set $\{u \in X \mid \partial(u,v) = 1,\ \partial(u,w) = i+2\}$ are contained in $\mathcal{B}_2(v)$. On the other hand, vertices of the set $\{u \in X \mid \partial(u,v) = i+2,\ \partial(u,w) = 1\}$ are contained in $\mathcal{A}_{i+1}$ (there is τ of these vertices and all are at distance $i+2$ from z), and in $\mathcal{B}_{i+1}(z)$ (there is $b_{i+1} - \tau$ of these vertices and all are at distance i from z). Taking the inner product of (9) with $E\hat{z}$, using Lemma 3 and the above comments, we get (after multiplying by $|V\Gamma|$)

$$\tau\theta^*_{i+2} + (b_{i+1} - \tau)\theta^*_i - b_{i+1}\theta^*_3 = b_{i+1} \frac{\theta^*_{i+2} - \theta^*_1}{\theta^*_0 - \theta^*_{i+1}} (\theta^*_2 - \theta^*_{i+1}).$$

Evaluating the above line using $\theta^*_i \neq \theta^*_{i+2}$ we obtain

$$\tau = b_{i+1} \frac{(\theta^*_0 - \theta^*_{i+1})(\theta^*_3 - \theta^*_i) - (\theta^*_1 - \theta^*_{i+2})(\theta^*_2 - \theta^*_{i+1})}{(\theta^*_0 - \theta^*_{i+1})(\theta^*_{i+2} - \theta^*_i)}.$$

The assertion now follows. □

Lemma 27. *With reference to Definition 13, let E denote a nontrivial primitive idempotent of Γ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then for $3 \leqslant i \leqslant D$ and for $w \in \mathcal{C}_i$ the following* (i), (ii) *hold.*

(i)
$$|\Gamma(w) \cap \mathcal{C}_{i-1}| = c_{i-1} \frac{(\theta^*_0 - \theta^*_{i-1})(\theta^*_3 - \theta^*_i) - (\theta^*_1 - \theta^*_{i-2})(\theta^*_2 - \theta^*_{i-1})}{(\theta^*_0 - \theta^*_{i-1})(\theta^*_{i-2} - \theta^*_i)}.$$

(ii)
$$|\Gamma(w) \cap \mathcal{C}_{i+1}| = b_i \frac{(\theta^*_0 - \theta^*_i)(\theta^*_3 - \theta^*_{i-1}) - (\theta^*_1 - \theta^*_{i+1})(\theta^*_2 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i+1} - \theta^*_{i-1})},$$

where $\mathcal{C}_{D+1} = \emptyset$.

Proof. This follows from Lemma 16 and Lemma 26. □

Theorem 28. *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 3$ and intersection number $c_2 = 2$. Then with reference to Definition 13, the partition of $V\Gamma$ into nonempty sets $D^{i-1}_{i+1}, D^{i+1}_{i-1}$ $(1 \leqslant i \leqslant D-1)$, $\mathcal{A}_i$ $(2 \leqslant i \leqslant D-1)$, $\mathcal{B}_i(z), \mathcal{B}_i(v)$ $(1 \leqslant i \leqslant D-1)$ and $\mathcal{C}_i$ $(3 \leqslant i \leqslant D)$ is equitable. Moreover the corresponding parameters are independent of x, y.*

Proof. Immediate from Lemma 20, Lemma 21, Lemma 22, Lemma 23, Lemma 24, Lemma 25, Lemma 26, and Lemma 27. □

7 Case $D = 4$

In this section we consider Q-polynomial bipartite distance-regular graph Γ with intersection number $c_2 \leqslant 2$, valency $k \geqslant 3$ and diameter $D = 4$. We show that Γ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube. For the case $c_2 = 1$ we have the following result.

Theorem 29. ([6, Theorem 6.1]) *There does not exist a Q-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geqslant 3$ and intersection number $c_2 = 1$.*

From now on we assume $c_2 = 2$.

Lemma 30. *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geqslant 3$ and intersection number $c_2 = 2$. With reference to Definition 13 the following* (i), (ii) *hold.*

(i) $|\mathcal{A}_2| = (k-2)(c_3-3)/2$;

(ii) $c_3 \geqslant 4$ *if and only if* $\mathcal{A}_2 \neq \emptyset$.

Proof. (i) Immediately from Lemma 17(iii) and Lemma 2(ii).
(ii) Immediately from (i) above. □

Lemma 31. *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D = 4$ and intersection numbers $c_2 = 2$, $k \geqslant c_3 \geqslant 4$. Assume Γ is not the 4-dimensional hypercube or the antipodal quotient of the 8-dimensional hypercube. With reference to Definition 13, pick $w \in \mathcal{A}_2$ and let λ denote the number of neighbours of w in $\mathcal{A}_3$. Then the following* (i), (ii) *hold.*

(i)
$$\lambda = \frac{(k-2)b_3(b_3-1)}{(k-2)(k-3)-2b_3}.$$

(ii) $(k-2)(k-3)-2b_3$ *divides* $(k-2)b_3(b_3-1)$.

Proof. (i) Let scalars s^*, q be as in Lemma 6. First note that by Lemma 6(iii) we have

$$c_2 - 2 = -\frac{(q-1)(q^{10}(s^*)^2 + s^*(q^7 + 2q^6 - 2q^4 - q^3) - 1)}{(1 - s^*q^5)(1 - s^*q^6)},$$

which implies

$$h(q, s^*) = q^{10}(s^*)^2 + s^*(q^7 + 2q^6 - 2q^4 - q^3) - 1 = 0. \tag{10}$$

By Lemma 26 we have

$$\lambda = b_3 \frac{(\theta_0^* - \theta_3^*)(\theta_3^* - \theta_2^*) - (\theta_1^* - \theta_4^*)(\theta_2^* - \theta_3^*)}{(\theta_0^* - \theta_3^*)(\theta_4^* - \theta_2^*)},$$

and by Lemma 6(ii),(iii) we find

$$\lambda = \frac{q^3(1 - s^*q^3)(1 - s^*q^5)}{(1 - s^*q^7)^2}. \tag{11}$$

Consider now the number

$$\frac{\lambda(k^2 - 5k + 4)}{b_3 - 1} - \frac{\lambda(k^2 - 5k + 6)}{b_3} - k + 2. \tag{12}$$

Note that $b_3 \neq 1$. Indeed, if $b_3 = 1$, then by Lemma 6(i),(iii) we have $s^*q^5 = -1$, and so $c_2 = (q^2 + 1)^2/(2q^2)$. But now $c_2 = 2$ implies $q = \pm 1$, a contradiction. Using Lemma 6 we find that (12) is equal to

$$\alpha \cdot (q^{10}(s^*)^2 + s^*(q^7 + 2q^6 - 2q^4 - q^3) - 1) = \alpha \cdot h(q, s^*),$$

where

$$\alpha = \frac{(s^*)^2(q^{12} - 2q^{11} - q^{10}) + s^*(q^9 + q^8 + q^7 - 2q^6 + q^5 + q^4 + q^3) - q^2 - 2q + 1}{(1 - s^*q^4)(1 + s^*q^5)(1 - s^*q^6)(1 - s^*q^7)}.$$

By (10) we therefore have

$$\lambda = \frac{(k-2)b_3(b_3 - 1)}{(k-2)(k-3) - 2b_3}.$$

(ii) This follows immediately from (i) above. □

Lemma 32. *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D = 4$ and intersection numbers $c_2 = 2$, $k \geqslant c_3 \geqslant 4$. Assume Γ is not the 4-dimensional hypercube or the antipodal quotient of the 8-dimensional hypercube. With reference to Definition 13, let λ be as in Lemma 31. Then the following* (i), (ii) *hold.*

(i) *Each vertex in $\mathcal{B}_3(v)$ has exactly*

$$\frac{(c_3 - 3)(b_3 - \lambda)}{b_3}$$

neighbours in $\mathcal{A}_2$.

(ii) $(k-2)(k-3)-2b_3$ *divides* $(k-4)b_3(b_3-1)$.

Proof. (i) By Lemma 17(ii),(iii) and Lemma 2 we find

$$|\mathcal{A}_2| = \frac{(k-2)(c_3-3)}{2}, \qquad |\mathcal{B}_3(v)| = \frac{b_3(k-2)}{2}.$$

By Lemma 22(e), every vertex from $\mathcal{A}_2$ has $b_3-\lambda$ neighbours in $\mathcal{B}_3(v)$. The result follows from the above comments and by counting the edges between $\mathcal{A}_2$ and $\mathcal{B}_3(v)$ in two different ways.

(ii) Consider the number $(c_3-3)(b_3-\lambda)/b_3$. Observe that, by Lemma 31(i), we have

$$\frac{(c_3-3)(b_3-\lambda)}{b_3} = k-2b_3-2+\frac{b_3(b_3-1)(k-4)}{(k-2)(k-3)-2b_3}.$$

As $(c_3-3)(b_3-\lambda)/b_3$ is integer by (i) above, the result follows. □

Lemma 33. *Let* Γ *denote a* Q*-polynomial bipartite distance-regular graph with diameter* $D=4$ *and intersection numbers* $c_2=2$, $k \geqslant c_3 \geqslant 4$. *Assume* Γ *is not the* 4*-dimensional hypercube or the antipodal quotient of the* 8*-dimensional hypercube. Let* λ *be as in Lemma 31. Then the following* (i)-(iii) *hold.*

(i) $(k-2)(k-3)-2b_3$ *divides* $2b_3(b_3-1)$;

(ii) $(k-2)(k-3)=2b_3^2$;

(iii) $\lambda=(k-2)/2$.

Proof. (i) Immediately from Lemma 31(ii) and Lemma 32(ii).

(ii) It follows from (i) above that $2b_3(b_3-1)=\ell((k-2)(k-3)-2b_3)$ for some nonnegative integer ℓ. We will show that $\ell=1$. If $\ell=0$, then $b_3=1$. By Lemma 6(i),(iii) we have $s^*q^5=-1$, and so $c_2=(q^2+1)^2/(2q^2)$. But now $c_2=2$ implies $q=\pm 1$, a contradiction. Therefore, $\ell \geqslant 1$. Assume $\ell \geqslant 2$. Then $2b_3(b_3-1) \geqslant 2((k-2)(k-3)-2b_3)$, which implies $(k-2)(k-3) \leqslant b_3(b_3+1)$. Recall that $c_3 \geqslant 4$, and so $b_3 \leqslant k-4$. But then $(k-2)(k-3) \leqslant b_3(b_3+1) \leqslant (k-4)(k-3)$, a contradiction. Therefore $2b_3(b_3-1)=(k-2)(k-3)-2b_3$ and the result follows.

(iii) Immediately from Lemma 31(i) and (ii) above. □

Lemma 34. *Let* Γ *denote a* Q*-polynomial bipartite distance-regular graph with diameter* $D=4$, *and intersection numbers* $c_2=2$, $k \geqslant c_3 \geqslant 4$. *Assume* Γ *is not the* 4*-dimensional hypercube or the antipodal quotient of the* 8*-dimensional hypercube. Then the following* (i), (ii) *hold.*

(i) $q=-(\sqrt{5}+3)/2$.

(ii) $s^*=72\sqrt{5}-161$.

Proof. (i) Let λ be as in Lemma 31. By (11) and by Lemma 33(iii) we find

$$\frac{k-2}{2} - \frac{q^3(1-s^*q^3)(1-s^*q^5)}{(1-s^*q^7)^2} = 0.$$

Observe that by Lemma 6(iii) we have

$$\frac{k-2}{2} - \frac{q^3(1-s^*q^3)(1-s^*q^5)}{(1-s^*q^7)^2} = \frac{(q-1)^2(q+1)f(q,s^*)}{2(1-s^*q^6)(1-s^*q^7)^2},$$

where

$$f(q,s^*) = q^{17}(s^*)^3 + q^{10}(s^*)^2(q^4+2q^3+4q^2+2q+2) - q^3s^*(2q^4+2q^3+4q^2+2q+1) - 1.$$

By Lemma 6 and comments above, we have $f(q,s^*)=0$. Recall polynomial $h(q,s^*)$ from (10). Recall also that $h(q,s^*)=0$. Note that

$$f(q,s^*) = h(q,s^*)(q^7s^*+4q^2+4q+3) - 2(q^3s^*(2q^6+6q^5+6q^4-4q^2-4q-1)-2q^2-2q-1).$$

As $f(q,s^*)=h(q,s^*)=0$, we also have $q^3s^*(2q^6+6q^5+6q^4-4q^2-4q-1)-2q^2-2q-1=0$, and so

$$s^* = \frac{2q^2+2q+1}{q^3(2q^6+6q^5+6q^4-4q^2-4q-1)}. \tag{13}$$

Using (13) together with $h(q,s^*)=0$ we get

$$-\frac{2(q-1)q^2(q+1)(q^2+q+1)^2(q^2+3q+1)}{(2q^6+6q^5+6q^4-4q^2-4q-1)^2} = 0.$$

As by Lemma 6 q is real and $|q|>1$, we get $q=-(\sqrt{5}+3)/2$.
(ii) Immediately from (13) and (i) above. □

Theorem 35. *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D=4$, valency $k \geqslant 3$ and intersection number $c_2=2$. Then Γ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.*

Proof. Assume first that $c_3 \geqslant 4$. Then by Lemma 34 we have $q=-(\sqrt{5}+3)/2$ and $s^*=72\sqrt{5}-161$. Lemma 6(iii) now implies $k=-6$, a contradiction. Therefore $c_3=3$. But now [4, Theorem 4.6] implies that Γ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube. □

We finish the paper with the proof of our main theorem.

Proof of Theorem 1. Immediately from Theorem 7, Theorem 29 and Theorem 35. □

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