Characterizations of regularity for certain Q-polynomial association schemes

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Abstract

It was shown that linked systems of symmetric designs with $a_1^*=0$ and mutually unbiased bases (MUBs) are triply regular association schemes. In this paper, we characterize triple regularity of linked systems of symmetric designs by its Krein number. And we prove that a maximal set of MUBs carries a quadruply regular association scheme and characterize the quadruple regularity of MUBs by its parameter.

Keywords: Q-polynomial association scheme; linked systems of symmetric designs; real mutually unbiased bases; quadruple regularity

1 Introduction

Q-polynomial association schemes are defined by Delsarte in [5] as a framework to study design theory including such as combinatorial t-designs or orthogonal arrays. As a continuous analogue of designs in Q-polynomial association schemes, Delsarte, Goethals and Seidel introduced the concept of spherical designs in [6]. Several combinatorial designs, spherical designs and mutually unbiased bases, which is considered in quantum information theory, have the structure of Q-polynomial association schemes of small class as follows: symmetric designs and linked systems of symmetric designs for 3 class Q-antipodal case [4, 8, 11], certain equiangular line sets for 3 class Q-bipartite case [8, 11], real mutually unbiased bases for 4 class, Q-antipodal and Q-bipartite case [1, 7, 11]. Much recent effort has focused on these imprimitive families and the research of imprimitive Q-polynomial association schemes of small class is important for design theory.

It was shown in [11] that 3-class Q-antipodal association schemes satisfying $a_1^*=0$ and 4-class association schemes which are both Q-antipodal and Q-bipartite are triply regular.

In this paper we characterize the triple regularity for 3-class Q-antipodal association schemes and the quadruple regularity for 4-class association schemes which are both Q-antipodal and Q-bipartite.

The present paper is organized as follows. In Section 2, we prepare the notion and lemmas on association schemes, spherical designs and triple and quadruple regularity for association schemes needed later.

In Section 3, we consider linked systems of symmetric designs. Systems of projective designs, that were defined by Cameron [3], are the combinatorial object of finite doubly transitive groups which have more than two pairwise inequivalent permutation representations with the same permutation character. We call it linked systems of symmetric designs, if symmetric designs appearing in systems of projective designs have all same parameters. Noda [10] showed several inequalities concerning the parameters of linked systems of symmetric designs and Mathon [9] showed every linked system of symmetric designs carries a 3-class association scheme and calculated its eigenmatrices. It implies that these association schemes are Q-polynomial with the Q-antipodal property. Conversely van Dam [4] showed every 3-class Q-antipodal association scheme arises from a linked system of symmetric designs. The author [11] proved that every linked system of symmetric designs with $a_1^* = 0$ is a triply regular association scheme. In this section we will show the converse proposition, that is, if a linked system of symmetric designs is triply regular, then $a_1^* = 0$. This proof is essentially due to [10, Theorem 2].

In Section 4, we consider the real mutually unbiased bases (MUBs). One important problem of real MUBs is to determine the maximal number of real MUBs in $\mathbb{R}^d$. It is well known that its number is at most $d/2 + 1$. A set of real MUBs is said to be maximal if equality holds. Recently Martin et al. [7] showed that there is a one-to-one correspondence between real MUBs and 4-class association schemes which are both Q-bipartite and Q-antipodal. Moreover Martin et al. [8] had shown that a 4-class association schemes which are both Q-bipartite and Q-antipodal is obtained by the extended Q-bipartite double of a linked system of symmetric design with certain parameters. The author proved in [11] that every set of MUBs carries a triply regular association scheme. The main theorem in this section is that a set of MUBs carries a quadruply regular association scheme if and only if a set of MUBs is maximal.

Finally, in Section 5, we discuss the quadruple regularity of linked systems of symmetric designs.

2 Preliminaries

Let X be a finite set, we define $\mathrm{diag}(X \times X) = \{(x, x) \mid x \in X\}$. Let $\{R_i\}_{i \in I}$ be a set of relations on X, we define $R_i^t = \{(y, x) \mid (x, y) \in R_i\}$. A pair $(X, \{R_i\}_{i \in I})$ is a coherent configuration if the following properties are satisfied:

1. $\{R_i\}_{i \in I}$ is a partition of $X \times X$,

2. $R_i^t = R_{i^*}$ for some $i^* \in I$,

3. $R_i \cap \mathrm{diag}(X \times X) \neq \emptyset$ implies $R_i \subset \mathrm{diag}(X \times X)$,

4. for $i, j, k \in I$, the number $|\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}|$ is independent of the choice of $(x,y) \in R_k$.

If moreover $R_0 = \mathrm{diag}(X \times X)$ and $i^* = i$ for all $i \in I$, then we call $(X, \{R_i\}_{i \in I})$ a symmetric association scheme.

Let S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. Let $X_1, \ldots, X_n$ be finite subsets of S^{d-1}. We denote by $\coprod_{i=1}^n X_i$ the disjoint union of $X_1, \ldots, X_n$. We denote by $\langle x, y \rangle$ the inner product of $x, y \in \mathbb{R}^d$. We define the nontrivial angle set $A(X_i, X_j)$ between X_i and X_j by

$$A(X_i, X_j) = \{\langle x, y \rangle \mid x \in X_i, y \in X_j, x \neq \pm y\},$$

and the angle set $A'(X_i, X_j)$ between X_i and X_j by

$$A'(X_i, X_j) = \{\langle x, y \rangle \mid x \in X_i, y \in X_j, x \neq y\}.$$

If $i = j$, then $A(X_i, X_i)$ (resp. $A'(X_i, X_i)$) is abbreviated $A(X_i)$ (resp. $A'(X_i)$).

We define the intersection numbers on X_j for $x, y \in S^{d-1}$ by

$$p^j_{\alpha,\beta}(x,y) = |\{z \in X_j \mid \langle x, z \rangle = \alpha, \langle y, z \rangle = \beta\}|.$$

For a positive integer t, a finite non-empty set X in the unit sphere S^{d-1} is called a spherical t-design in S^{d-1} if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) d\sigma(x)$$

for all polynomials $f(x) = f(x_1, \ldots, x_d)$ of degree not exceeding t. Here $|S^{d-1}|$ denotes the volume of the sphere S^{d-1}. When X is a t-design and not a $(t+1)$-design, we call t its strength.

We define the Gegenbauer polynomials $\{Q_k(x)\}_{k=0}^{\infty}$ on S^{d-1} by

$$Q_0(x) = 1, \quad Q_1(x) = dx,$$
$$\frac{k+1}{d+2k} Q_{k+1}(x) = x Q_k(x) - \frac{d+k-3}{d+2k-4} Q_{k-1}(x).$$

A criterion for t-designs using Gegenbauer polynomials is known [6, Theorem 5.3, 5.5].

Lemma 1. *Let X be a finite set in S^{d-1}. The following conditions are equivalent:*

1. X is a t-design,

2. $\sum_{x,y \in X} Q_k(\langle x, y \rangle) = 0$ for any $k \in \{1, \ldots, t\}$.

We define $\{f_{\lambda,l}\}_{l=0}^{\lambda}$ as the coefficients of Gegenbauer expansion of x^λ for any nonnegative integers λ, i.e., $x^\lambda = \sum_{l=0}^{\lambda} f_{\lambda,l} Q_l(x)$, and let $F_{\lambda,\mu}(x) = \sum_{l=0}^{\min\{\lambda,\mu\}} f_{\lambda,l} f_{\mu,l} Q_l(x)$, where λ, μ are nonnegative integers.

Next, we consider triple or quadruple regularity of a symmetric association scheme.

Definition 2. Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. Then the association scheme X is said to be triply regular if, for all $i, j, k, l, m, n \in \{0, 1, \dots, d\}$, and for all $x, y, z \in X$ such that $(x, y) \in R_i, (y, z) \in R_j, (z, x) \in R_k$, the number $p_{l,m,n}^{i,j,k} := |R_m(x) \cap R_n(y) \cap R_l(z)|$ depends only on i, j, k, l, m, n and not on x, y, z. We call $p_{l,m,n}^{i,j,k}$ triple intersection numbers.

Definition 3. Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. Then the association scheme X is said to be quadruply regular if, for all $I = (i_1, i_2, i_3, i_4) \subset \{0, 1, \dots, d\}^4$, $J = (j_{\alpha,\beta})_{1 \leqslant \alpha < \beta \leqslant 4} \subset \{0, 1, \dots, d\}^6$ and $x_1, \dots, x_4 \in X$ such that $(x_k, x_l) \in R_{j_{k,l}}$ for any $1 \leqslant k < l \leqslant 4$, the number

$$|R_{i_1}(x_1) \cap R_{i_2}(x_2) \cap R_{i_3}(x_3) \cap R_{i_4}(x_4)|$$

depends only on I, J and not on $x_1, \dots, x_4$.

Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. We define the i-th subconstituent with respect to $z \in X$ by $R_i(z) := \{y \in X \mid (z, y) \in R_i\}$ and the (i, j)-th subconstituent with respect to $(z_1, z_2) \in X \times X$ by $R_{i,j}(z_1, z_2) := R_i(z_1) \cap R_j(z_2)$. We denote by $R_{i,j,k,l}^{m,n}(z_1, z_2)$ the restriction R_n to $R_{i,j}(z_1, z_2) \times R_{k,l}(z_1, z_2)$ for $(z_1, z_2) \in R_m$. Quadruple regularity is characterized by the concept of coherent configuration. We omit easy proof of the following lemma.

Lemma 4. *A symmetric association scheme $(X, \{R_i\}_{i=0}^d)$ is quadruply regular if and only if $(X, \{R_i\}_{i=0}^d)$ is triply regular and for all $m \in \{1, \dots, d\}$ and $z_1, z_2 \in X$ with $(z_1, z_2) \in R_m$,*

$$(\bigcup_{i,j=1}^d R_{i,j}(z_1, z_2), \{R_{i,j,k,l}^{m,n}(z_1, z_2) \mid 1 \leqslant i, j, k, l, n \leqslant d, p_{i,j,m}^{l,k,n} \neq 0\})$$

is a coherent configuration whose parameters depend only on m, not on the choice of z_1, z_2 with $(z_1, z_2) \in R_m$.

Remark 5. For $(z_1, z_2) \in R_m$ with $m = 0$ namely $z_1 = z_2$,

$$\begin{aligned}&(\bigcup_{i,j=1}^d R_{i,j}(z_1, z_2), \{R_{i,j,k,l}^{m,n}(z_1, z_2) \mid 1 \leqslant i, j, k, l, m \leqslant d, p_{i,j,m}^{l,k,n} \neq 0\}) \\ &= (\bigcup_{i=1}^d R_i(z_1), \{R_{i,j}^k(z_1) \mid 1 \leqslant i, j \leqslant d, 0 \leqslant k \leqslant d, p_{i,j}^k \neq 0\})\end{aligned}$$

holds. The condition that $(\bigcup_{i,j=1}^d R_{i,j}(z_1, z_2), \{R_{i,j,k,l}^{m,n}(z_1, z_2) \mid 1 \leqslant i, j, k, l, m \leqslant d, p_{i,j,m}^{l,k,n} \neq 0\})$ is a coherent configuration whose parameters depend only on m, not on the choice of z_1, z_2 with $(z_1, z_2) \in R_m$ with $m = 0$ is equivalent that the association scheme is triply regular.

Let X be a finite subset in S^{d-1} with degree s, and $A(X)=\{\alpha_1,\dots,\alpha_s\}$. For $z_1,z_2\in X$ with $\langle z_1,z_2\rangle=\alpha_m\neq\pm1$, $X_{i,j}^m=X_{i,j}^m(z_1,z_2)$ will denote the orthogonal projection of $\{y\in X\mid\langle y,z_1\rangle=\alpha_i,\langle y,z_2\rangle=\alpha_j\}$ to $\langle z_1,z_2\rangle^\perp=\{y\in\mathbb{R}^d\mid\langle y,z_1\rangle=\langle y,z_1\rangle=0\}$, rescaled to lie in S^{d-3}. If $\langle x,z_1\rangle=\alpha_i$, $\langle x,z_2\rangle=\alpha_j$, $\langle y,z_1\rangle=\alpha_k$, $\langle y,z_2\rangle=\alpha_l$ and $\langle x,y\rangle=\alpha_n$, then the inner product of the orthogonal projections of x,y to $\langle z_1,z_2\rangle^\perp$ rescaled to lie in S^{d-3} is

$$\alpha_{i,j,k,l}^{m,n}:=\frac{(\alpha_n-\alpha_i\alpha_k)(1-\alpha_m^2)-(\alpha_j-\alpha_i\alpha_m)(\alpha_l-\alpha_k\alpha_m)}{\sqrt{(1-\alpha_i^2-\alpha_j^2-\alpha_m^2+2\alpha_i\alpha_j\alpha_m)(1-\alpha_k^2-\alpha_l^2-\alpha_m^2+2\alpha_k\alpha_l\alpha_m)}}.$$

We denote $p_{\alpha,\beta}^{(i,j,m)}(x,y)=|\{z\in X_{i,j}^m\mid\langle x,z\rangle=\alpha,\langle y,z\rangle=\beta\}|$.

Lemma 6. *Let $X\subset S^{d-1}$ be a finite set and $A'(X)=\{\alpha_1,\dots,\alpha_s\}$. Assume that $(X,\{R_k\}_{k=0}^s)$ is a symmetric association scheme, where $R_k=\{(x,y)\in X\times X\mid\langle x,y\rangle=\alpha_k\}$ $(0\leqslant k\leqslant s)$ and $\alpha_0=1$. Then $|\{(i,j)\in\{1,\dots,s\}^2\mid X_{i,j}^m(z_1,z_2)\neq\emptyset\}|=|\{(i,j)\in\{1,\dots,s\}^2\mid p_{i,j}^m\neq0\}|$ for $\langle z_1,z_2\rangle=\alpha_m$.*

Proof. Immediate from definition. □

The following theorem is used to prove Corollary 9.

Theorem 7 ([11, Theorem 2.6])**.** *Let $X_i\subset S^{d-1}$ be a spherical t_i-design for $i\in\{1,\dots,n\}$. Assume that $X_i\cap X_j=\emptyset$ or $X_i=X_j$, and $X_i\cap(-X_j)=\emptyset$ or $X_i=-X_j$ for $i,j\in\{1,\dots,n\}$. Let $s_{i,j}=|A(X_i,X_j)|$, $s_{i,j}^*=|A'(X_i,X_j)|$ and $A(X_i,X_j)=\{\alpha_{i,j}^1,\dots,\alpha_{i,j}^{s_{i,j}}\}$, $\alpha_{i,j}^0=1$, when $-1\in A'(X_i,X_j)$, we define $\alpha_{i,j}^{s_{i,j}^*}=-1$. We define $R_{i,j}^k=\{(x,y)\in X_i\times X_j\mid\langle x,y\rangle=\alpha_{i,j}^k\}$. If one of the following holds depending on the choice of $i,j,k\in\{1,\dots,n\}$:*

1. *$s_{i,j}+s_{j,k}-2\leqslant t_j$,*
2. *$s_{i,j}+s_{j,k}-3=t_j$ and for any $\gamma\in A(X_i,X_k)$ there exist $\alpha\in A(X_i,X_j),\beta\in A(X_j,X_k)$ such that the number $p_{\alpha,\beta}^j(x,y)$ is independent of the choice of $x\in X_i,y\in X_k$ with $\gamma=\langle x,y\rangle$,*
3. *$s_{i,j}+s_{j,k}-4=t_j$ and for any $\gamma\in A(X_i,X_k)$ there exist $\alpha,\alpha'\in A(X_i,X_j),\beta,\beta'\in A(X_j,X_k)$ such that $\alpha\neq\alpha'$, $\beta\neq\beta'$ and the numbers $p_{\alpha,\beta}^j(x,y)$, $p_{\alpha,\beta'}^j(x,y)$ and $p_{\alpha',\beta}^j(x,y)$ are independent of the choice of $x\in X_i,y\in X_k$ with $\gamma=\langle x,y\rangle$,*

then $(\coprod_{i=1}^n X_i,\{R_{i,j}^k\mid 1\leqslant i,j\leqslant n,1-\delta_{X_i,X_j}\leqslant k\leqslant s_{i,j}^*\})$ is a coherent configuration. The parameters of this coherent configuration are determined by $A(X_i,X_j)$, $|X_i|$, t_i, δ_{X_i,X_j}, $\delta_{X_i,-X_j}$, and when $s_{i,j}+s_{j,k}-3=t_j$ (resp. $s_{i,j}+s_{j,k}-4=t_j$), the numbers $p_{\alpha,\beta}^j(x,y)$ (resp. $p_{\alpha,\beta}^j(x,y)$, $p_{\alpha',\beta}^j(x,y)$, $p_{\alpha,\beta'}^j(x,y)$) which are assumed be independent of (x,y) with $\langle x,y\rangle=\gamma$.

The following lemma shows the antipodal double cover of coherent configurations obtained from finite subsets in S^{d-1} are also coherent configurations.

Lemma 8. *Let $X_i^+, X_i^- \subset S^{d-1}$ be a finite subset such that $X_i^+ = -X_i^-$ for $i \in \{1, \ldots, n\}$. If $\{X_i^+\}_{i=1}^n$ carries a coherent configuration, then $\{X_i^+, X_i^-\}_{i=1}^n$ carries also a coherent configuration.*

Proof. We define $X_i^\varepsilon(x, \alpha) = \{w \in X_i^\varepsilon \mid \langle x, w\rangle = \alpha\}$, and $X_i^\varepsilon(x, \alpha; y, \beta) = X_i^\varepsilon(x, \alpha) \cap X_i^\varepsilon(y, \beta)$ for $x \in S^{d-1}$, $\varepsilon = +$ or $-$. Then the following equalities hold:

1. $X_i^+(x, -\alpha) = X_i^+(-x, \alpha)$,
2. $X_i^+(x, \alpha) = -X_i^-(x, -\alpha)$.

By (1), $X_i^+(x, \alpha; y, \beta) = X_i^+(-x, -\alpha; y, \beta) = X_i^+(x, \alpha; -y, -\beta)$ holds. Then by (2), $X_i^+(x, \alpha; y, \beta) = -X_i^-(x, -\alpha; y, -\beta)$ holds. Therefore

$$|X_i^+(x, \alpha; y, \beta)| = |X_i^+(-x, -\alpha; y, \beta)| = |X_i^+(x, \alpha; -y, -\beta)| = |X_i^-(x, -\alpha; y, -\beta)|$$

holds. This implies that intersection numbers on $\{X_i^+, X_i^-\}_{i=1}^n$ is determined by the coherent configuration $\{X_i^+\}_{i=1}^n$. □

The following corollary gives a sufficient condition for association schemes obtained from an antipodal finite subset of sphere to be quadruple regular of triply regular. Its proof follows from the same argument of [11, Corollary 2.9].

Corollary 9. *Let $X \subset S^{d-1}$ be an antipodal finite subset and $A'(X) = \{\alpha_1, \ldots, \alpha_s\}$ with $\alpha_1 > \cdots > \alpha_s = -1$. Assume that $(X, \{R_k\}_{k=0}^s)$ is a triply regular symmetric association scheme, where $R_k = \{(x, y) \in X \times X \mid \langle x, y\rangle = \alpha_k\}$ $(0 \leqslant k \leqslant s)$ and $\alpha_0 = 1$. Then for $1 \leqslant i, j, k, l, m \leqslant s - 1$ such that $p_{i,j}^m \neq 0$ and $p_{k,l}^m \neq 0$,*

1. *$A(X_{i,j}^m(z_1, z_2), X_{k,l}^m(z_1, z_2)) = \{\alpha_{i,j,k,l}^{m,n} \mid 0 \leqslant n \leqslant s, p_{i,j,m}^{l,k,n} \neq 0, \alpha_{i,j,k,l}^{m,n} \neq \pm 1\}$.*
2. *$X_{i,j}^m(z_1, z_2) = X_{k,l}^m(z_1, z_2)$ or $X_{i,j}^m(z_1, z_2) \cap X_{k,l}^m(z_1, z_2) = \emptyset$, and similarly $X_{i,j}^m(z_1, z_2) = -X_{k,l}^m(z_1, z_2)$ or $X_{i,j}^m(z_1, z_2) \cap -X_{k,l}^m(z_1, z_2) = \emptyset$ for any $z_1, z_2 \in X$ with $\langle z_1, z_2\rangle = \alpha_m$. And $\delta_{X_{i,j}^m(z_1,z_2), X_{k,l}^m(z_1,z_2)}$, $\delta_{X_{i,j}^m(z_1,z_2), -X_{k,l}^m(z_1,z_2)}$ are independent of $z_1, z_2 \in X$ with $\alpha_m = \langle z_1, z_2\rangle$.*
3. *$X_{i,j}^m(z_1, z_2)$ has the same strength for all $z_1, z_2 \in X$ with $\alpha_m = \langle z_1, z_2\rangle$.*

Moreover if the assumption (1), (2) or (3) of Theorem 7 is satisfied for $\{X_{i,j}^m(z_1, z_2) \mid 1 \leqslant i \leqslant \frac{s-1}{2}, 1 \leqslant j \leqslant s-1, p_{i,j}^m \neq 0\} \cup \{X_{i,j}^m(z_1, z_2) \mid \frac{s-1}{2} \leqslant i \leqslant \frac{s}{2}, 1 \leqslant j \leqslant \frac{s}{2}, p_{i,j}^m \neq 0\}$ with $m \neq 0$ or s, and when $((i, j), (k, l), (m, n))$ satisfies (2) (resp. (3)) the numbers $p_{\alpha,\beta}^{(k,l,m)}(x, y)$ (resp. $p_{\alpha,\beta}^{(k,l,m)}(x, y)$, $p_{\alpha,\beta'}^{(k,l,m)}(x, y)$, $p_{\alpha',\beta}^{(k,l,m)}(x, y)$) which are assumed to be independent of (x, y) with $\gamma = \langle x, y\rangle$ are independent of the choice of z_1, z_2 with $\alpha_m = \langle z_1, z_2\rangle$, then $(X, \{R_k\}_{k=0}^s)$ is a quadruply regular association scheme.

Proof. (1), (2), (3) follow from arguments similar to that in [11, Corollary 2.9].

Fix $z_1, z_2 \in X$ with $\alpha_m = \langle z_1, z_2 \rangle$.

If $m = 0$ or s, then $\bigcup_{i,j=1}^{s} R_i(z_1) \cap R_j(z_2) = \bigcup_{i=1}^{s} R_i(z_1)$. The triple regularity of $(X, \{R_k\}_{k=0}^{s})$ is equivalent that $\bigcup_{i,j=1}^{s} R_i(z_1) \cap R_j(z_2)$ is a coherent configuration whose parameters are independent of z_1, z_2 with $\langle z_1, z_2 \rangle = \pm 1$.

If $1 \leqslant m \leqslant s-1$, then $X_{i,s}^{m}(z_1, z_2) \neq \emptyset$ if and only if $X_{s,i}^{m}(z_1, z_2) \neq \emptyset$ if and only if $i = s - m$ hold, and then $X_{s,m-s}^{m}(z_1, z_2) = \{-z_1\}$, $X_{s-m,s}^{m}(z_1, z_2) = \{-z_2\}$ hold. Moreover $X_{i,j}^{m} = -X_{s-i,s-j}^{m}$ hold for any $1 \leqslant i, j \leqslant s-1$. By Lemma 8, it is sufficient to show that $\{X_{i,j}^{m}(z_1, z_2) \mid 1 \leqslant i \leqslant \frac{s-1}{2}, 1 \leqslant j \leqslant s-1, p_{i,j}^{m} \neq 0\} \cup \{X_{i,j}^{m}(z_1, z_2) \mid \frac{s-1}{2} \leqslant i \leqslant \frac{s}{2}, 1 \leqslant j \leqslant \frac{s}{2}, p_{i,j}^{m} \neq 0\}$ carries a coherent configuration whose parameters are independent of z_1, z_2 with $\alpha_m = \langle z_1, z_2 \rangle$, and the rest of the proof follows from the similar argument of that in [11, Corollary 2.9]. □

3 Linked systems of symmetric designs

In this section we characterize the triple regularity for association schemes obtained from linked systems of symmetric designs in terms of one Krein parameter.

Definition 10. Let $(X_i, X_j, I_{i,j})$ be an incidence structure satisfying $X_i \cap X_j = \emptyset$, $I_{j,i}^{t} = I_{i,j}$ for any distinct integers $i, j \in \{1, \ldots, f\}$. We put $X = \bigcup_{i=1}^{f} X_i$, $I = \bigcup_{i \neq j} I_{i,j}$. The pair (X, I) is called a linked system of symmetric (v, k, λ) designs if the following conditions hold:

1. for any distinct integers $i, j \in \{1, \ldots, f\}$, $(X_i, X_j, I_{i,j})$ is a symmetric (v, k, λ) design,
2. for any distinct integers $i, j, l \in \{1, \ldots, f\}$, and for any $x \in X_i, y \in X_j$, the number of $z \in X_l$ incident with both x and y depends only on whether x and y are incident or not, and does not depend on i, j, l.

We define the integers σ, τ by

$$|\{z \in X_l \mid (x, z) \in I_{i,l}, (y, z) \in I_{j,l}\}| = \begin{cases} \sigma & \text{if } (x, y) \in I_{i,j}, \\ \tau & \text{if } (x, y) \notin I_{i,j}, \end{cases}$$

where $i, j, l \in \{1, \ldots, f\}$ are distinct and $x \in X_i$, $y \in X_j$. Theorem 1 in [3] shows

$$(\sigma, \tau) = \left(\frac{1}{v}(k^2 \mp \sqrt{n}(v-k)), \frac{k}{v}(k \pm \sqrt{n}) \right),$$

where $n = k - \lambda$. Considering complement designs $(X_i, X_j, \overline{I_{i,j}})$ for any distinct integers $i, j \in \{1, \ldots, f\}$, we can assume either $(\sigma, \tau) = (\frac{1}{v}(k^2 - \sqrt{n}(v-k)), \frac{k}{v}(k + \sqrt{n}))$ or $(\frac{1}{v}(k^2 + \sqrt{n}(v-k)), \frac{k}{v}(k - \sqrt{n}))$.

From a linked system of symmetric designs, we obtain a 3-class Q-antipodal association scheme $(X, \{R_i\}_{i=0}^3)$ where

$$\begin{aligned} R_0 &= \{(x, x) \mid x \in X\}, \\ R_1 &= \{(x, y) \mid x \in X_i, y \in X_j, (x, y) \in I_{i,j} \text{ for some } i \neq j\}, \\ R_2 &= \{(x, y) \mid x, y \in X_i, x \neq y \text{ for some } i\}, \\ R_3 &= \{(x, y) \mid x \in X_i, y \in X_j, (x, y) \notin I_{i,j} \text{ for some } i \neq j\}. \end{aligned}$$

Conversely every 3-class Q-antipodal association scheme with equivalence relation $R_0 \cup R_2$ arises from a linked system of symmetric designs in [4, Theorem 5.8].

The following is the main result in this section. The implication (2) ⇒ (1) was shown in [11].

Theorem 11. *Let $(X, \{R_i\}_{i=0}^3)$ be a Q-polynomial association scheme which is Q-antipodal with equivalence relation $R_0 \cup R_2$. Then the following are equivalent:*

1. *$(X, \{R_i\}_{i=0}^3)$ is triply regular,*
2. *$a_1^* = 0$.*

Proof. (2)⇒(1): See [11, Corollary 6.2].

(1)⇒(2):Let $\{X_i, \ldots, X_f\}$ be a system of imprimitivity with respect to the equivalence relation $R_0 \cup R_2$ and (X, R_1) a linked system of symmetric (v, k, λ) designs. Assume that

$$\sigma = \frac{1}{v}(k^2 - \sqrt{n}(v - k)), \quad \tau = \frac{k}{v}(k + \sqrt{n}).$$

By the assumption of triple regularity, the following number

$$|\{w \in X \mid (x, w), (y, w), (z, w) \in R_1\}|$$

for distinct points $x, y, z \in X_1$ does not depend on $x, y, z \in X_1$. This implies that a pair $(X_1, \bigcup_{i=2}^f X_i)$ is a 3-design, therefore equality holds in [10, Theorem 2] (see also Remark 12 below). It follows that

$$f - 1 = \frac{(v-2)\sqrt{k(v-k)}}{(v-2k)\sqrt{v-1}}.$$

This implies $a_1^* = 0$ (See [11, p.14]). □

Remark 12. Mathon [9] pointed out that the inequality in [10, Theorem 2], which is obtained by the counting argument,

$$\begin{aligned} &(v-2)\left((v-1)\binom{\lambda}{3} + \binom{k}{3} - \left(k\binom{\sigma}{3} + (v-k)\binom{\tau}{3}\right)\right) \\ &\geqslant (f-1)\left((k-2)\lambda\binom{k}{3} - (v-2)\left(k\binom{\sigma}{3} + (v-k)\binom{\tau}{3}\right)\right) \end{aligned}$$

is equivalent to $a_1^* \geqslant 0$.

4 Real mutually unbiased bases

In this section we characterize the quadruple regularity for association schemes obtained from real MUBs in terms of its Krein parameter.

Definition 13. Let $M = \{M_i\}_{i=1}^{f}$ be a collection of orthonormal bases of $\mathbb{R}^d$. M is called a set of real mutually unbiased bases (MUBs) if any two vectors x and y from different bases satisfy $\langle x, y\rangle = \pm 1/\sqrt{d}$.

Let $M = \{M_i\}_{i=1}^{f}$ be a set of MUBs, and put $X = M \cup (-M)$. The angle set of X is

$$A'(X) = \{\frac{1}{\sqrt{d}}, 0, -\frac{1}{\sqrt{d}}, -1\}.$$

We set

$$\alpha_0 = 1, \quad \alpha_1 = \frac{1}{\sqrt{d}}, \quad \alpha_2 = 0, \quad \alpha_3 = -\frac{1}{\sqrt{d}}, \quad \alpha_4 = -1,$$

and we define $R_k = \{(x, y) \in X \times X \mid \langle x, y\rangle = \alpha_k\}$. Then $(X, \{R_k\}_{k=0}^4)$ is a Q-polynomial association scheme which is both Q-antipodal and Q-bipartite [7, Theorem 4.1].

Conversely let $(X, \{R_k\}_{k=0}^4)$ be a Q-polynomial association scheme which is both Q-antipodal and Q-bipartite, then the image of the embedding into first eigenspace by primitive idempotent E_1 is $M \cup (-M)$, where M is a set of MUBs [7, Theorem 4.2].

Applying [2, Theorem 4.8] to the above scheme for $i = j = 1$ using the parameters in [7, Appendix], we obtain the inequality $f \leqslant \frac{d}{2} + 1$. We call M a maximal set of MUBs if this upper bound is attained.

We now show that every set of f MUBs gives rise to a linked system of symmetric designs with $f - 1$ Q-antipodal classes.

Lemma 14. *Let $(X, \{R_i\}_{i=0}^4)$ be a Q-polynomial association scheme which is both Q-antipodal and Q-bipartite with f Q-antipodal classes of size $2d$. Assume $f \geqslant 3$. Then for $z \in X$ and $j = 1, 3$ $(R_j(z), \{R_i \cap (R_j(z) \times R_j(z))\}_{i=0}^3)$ is a Q-polynomial association scheme which is Q-antipodal with $(f-1)$ Q-antipodal classes of size d and $a_1^* = \frac{d}{f-1} - 2$.*

Proof. It was shown in [11, Section 5] that $(X, \{R_i\}_{i=0}^4)$ is triply regular, in particular $R_j(z)$ carries an association scheme for any $z \in X$, $j \in \{1, 3\}$. Let $X_j(z)$ be a derived design in S^{d-2} of X with respect to z, α_j. We determine the intersection numbers of $X_j(z)$. For $x, y \in X_j(z)$, we set

$$p_{\alpha,\beta}(x, y) = |\{w \in X_j(z) \mid \langle x, w\rangle = \alpha, \langle w, y\rangle = \beta\}|.$$

The angle set of $X_j(z)$ is

$$A(X_j(z)) = \left\{\alpha_1 := \frac{\sqrt{d}-1}{d-1}, \alpha_2 := \frac{-1}{d-1}, \alpha_3 := \frac{-\sqrt{d}-1}{d-1}\right\},$$

$X_j(z)$ is a 3-distance set and a 2-design in S^{d-2}. So set the degree $s = 3$ and the strength $t = 2$ and observe $t = 2s - 4$ here. And for any $\gamma = \langle x, y\rangle$, the intersection numbers

$p_{\alpha_2,\alpha_2}(x,y)$, $p_{\alpha_2,\alpha_1}(x,y)$, $p_{\alpha_1,\alpha_2}(x,y)$ are independent of the choice of $x,y \in X_j(z)$ with $\gamma = \langle x,y\rangle$ as follows:

$$p_{\alpha_2,\alpha_2}(x,y) = \begin{cases} 0 & \text{if } \langle x,y\rangle = \alpha_1, \\ d-2 & \text{if } \langle x,y\rangle = \alpha_2, \\ 0 & \text{if } \langle x,y\rangle = \alpha_3, \end{cases}$$

$$p_{\alpha_2,\alpha_1}(x,y) = p_{\alpha_1,\alpha_2}(x,y) = \begin{cases} \frac{d+\sqrt{d}}{2} - 1 & \text{if } \langle x,y\rangle = \alpha_1, \\ 0 & \text{if } \langle x,y\rangle = \alpha_2, \\ \frac{d+\sqrt{d}}{2} & \text{if } \langle x,y\rangle = \alpha_3. \end{cases}$$

For $0 \leqslant \lambda \leqslant 2$, $0 \leqslant \mu \leqslant 2$ and $(\lambda,\mu) \neq (1,2),(2,1),(2,2)$, we obtain a system of 6 linear equations

$$\begin{aligned}\sum_{\substack{1\leqslant l\leqslant 3\\ 1\leqslant m\leqslant 3\\ (l,m)\neq(2,2),(2,1),(1,2)}} \alpha_l^\lambda \beta_m^\mu p_{\alpha_l,\alpha_m}(x,y) = |X_j(z)|F_{\lambda,\mu}(\langle x,y\rangle) - \langle x,y\rangle^\lambda - \langle x,y\rangle^\mu - \alpha_2^\lambda\alpha_2^\mu p^j_{\alpha_2,\alpha_2}(x,y) \\ - \alpha_2^\lambda\alpha_1^\mu p^j_{\alpha_2,\alpha_1}(x,y) - \alpha_1^\lambda\alpha_2^\mu p^j_{\alpha_1,\alpha_2}(x,y),\end{aligned}$$

where $F_{\lambda,\mu}(t)$ is defined in [6, Section 7].

$$\{p_{\alpha_i,\alpha_j}(x,y) \mid 1 \leqslant i,j \leqslant 3, (i,j) \neq (2,2),(2,1),(1,2)\}$$

is uniquely determined by Theorem 7. The intersection matrices B_i and the second eigenmatrix Q are as follows:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(f-2)(d+\sqrt{d})}{2} & \frac{(f-3)(d+3\sqrt{d})}{4} & \frac{d+2\sqrt{d}}{4} & \frac{d+\sqrt{d}}{4} \\ 0 & \frac{d+\sqrt{d}-2}{2} & 0 & \frac{d+\sqrt{d}}{2} \\ 0 & \frac{(f-3)(d-\sqrt{d})}{4} & \frac{(f-2)d}{4} & \frac{(f-3)(d+\sqrt{d})}{4} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{d+\sqrt{d}-2}{2} & 0 & \frac{d+\sqrt{d}}{2} \\ d-1 & 0 & d-2 & 0 \\ 0 & \frac{d-\sqrt{d}}{2} & 0 & \frac{d-\sqrt{d}-2}{2} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{(f-3)(d-\sqrt{d})}{4} & \frac{(f-2)d}{4} & \frac{(f-3)(d+\sqrt{d})}{4} \\ 0 & \frac{d-\sqrt{d}}{2} & 0 & \frac{d-\sqrt{d}-2}{2} \\ \frac{(f-2)(d-\sqrt{d})}{2} & \frac{(f-3)(d-\sqrt{d})}{4} & \frac{(f-2)(d-2\sqrt{d})}{4} & \frac{(f-3)(d-3\sqrt{d})}{4} \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & d-1 & (f-1)(d-1) & f-1 \\ 1 & \sqrt{d}-1 & -\sqrt{d}+1 & -1 \\ 1 & -1 & -f+1 & f-1 \\ 1 & -\sqrt{d}-1 & \sqrt{d}+1 & -1 \end{pmatrix},$$

and hence the Krein matrix B_1^* is given as follows:

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ d-1 & \frac{d}{f-1}-2 & \frac{d}{f-1} & 0 \\ 0 & \frac{(f-2)d}{f-1} & \frac{(f-2)d}{f-1}-2 & d-1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore $X_1(z)$ is a Q-polynomial association scheme which is Q-antipodal. □

The following theorem shows that for MUBs, quadruple regularity is equivalent to their maximality.

Theorem 15. *Suppose that $(X, \{R_i\}_{i=0}^4)$ is a Q-polynomial association scheme which is both Q-antipodal and Q-bipartite. Then the following conditions are equivalent:*

1. *$(X, \{R_i\}_{i=0}^4)$ is quadruply regular,*

2. *$f = \frac{d}{2} + 1$.*

Proof. (1)⇒(2): Assume $(X, \{R_i\}_{i=0}^4)$ is quadruply regular. Then $X_1(z)$ is triply regular for any $z \in X$. By Lemma 14 and Theorem 11, $\frac{d}{f-1} - 2 = 0$. Therefore $f = \frac{d}{2} + 1$ holds.

(2)⇒(1): By [11, Corollary 5.3] it is sufficient to show that the assumption of Corollary 9 is satisfied.

(i) When $\langle z_1, z_2\rangle = \alpha_2$, $\{(i,j) \mid 1 \leqslant i \leqslant \frac{s-1}{2}, 1 \leqslant j \leqslant s-1, p_{i,j}^m \neq 0\} \cup \{(i,j) \mid \frac{s-1}{2} \leqslant i \leqslant \frac{s+1}{2}, 1 \leqslant j \leqslant \frac{s+1}{2}, p_{i,j}^m \neq 0\}$ is $\{(1,1),(1,3),(2,2)\}$. $X_{i,j}^2 = X_{i,j}^2(z_1, z_2)$ is a 3-design in S^{d-3} for $(i,j) \in \{(1,1),(1,3),(2,2)\}$. Indeed $X_{2,2}^2$ is a cross polytope in S^{d-3}. $|X_{1,1}^2| = p_{1,1}^2 = \frac{d^2}{4}$, $|X_{1,3}^2| = p_{1,3}^2 = \frac{d^2}{4}$ where $p_{1,1}^2$ and $p_{1,3}^2$ are the intersection numbers of X in [1, 6 Appendix]. And the angle sets $A(X_{1,1}^2) = A(X_{1,3}^2) = \{\frac{\sqrt{d}-2}{d-2}, \frac{-2}{d-2}, \frac{-\sqrt{d}-2}{d-2}\}$ hold, so Gegenbauer polynomial expansion of their annihilator polynomial $F(x) := \prod_{\alpha \in A'(X_{i,j}^2)} \frac{x-\alpha}{1-\alpha}$ is

$$F(x) = \frac{4}{d^2}Q_0(x) + \frac{2(d^2+6)(d-2)}{d^3(d-1)}Q_1(x) + \frac{(d-2)^3(d+3)}{d^3(d-1)}Q_2(x) + \frac{6(d-2)(d-3)}{d^2(d-1)}Q_3(x),$$

therefore $X_{1,1}^2$ and $X_{1,3}^2$ are 3-designs in S^{d-3} by [6, Theorem 6.5]. We renumber as follows:

$$X_1 = X_{2,2}^2, \quad X_2 = X_{1,1}^2, \quad X_3 = X_{1,3}^2.$$

We define $s_{i,j} = |A(X_i, X_j)|$. Then the matrix $(s_{i,j})$ is

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}.$$

If $s_{i,j} + s_{j,k} - 2 \leqslant 3$, that is, when one of the i, j, k is equal to 1, then the assumption (1) of Theorem 7 holds.

If $s_{i,j}+s_{j,k}-3=3$, that is, when

$$(i,j,k)\in\{(l,m,n)\mid 2\leqslant l,m,n\leqslant 3\}, \tag{1}$$

And $X_2\cup X_3$ carries a subconstituent association scheme $R_1(z_1)$ of X whose parameters are independent of z_1 by Lemma 14, therefore those for $(2,3,3)$ (respectively $(2,3,2)$, $(3,2,3)$) are determined by those for $(2,2,3)$ (respectively $(2,2,2)$, $(3,3,3)$). The intersection numbers $\{p^j_{\alpha,\beta}\mid \alpha=\alpha^2_{i,j}\text{ or }\beta=\alpha^2_{j,k}\}$ for $x\in X_i$, $y\in X_k$ and $(i,j,k)\in\{(2,2,2),(3,3,3),(2,2,3)\}$ are given in Table 1. These numbers are independent of $z_1,z_2\in X$ with $\langle z_1,z_2\rangle=\alpha_2$. Hence the assumption of (2) of Theorem 7 holds for i,j,k (i,j,k) in (1).

(ii) When $\langle z_1,z_2\rangle=\alpha_1$, $\{X^m_{i,j}(z_1,z_2)\mid 1\leqslant i\leqslant \frac{s-1}{2},1\leqslant j\leqslant s-1,p^m_{i,j}\neq 0\}\cup\{X^m_{i,j}(z_1,z_2)\mid \frac{s-1}{2}\leqslant i\leqslant\frac{s+1}{2},1\leqslant j\leqslant\frac{s+1}{2},p^m_{i,j}\neq 0\}$ is $\{X^1_{1,1},X^1_{1,2},X^1_{1,3},X^1_{2,1}\}$. $X^1_{i,j}=X^1_{i,j}(z_1,z_2)$ is a 2-design in S^{d-3} for $(i,j)\in\{(1,1),(1,2),(1,3),(2,1)\}$. Indeed $X^1_{1,2}$, $X^1_{2,1}$ are regular simplexes in S^{d-3}. And $X^1_{1,1}$ and $X^1_{1,3}$ are subconstituents of $X_1(z_1)$ with respect to $z_2\in X_1(z_1)$. $X_1(z_1)$ is a Q-polynomial association scheme by Theorem 14 with $a^*_1=0$, so [11, Lemma 4.2] implies that $X^1_{1,1}$ and $X^1_{1,3}$ are 2-designs in S^{d-3}. We renumber as follows:

$$X_1=X^1_{2,1},\quad X_2=X^1_{1,2},\quad X_3=X^1_{1,1},\quad X_4=X^1_{1,3}.$$

We define $s_{i,j}=|A(X_i,X_j)|$. Then the matrix $(s_{i,j})$ is

$$\begin{pmatrix}1&2&2&2\\2&1&2&2\\2&2&3&3\\2&2&3&3\end{pmatrix}.$$

If $s_{i,j}+s_{j,k}-2\leqslant 2$, that is, when

$$(i,j,k)\in\{(l,m,n)\mid 1\leqslant m\leqslant 2,1\leqslant l,n\leqslant 4\text{ or }3\leqslant m\leqslant 4,1\leqslant l,n\leqslant 2\},$$

then the assumption (1) of Theorem 7 holds.

If $s_{i,j}+s_{j,k}-3=2$, that is, when

$$(i,j,k)\in\{(l,m,n)\mid 1\leqslant l\leqslant 2,3\leqslant m,n\leqslant 4\text{ or }3\leqslant l,m\leqslant 4,1\leqslant n\leqslant 2\}, \tag{2}$$

or if $s_{i,j}+s_{j,k}-4=2$, that is, when

$$(i,j,k)\in\{(l,m,n)\mid 3\leqslant l,m,n\leqslant 4\}, \tag{3}$$

we directly verify that the intersection numbers on X_j for $x\in X_i$, $y\in X_k$ are independent of x,y and of z_1,z_2 by using the triple regularity of subconstituents of X in stead of showing that the (i,j,k) in (2) (respectively (3)) satisfy the assumption (2) (respectively (3)) of Theorem 7. By Theorem 11, Lemma 14 and the assumption $f=d/2+1$, $R_1(z_1)$ is a triply regular association scheme and its parameters are independent of z_1. Since

X_2, X_3 and X_4 are the subconstituents of $R_1(z_1)$ with respect to z_2, $X_2 \cup X_3 \cup X_4$ carries a coherent configuration with fibers X_2, X_3 and X_4. Moreover the parameters of its coherent configuration are independent of z_1 with $\langle z_1, z_2\rangle = \alpha_1$. Interchanging z_1 with z_2 and using $X_4 = -X^1_{3,1}$, it holds that $X_1 \cup X_3 \cup X_4$ carries a coherent configuration with fibers X_2, X_3 and X_4 whose parameters are independent of z_1, z_2 with $\langle z_1, z_2\rangle = \alpha_1$. Thus the intersection numbers on X_j for $x \in X_i$, $y \in X_k$ are independent of x, y and of z_1, z_2 for i, j, k satisfying (2) or (3).

(iii) The case $\langle z_1, z_2\rangle = \alpha_3$ is similar to the case $\langle z_1, z_2\rangle = \alpha_1$.

By Corollary 9, we obtain the desired result. □

Table 1: the values of $p^j_{\alpha,\beta}(x, y)$, where $x \in X_i(z)$, $y \in X_k(z)$

(i, j, k)	(α, β)	$p^j_{\alpha,\beta}(x, y)$
	$(\alpha^2_{i,j}, \alpha^2_{j,k})$	$\begin{cases} 0 & \langle x, y\rangle = \alpha^1_{i,k} \\ \frac{d}{2} - 1 & \langle x, y\rangle = \alpha^2_{i,k} \\ 0 & \langle x, y\rangle = \alpha^3_{i,k} \end{cases}$
$(2, 2, 2)$ $(3, 3, 3)$	$(\alpha^2_{i,j}, \alpha^1_{j,k})$ $(\alpha^1_{i,j}, \alpha^2_{j,k})$	$\begin{cases} \frac{d+2\sqrt{d}}{4} - 1 & \langle x, y\rangle = \alpha^1_{i,k} \\ 0 & \langle x, y\rangle = \alpha^2_{i,k} \\ \frac{d+2\sqrt{d}}{4} & \langle x, y\rangle = \alpha^3_{i,k} \end{cases}$
	$(\alpha^2_{i,j}, \alpha^3_{j,k})$ $(\alpha^3_{i,j}, \alpha^2_{j,k})$	$\begin{cases} \frac{d-2\sqrt{d}}{4} & \langle x, y\rangle = \alpha^1_{i,k} \\ 0 & \langle x, y\rangle = \alpha^2_{i,k} \\ \frac{d-2\sqrt{d}}{4} - 1 & \langle x, y\rangle = \alpha^3_{i,k} \end{cases}$
	$(\alpha^2_{2,2}, \alpha^2_{2,3})$	$\begin{cases} 0 & \langle x, y\rangle = \alpha^1_{2,3} \\ \frac{d}{2} - 1 & \langle x, y\rangle = \alpha^2_{2,3} \\ 0 & \langle x, y\rangle = \alpha^3_{2,3} \end{cases}$
	$(\alpha^2_{2,2}, \alpha^1_{2,3})$ $(\alpha^2_{2,2}, \alpha^3_{2,3})$	$\begin{cases} \frac{d}{4} - 1 & \langle x, y\rangle = \alpha^1_{2,3} \\ 0 & \langle x, y\rangle = \alpha^2_{2,3} \\ \frac{d}{4} & \langle x, y\rangle = \alpha^3_{2,3} \end{cases}$
$(2, 2, 3)$	$(\alpha^1_{2,2}, \alpha^2_{2,3})$	$\begin{cases} \frac{d+2\sqrt{d}}{4} & \langle x, y\rangle = \alpha^1_{2,3} \\ 0 & \langle x, y\rangle = \alpha^2_{2,3} \\ \frac{d+2\sqrt{d}}{4} & \langle x, y\rangle = \alpha^3_{2,3} \end{cases}$
	$(\alpha^3_{2,2}, \alpha^2_{2,3})$	$\begin{cases} \frac{d-2\sqrt{d}}{4} & \langle x, y\rangle = \alpha^1_{2,3} \\ 0 & \langle x, y\rangle = \alpha^2_{2,3} \\ \frac{d-2\sqrt{d}}{4} & \langle x, y\rangle = \alpha^3_{2,3} \end{cases}$

Remark 16. Let M be a maximal set of MUBs and $X = M \cup (-M)$. It was already shown in [1, Theorem 5] that $\{x \in X \mid \langle x, z_1\rangle = \langle x, z_2\rangle = \frac{1}{\sqrt{d}}\}$ for $z_1, z_2 \in X$ such that $\langle z_1, z_2\rangle = 0$ carries an association scheme.

Remark 17. One might wonder if the integrality of intersection numbers of the above quadruply regular association scheme implies new necessary condition for existing maximal mutually unbiased bases, but the conditions of the integrality show $d = \frac{k^2}{16}$, which is already known.

5 Quadruple regularity of linked systems of symmetric designs

Finally we consider whether the linked of symmetric designs with $a_1^* = 0$ could become quadruply regular or not. We denote the collection of all k-subsets of Ω by $\binom{\Omega}{k}$. Let $(X_i, X_j, I_{i,j})$ be an incidence structure satisfying $X_i \cap X_j = \emptyset$, $I_{j,i}^t = I_{i,j}$ for any distinct integers $i, j \in \{1, \dots, f\}$. We put $X = \bigcup_{i=1}^f X_i$, $I = \bigcup_{i \neq j} I_{i,j}$. Let (X, I) be a linked system of symmetric designs with $1 < k < v - 1$. By [3, Theorem 1], $n = k - \lambda$ is a square number. Since $k < v - 1$, we have $n \neq 1$. Hence $n \geqslant 4$ and we have $v \geqslant 15$. We define

$$\alpha(S) = |R_1(x_1) \cap R_1(x_2) \cap R_1(x_3) \cap R_1(x_4)|,$$

for $S = \{x_1, x_2, x_3, x_4\} \in \binom{X_1}{4}$.

Counting in two ways the numbers of these sets

$$\{(S, y) \in \binom{X_1}{4} \times \bigcup_{i=2}^f X_i \mid (x, y) \in R_1 \text{ for any } x \in S\},$$

$$\{(S, T) \in \binom{X_1}{4} \times \binom{\bigcup_{i=2}^f X_i}{2} \mid (x, y) \in R_1 \text{ for any } x \in S, y \in T\},$$

we have the following equalities:

$$\sum_{S \in \binom{X_1}{4}} \alpha(S) = (f-1)v\binom{k}{4}, \tag{4}$$

$$\sum_{S \in \binom{X_1}{4}} \binom{\alpha(S)}{2} = \frac{1}{2}(f-1)v\left((f-2)k\binom{\sigma}{4} + (v-1)\binom{\lambda}{4} + (f-2)(v-k)\binom{\tau}{4}\right). \tag{5}$$

Using Cauchy-Schwarts inequality with (4) and (5), we obtain

$$\frac{(k-1)k^2(v-k)^2(v-k-1)(v-2)}{(v-2k)(v-1)^2 v}(k(v-3)(v-k)(v-2k) + \sqrt{n}(v-1)(v^2 - 6kv + v + 6k^2)) \geqslant 0.$$

If equality holds, then we have $k(v-3)(v-k)(v-2k) + \sqrt{n}(v-1)(v^2 - 6kv + v + 6k^2) = 0$. Multiplying $k(v-3)(v-k)(v-2k) - \sqrt{n}(v-1)(v^2 - 6kv + v + 6k^2)$ and dividing by $vk(k-1)(v-k-1)(v-k) \neq 0$, we obtain

$$v(v+1)^2 + 4k^2(v+3) - 4kv(v+3) = 0.$$

Solving this quadratic equation by k, we obtain

$$k = \frac{v(v+3) \pm \sqrt{v(v-1)(v+3)}}{2(v+3)}.$$

Then it is a necessary condition that $v(v-1)(v+3)$ is a square number. The elliptic curve $y^2 = v(v-1)(v+3)$ has rank 0 and only 6 integral points $(v, y) = (-2, 6)$, $(-1, 4)$, $(3, 36)$, $(0, 0)$, $(1, 0)$, $(-3, 0)$. It contradicts $v \geqslant 15$. Therefore $(X_1, \bigcup_{i=2}^{f} X_i)$ does not become a 4-design. Hence a linked system of symmetric designs does not carry a quadruply regular association scheme and a maximal set of MUBs does not carry a quintuply regular association scheme. However, let $(X, \{R_i\}_{i=0}^4)$ be the association scheme derived from a maximal set of MUBs with $d > 4$, the numbers

$$|R_{i_1}(x_1) \cap R_{i_2}(x_2) \cap R_{i_3}(x_3) \cap R_{i_4}(x_4) \cap R_{i_5}(x_5)| \tag{6}$$

for $x_k \in X_1$ such that $(x_k, x_l) \in R_2$, $i_m \in \{1, 3\}$ and $1 \leqslant k, l, m \leqslant 5$ are uniquely determined as follows. Let $M = \{M_1, \dots, M_{d/2+1}\}$ be a maximal set of MUBs. Consider the orthogonal transformation on M given by M_1 to the standard basis. Then the elements of $\bigcup_{i=2}^{\frac{d}{2}+1} X_i$ have form $\frac{1}{\sqrt{d}}(\pm 1, \dots, \pm 1)$ since M_1 and M_i are mutually unbiased. The binary code $\mathcal{C}$ is defined corresponding to the elements of $\bigcup_{i=2}^{\frac{d}{2}+1} X_i$ as follows: $c = (c_i) \in \mathcal{C}$ corresponds to $x = (x_i) \in \bigcup_{i=2}^{\frac{d}{2}+1} X_i$, then $c_i = 0, 1$ according to whether $x_i = \frac{1}{\sqrt{d}}, -\frac{1}{\sqrt{d}}$, respectively. $\mathcal{C}$ is said to be a Kerdock-like code in [1]. The weight enumerator of $\mathcal{C}$ is $W_{\mathcal{C}}(x, y) = x^d + \frac{d(d-2)}{2} x^{\frac{d+\sqrt{d}}{2}} y^{\frac{d-\sqrt{d}}{2}} + 2(d-1) x^{\frac{d}{2}} y^{\frac{d}{2}} + \frac{d(d-2)}{2} x^{\frac{d-\sqrt{d}}{2}} y^{\frac{d+\sqrt{d}}{2}} + y^d$. Then

$$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{d^2} W_{\mathcal{C}}(x + y, x - y) = x^d + \frac{d(d-1)(d-2)(d-4)}{360} x^{d-6} y^6 + \cdots .$$

Hence $\mathcal{C}$ is an orthogonal array whose strength is 5 if $d > 4$. This implies that (6) are uniquely determined for $x_k \in X_1$ such that $(x_k, x_l) \in R_2$, $i_m \in \{1, 3\}$ and $1 \leqslant k, l, m \leqslant 5$.

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