Spanning k-trees of Bipartite Graphs

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Abstract

A tree is called a k-tree if its maximum degree is at most k. We prove the following theorem. Let $k \geqslant 2$ be an integer, and G be a connected bipartite graph with bipartition (A, B) such that $|A| \leqslant |B| \leqslant (k-1)|A| + 1$. If $\sigma_k(G) \geqslant |B|$, then G has a spanning k-tree, where $\sigma_k(G)$ denotes the minimum degree sum of k independent vertices of G. Moreover, the condition on $\sigma_k(G)$ is sharp. It was shown by Win (Abh. Math. Sem. Univ. Hamburg, 43, 263–267, 1975) that if a connected graph H satisfies $\sigma_k(H) \geqslant |H| - 1$, then H has a spanning k-tree. Thus our theorem shows that the condition becomes much weaker if the graph is bipartite.

Keywords: spanning k-tree, bipartite graph

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1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, let $N_G(v)$ denote the neighborhood of v in G, and denote the degree of v in G by $\deg_G(v)$, in particular, $\deg_G(v) = |N_G(v)|$. A set X of vertices of G is called an independent set if no two vertices of X are adjacent. For two vertices x and y of G, an edge joining them is denoted by xy or yx. For an integer $k \geqslant 2$, a tree is called a *k-tree* if its maximum degree is at most k. Let $\alpha(G)$ denote the independence number of G. The number $\sigma_k(G)$ is defined to be the minimum degree sum of k independent vertices of G. Namely, for an integer $k \geqslant 1$ with $\alpha(G) \geqslant k$, we define

$$\sigma_k(G) := \min_S \left\{ \sum_{x \in S} \deg_G(x) \ : \ S \text{ is an independent set of size } k \right\}$$

and $\sigma_k(G) := \infty$ if $\alpha(G) < k$.

We begin with some known results on spanning k-trees related to our theorem, and other results on a spanning k-tree can be found in the book [1], and papers [2], [3], [5] and others. In particular, a survey article [6] contains many current results on spanning trees including spanning k-trees.

The next theorem gives a sufficient condition using $\sigma_k(G)$ for a graph to have a spanning k-tree.

Theorem 1 (Win [7]). *Let $k \geqslant 2$ be an integer and G be a connected graph. If $\sigma_k(G) \geqslant |G| - 1$, then G has a spanning k-tree.*

Our main result of this paper is the following theorem, which shows that the condition on $\sigma_k(G)$ in the above Theorem 1 can be relaxed a lot for bipartite graphs.

Theorem 2. *Let $k \geqslant 2$ be an integer, and G be a connected bipartite graph with bipartition (A, B) such that $|A| \leqslant |B| \leqslant (k-1)|A| + 1$. If*

$$\sigma_k(G) \geqslant |B|,$$

then G has a spanning k-tree.

The above theorem with $k = 2$ was obtained by Moon and Moser.

Theorem 3 (Moon and Moser [4]). *Let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leqslant |B| \leqslant |A| + 1$. If $\sigma_2(G) \geqslant |B|$, then then G has a Hamiltonian path.*

Note that the condition $|B| \leqslant (k-1)|A| + 1$ is necessary for the bipartite graph G to have a spanning k-tree since if $|B| > (k-1)|A| + 1$, then G cannot have a spanning k-tree. The degree sum condition is sharp in the following sense. Let $k \geqslant 3$ and $s \geqslant 1$ be integers, and let A_1, A_2, B_1 and B_2 be disjoint sets of vertices such that $|A_1| = (k-2)s+1$,

$|A_2| = s$, $|B_1| = s$, $|B_2| = (k-1)s+1$. Then define a bipartite graph G with bipartition $(A_1 \cup A_2, B_1 \cup B_2)$ and edge set $E(G) = \{xy : x \in A_1, y \in B_1\} \cup \{xy : x \in B_1, y \in A_2\} \cup \{xy : x \in A_2, y \in B_2\}$. Then $|A_1 \cup A_2| = (k-1)s+1 < |B_1 \cup B_2| = ks+1$ and $\sigma_k(G) = ks = |B_1 \cup B_2| - 1$. Moreover, G has no spanning k-tree. Therefore the condition on $\sigma_k(G)$ in Theorem 2 is sharp.

2 Proof of Theorem 2

We begin with some notation. Let T be a tree. We denote the set of leaves of T by $Leaf(T)$. For two vertices u and v of T, there exists a unique path connecting u and v in T, and it is denoted by $P_T(u,v)$. Let T be a rooted tree with root w. For a vertex $v \in V(T) - \{w\}$, the vertex adjacent to v and lying on the path $P_T(v,w)$ is called the *parent* of v and denoted by v^-. A vertex whose parent is v is called a *child* of v. In particular, there are $\deg_T(v) - 1$ children of v, and the set of children of v is denoted by $Child(v)$. We define the *total excess* $te(G;k)$ *from* k of a graph G as

$$\mathrm{te}(G;k) := \sum_{v \in V(G)} \max\{\deg_G(v) - k, 0\}.$$

Thus a tree T has $\mathrm{te}(T;k) = 0$ if and only if T is a k-tree. We are ready to prove Theorem 2.

Proof of Theorem 2. By Theorem 3, we may assume that $k \geqslant 3$ though most part of the following proof holds even if $k = 2$. Let G be a connected bipartite graph with bipartition (A, B) that satisfies the following two conditions instead of the conditions of Theorem 2.

$$\max\{|A|, |B|\} \leqslant (k-1)\min\{|A|, |B|\} + 1, \quad \text{and} \tag{1}$$
$$\sigma_k(G) \geqslant \max\{|A|, |B|\}. \tag{2}$$

Notice that the above two conditions and the conditions of Theorem 2 are essentially equivalent, and by these new conditions, we can assume that $w \in A$ without loss of generality, which will soon be apparent, and decrease the number of cases in case analysis. Moreover, we do not use the sizes of two partite sets until the last stage of the proof.

Suppose that G has no spanning k-tree. Choose a spanning tree T of G so that

(T1) $\mathrm{te}(T;k)$ is as small as possible,
(T2) $|Leaf(T)|$ is as small as possible, subject to (T1) and,
(T3) $Leaf(T) \cap A \neq \emptyset$ and $Leaf(T) \cap B \neq \emptyset$ if possible, subject to (T2).

Since G has no spanning k-tree, there exists a vertex w such that $\deg_T(w) = l \geqslant k+1$. Let $D_1, D_2, \ldots, D_l$ be the components of $T - w$. For every $1 \leqslant i \leqslant l$, let v_i denote the vertex of D_i adjacent to w in T. For every $1 \leqslant i \leqslant k$, let $u_i \in V(D_i)$ be a leaf of T, and let $U := \{u_1, \ldots, u_k\}$.

Without loss of generality, we may assume that $w \in A$ as mentioned above. Assume that $u_i \in A$ for $1 \leqslant i \leqslant m$, and $u_i \in B$ for $m+1 \leqslant i \leqslant k$, where it might occur that

$m = 0$ or $m = k$. We regard D_i as a rooted tree with root u_i for $1 \leqslant i \leqslant k$, and with root v_i for $k+1 \leqslant i \leqslant l$ (see Figure 1). For every $1 \leqslant t \leqslant l$, let

$$X_t := \bigcup_{1 \leqslant j \leqslant m,\ j \neq t} (N_G(u_j) \cap V(D_t)),$$

$$Y_t := \bigcup_{m+1 \leqslant j \leqslant k,\ j \neq t} (N_G(u_j) \cap V(D_t)),$$

$$Z_t := \begin{cases} (N_G(u_t) \cap V(D_t)) \setminus X_t & \text{if} \quad 1 \leqslant t \leqslant m, \\ (N_G(u_t) \cap V(D_t)) \setminus Y_t & \text{if} \quad m+1 \leqslant t \leqslant k, \\ \emptyset & \text{if} \quad k+1 \leqslant t \leqslant l, \end{cases}$$

and

$$Z_t^1 := \begin{cases} \{z \in Z_t : \text{a vertex of } P_T(u_t, z^-) \text{ is adjacent to some } u_j \text{ in } G, \\ \qquad\qquad \text{where } 1 \leqslant j \leqslant m \text{ and } j \neq t\} \qquad \text{if} \quad 1 \leqslant t \leqslant m, \\ \{z \in Z_t : \text{a vertex of } P_T(u_t, z^-) \text{ is adjacent to some } u_j \text{ in } G, \\ \qquad\qquad \text{where } m+1 \leqslant j \leqslant k \text{ and } j \neq t\} \qquad \text{if} \quad m+1 \leqslant t \leqslant k, \\ \emptyset \qquad \text{if} \quad k+1 \leqslant t \leqslant l \qquad \text{(see Figure 1).} \end{cases}$$

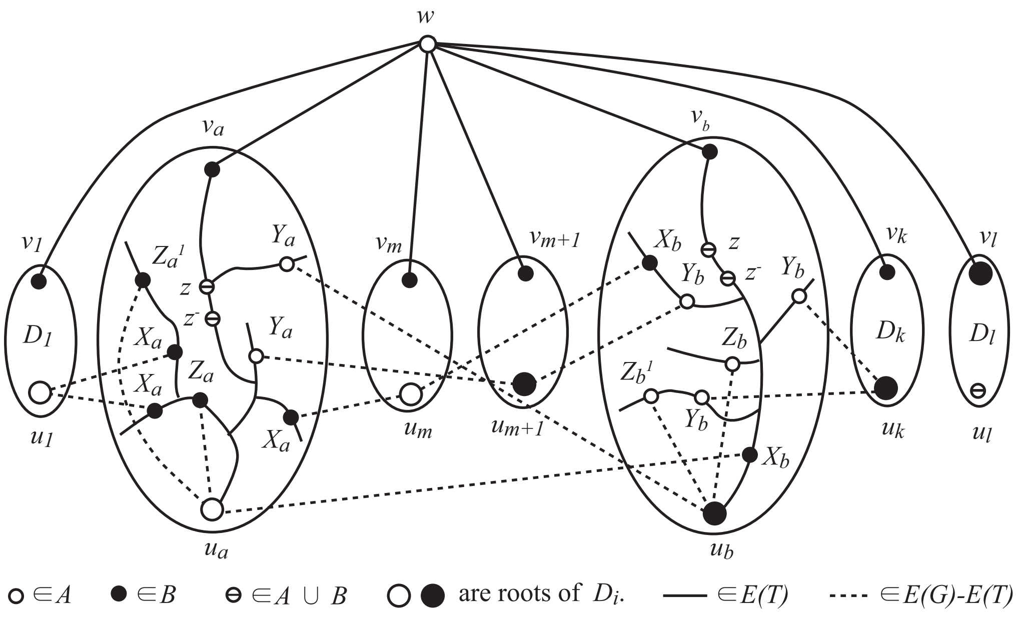

Figure 1: A spanning tree T of a bipartite graph G with bipartition $A \cup B$.

Then $X_i \subseteq B$ and $Y_i \subseteq A$ for all $1 \leqslant i \leqslant l$, and $Z_i \subseteq B$ for $1 \leqslant i \leqslant m$ and $Z_i \subseteq A$ for $m+1 \leqslant i \leqslant k$. We relabel indices i of D_i and rechoose u_i so that

(U1) $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$ if possible and,
(U2) $X_l \neq \emptyset$ if possible, subject to (U1).

Let $X := \bigcup_{1 \leqslant i \leqslant l} X_i$, $Y := \bigcup_{1 \leqslant i \leqslant l} Y_i$ and $Z := \bigcup_{1 \leqslant i \leqslant l} Z_i$. Then the following claim holds.

Claim 1. *For every integer $1 \leqslant t \leqslant l$, let w_t be any vertex of D_t such that $\deg_T(w_t) \leqslant k-1$ or $w_t = v_t$. Then for any two distinct integers $i, j \in \{1, 2, \ldots, l\}$, the following three statements hold.*

(i) *$w_i w_j \notin E(G)$.*

(ii) *If $v \in V(D_i)$ is adjacent to w_i in $G - E(T)$ and if $v' \in N_T(v) \cap P_T(v, w_i)$, then $v'w_j \notin E(G)$.*

(iii) *Assume that $w_i \neq v_i$, $w_j \neq v_j$ and $v \in N_G(w_i) \cap V(D_h)$ for some $h \in \{1, \ldots, l\} - \{i, j\}$. If $v' \in N_T(v)$, then $v'w_j \notin E(G)$.*

Proof. (i). If $w_i w_j \in E(G)$ and $w_j \neq v_j$, then $T_1 = T - wv_i + w_i w_j$ is a spanning tree of G and $\mathrm{te}(T_1; k) = \mathrm{te}(T; k) - 1$, which contradicts the choice (T1). If $w_i = v_i$ and $w_j = v_j$, then $w_i w_j \notin E(G)$ since v_i and v_j are both contained in B. Hence (i) holds.

(ii). If $v'w_j \in E(G)$ and $w_i \neq v_i$, then $T_2 = T - wv_j - vv' + w_i v + v'w_j$ is a spanning tree and $\mathrm{te}(T_2; k) = \mathrm{te}(T; k) - 1$, a contradiction. If $v'w_j \in E(G)$ and $w_j \neq v_j$, then $T_2 = T - wv_i - vv' + w_i v + v'w_j$ is a spanning tree and $\mathrm{te}(T_2; k) = \mathrm{te}(T; k) - 1$, a contradiction again. If $w_i = v_i$ and $w_j = v_j$, then $v'w_j \notin E(G)$ since all of v_i, v' and v_j are contained in B. Thus (ii) holds.

(iii). If $v'w_j \in E(G)$, then $T_3 = T - wv_h - vv' + w_i v + w_j v'$ is a spanning tree and $\mathrm{te}(T_3; k) = \mathrm{te}(T; k) - 1$, a contradiction. Therefore Claim 1 holds. □

By Claim 1 (i) and by choosing $w_t = u_t$, we have that U is an independent set of G. Furthermore, by Claim 1 (i), we obtain the following claim.

Claim 2. *For all $v \in X \cup Y$, $\deg_T(v) \geqslant k$.*

Claim 3. *For each $1 \leqslant i \leqslant l$, the following statements hold.*

(i) *$(\{v_i\} \cup Leaf(D_i)) \cap (X_i \cup Y_i) = \emptyset$.*

(ii) *$(\{v_i\} \cup Leaf(D_i)) \cap Z_i^1 = \emptyset$.*

Proof. (i). The statement (i) follows immediately from Claim 1 (i).

(ii). Suppose that there exists a vertex $v \in (\{v_p\} \cup Leaf(D_p)) \cap Z_p^1$ for some $1 \leqslant p \leqslant l$. Since $Z_i^1 = \emptyset$ for each $k+1 \leqslant i \leqslant l$, we have $p \in \{1, \ldots, k\}$. Assume $p \in \{1, \ldots, m\}$. Then $v \in B$, and there exist two vertices $x \in X_p \cap V(P_T(u_p, v^-))$ and $u_q \in U$ with $1 \leqslant q \neq p \leqslant m$ that are adjacent in G. Let $x' \in Child(x) \cap V(P_T(u_p, v^-))$. Then $T_1 = T - wv_p - xx' + u_p v + xu_q$ is a spanning tree of G and $\mathrm{te}(T; k) > \mathrm{te}(T_1; k)$ since $v \in \{v_p\} \cup Leaf(D_p)$. This contradicts (T1). We can similarly derive a contradiction in the case $p \in \{m+1, \ldots, k\}$. Hence Claim 3 holds. □

By Claim 2, it follows that $|Child(v)| \geqslant k-1$ for all $v \in X \cup Y$. For each $x \in X$, choose a set $Child_{k-1}(x)$ of $k-1$ children of x, and let

$$Q(x) := \{x\} \cup Child_{k-1}(x).$$

Similarly, for each $y \in Y$, we define

$$R(y) := \{y\} \cup Child_{k-1}(y),$$

where $Child_{k-1}(y)$ is a set of $k-1$ children of y. Then

$$|Q(x) \cap A| = |R(y) \cap B| = k-1 \quad \text{and} \quad |Q(x) \cap B| = |R(y) \cap A| = 1 \tag{3}$$

for every $x \in X$ and $y \in Y$. By Claim 3 (ii), $Child(z) \neq \emptyset$ for every $z \in Z_i^1$, where $1 \leqslant i \leqslant k$. For every $z \in Z_i$ with $1 \leqslant i \leqslant k$, let $z^* = z^-$ if $z \in Z_i - Z_i^1$; otherwise, let $z^* \in Child(z)$. Let

$$S(z) := \{z, z^*\} \qquad \text{for every} \quad z \in Z.$$

Claim 4. *For every $z \in Z$, $\deg_T(z^-) \leqslant 2$.*

Proof. Suppose that there exists a vertex $z \in Z_i$ such that $\deg_T(z^-) \geqslant 3$ for some i. Then $T_1 = T - zz^- + zu_i$ is a spanning tree of G and $\mathrm{te}(T;k) \geqslant \mathrm{te}(T_1;k)$ and $|Leaf(T)| > |Leaf(T_1)|$. This contradicts (T1) or (T2). Hence Claim 4 holds. $\square$

Claim 5. *For all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ with $x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2$, the following holds.*

(i) *$Q(x_1) \cap Q(x_2) = \emptyset$, $R(y_1) \cap R(y_2) = \emptyset$ and $S(z_1) \cap S(z_2) = \emptyset$.*

(ii) *$Q(x_1) \cap R(y_1) = \emptyset$, $Q(x_1) \cap S(z_1) = \emptyset$ and $R(y_1) \cap S(z_1) = \emptyset$.*

Proof. (i). Obviously, $Q(x_1) \cap Q(x_2) = \emptyset$ for all $x_1 \neq x_2 \in X$ since x_1 and x_2 are not adjacent in T. Similarly, $R(y_1) \cap R(y_2) = \emptyset$ for each $y_1 \neq y_2 \in Y$. Suppose that $S(z_1) \cap S(z_2) \neq \emptyset$ for some $z_1 \neq z_2 \in Z$, which implies $z_1^* = z_2^*$. By the definition of Z_i, we have $z_1, z_2 \in Z_h$ for some $1 \leqslant h \leqslant k$. By the symmetry of z_1 and z_2, we may assume that (a) $z_1, z_2 \in Z_h^1$, (b) $z_1 \in Z_h^1$ and $z_2 \in Z_h - Z_h^1$, or (c) $z_1, z_2 \in Z_h - Z_h^1$.

First, suppose that (a) holds. Then $z_1^* \in Child(z_1)$ and $z_2^* \in Child(z_2)$. Hence $z_1^* \neq z_2^*$, a contradiction. Next, suppose that (b) holds. Then $z_2^- = z_2^* = z_1^* \in Child(z_1)$, and hence $P_T(u_h, z_2^-) = P_T(u_h, z_1) + z_1 z_2^-$. Since a vertex of $P_T(u_h, z_1^-)$ is adjacent to some u_p in G, where $p \neq h$, it follows from the definition of Z_h^1 that $z_2 \in Z_h^1$. This contradicts $z_2 \in Z_h - Z_h^1$. Finally, suppose that (c) holds. Then $z_1^- = z_2^-$, which implies $\deg_T(z_1^-) \geqslant 3$. This contradicts Claim 4.

(ii). By Claim 1 (iii), $Q(x_1) \cap R(y_1) = \emptyset$ for each $x_1 \in X$ and $y_1 \in Y$. Suppose that $Q(x_1) \cap S(z_1) \neq \emptyset$ for some $x_1 \in X$ and $z_1 \in Z$. Since $Z_i = \emptyset$ for $k+1 \leqslant i \leqslant l$, it follows that $x_1 \in X_h$ and $z_1 \in Z_h$ for some $1 \leqslant h \leqslant k$. If $h \in \{1, \ldots, m\}$, then both x_1 and z_1 are contained in B, and so $z_1^* = z_1^- \in Child(x_1)$. But this implies that $z_1 \in Z_h^1$, and so $z_1^* \in Child(z_1)$, a contradiction.

Assume $h \in \{m+1, \dots, k\}$. Then $x_1 \in B$ and $z_1 \in A$, and x_1 is adjacent to some u_p in G with $1 \leqslant p \leqslant m$. By Claim 1 (ii), $z_1^- \neq x_1$. Hence $x_1 = z_1^* \in Child(z_1)$, which implies $z_1 \in Z_h^1$. Then there exist two vertices $y \in V(P_T(u_h, z_1^-))$ and u_q with $m+1 \leqslant q \neq h \leqslant k$ which are adjacent in G. Choose $y' \in Child(y) \cap V(P_T(y, z_1))$. Then $T_2 = T - wv_h - x_1z_1 - yy' + x_1u_p + u_hz_1 + u_qy$ is a spanning tree of G and $\mathrm{te}(T;k) > \mathrm{te}(T_2;k)$. This contradicts (T1). Hence $Q(x_1) \cap S(z_1) = \emptyset$ for each $x_1 \in X$ and $z_1 \in Z$. Similarly, we can show that $R(y_1) \cap S(z_1) = \emptyset$ for each $y_1 \in Y$ and $z_1 \in Z$. Hence Claim 5 is proved. □

For every $1 \leqslant i \leqslant l$, let $Q_i := \bigcup_{x \in X_i} Q(x)$, $R_i := \bigcup_{y \in Y_i} R(y)$, $S_i := \bigcup_{z \in Z_i} S(z)$ and $O_i := V(D_i) - (Q_i \cup R_i \cup S_i)$. Let

$$Q := \bigcup_{1 \leqslant i \leqslant l} Q_i, \quad R := \bigcup_{1 \leqslant i \leqslant l} R_i, \quad S := \bigcup_{1 \leqslant i \leqslant l} S_i \quad \text{and} \quad O := \bigcup_{1 \leqslant i \leqslant l} O_i.$$

Claim 6. (i) *If $X_i \neq \emptyset$ for some $1 \leqslant i \leqslant m$ or $k+1 \leqslant i \leqslant l$, then $O_i \cap A \neq \emptyset$.*

(ii) *$O_i \cap B \neq \emptyset$ for each $m+1 \leqslant i \leqslant l$.*

Proof. (i) Suppose that $X_i \neq \emptyset$ for some $1 \leqslant i \leqslant m$ or $k+1 \leqslant i \leqslant l$. Let r_i be the root of D_i, that is, $r_i = u_i$ for $1 \leqslant i \leqslant m$, and $r_i = v_i$ for $k+1 \leqslant i \leqslant l$. Choose $x_i \in X_i$ so that $|P_T(r_i, x_i)|$ is as small as possible. Note that $x_i \neq r_i$ by Claim 3 (i). Recall that $x_i \in X_i \subset B$, and so $x_i^- \in A$. By Claim 1 (iii), we obtain $x_i^- \notin R_i$. By the minimality of $|P_T(r_i, x_i)|$, we have $x_i^- \notin Q_i \cup S_i$. Thus we obtain that $x_i^- \in O_i \cap A$, and hence $O_i \cap A \neq \emptyset$.

(ii) First, assume $i \in \{k+1, \dots, l\}$. By Claim 3 (i), we have $v_i \notin Q_i$. Since v_i is a root of D_i, we have $v_i \notin R_i$. Since $Z_i = \emptyset$, it follows that $v_i \in O_i \cap B$. Next, assume $i \in \{m+1, \dots, k\}$. If $Y_i \neq \emptyset$, then we can also prove the statement (ii) by a similar argument as in the statement (i). Hence we may assume $Y_i = \emptyset$. Thus $Z_i^1 = \emptyset$, and so $S(z) = \{z, z^-\}$ for any $z \in Z_i$. Suppose that $z^- = v_i$ for some $z \in Z_i$. Then $d_T(z^-) \geqslant 3$. This contradicts Claim 4. Hence $v_i \notin S_i$. On the other hand, by Claim 3 (i), we have $v_i \notin Q_i$. Thus we obtain $v_i \in O_i \cap B$. □

Claim 7. (i) $|A| = (k-1)|X| + |Y| + |Z| + |O \cap A| + 1$

(ii) $|B| = |X| + (k-1)|Y| + |Z| + |O \cap B|$

Proof. By (3) and Claim 5 (i), we have $|Q \cap A| = (k-1)|X|$, $|R \cap A| = |Y|$ and $|S \cap A| = |Z|$. By Claim 5 (ii), Q, R, S, O and $\{w\}$ are pairwise disjoint. Thus we deduce

$$\begin{aligned} |A| &= |Q \cap A| + |R \cap A| + |S \cap A| + |O \cap A| + |\{w\}| \\ &= (k-1)|X| + |Y| + |Z| + |O \cap A| + 1. \end{aligned}$$

Similarly, we can obtain the desired equality (ii). □

We now prove the theorem by considering three cases.

Case 1. $|X| \geqslant |Y| + 1$.

It follows that $U \cap A \neq \emptyset$ since $X \neq \emptyset$. Moreover, the following claim holds in this case.

Claim 8. $O \cap A \neq \emptyset$.

By the assumption of Case 1, we have $X \neq \emptyset$. If $X_i \neq \emptyset$ for some $1 \leqslant i \leqslant m$ or $k+1 \leqslant i \leqslant l$, then Claim 6 (i) implies $O \cap A \neq \emptyset$. Therefore we may assume that $X_i = \emptyset$ for all $1 \leqslant i \leqslant m$ and $k+1 \leqslant i \leqslant l$, and thus $X_h \neq \emptyset$ for some $m+1 \leqslant h \leqslant k$. If $Leaf(D_l) \cap B \neq \emptyset$, then, by exchanging the role of D_h with u_h and D_l with $u_l \in Leaf(D_l) \cap B$, we can obtain a contradiction to (U2). Thus we have $Leaf(D_l) \subseteq A$. It follows from Claim 3 (i) that $Leaf(D_l) \cap Y_l = \emptyset$, and since $X_l = \emptyset$ and $Z_l = \emptyset$ (by the definition of Z_l), we have $Leaf(D_l) \subseteq O \cap A$, and hence $O \cap A \neq \emptyset$.

Assume first $m \leqslant k-1$. By the assumption of Case 1 and Claims 7 (i) and 8, we obtain

$$\begin{aligned}
|A| &= (k-1)|X| + |Y| + |Z| + |O \cap A| + 1 \\
&= m|X| + (k-m-1)|X| + |Y| + |Z| + |O \cap A| + 1 \\
&\geqslant m|X| + (k-m-1)(|Y|+1) + |Y| + |Z| + 2 \\
&= m|X| + (k-m)|Y| + |Z| + (k-m) + 1 \\
&\geqslant \sum_{1 \leqslant i \leqslant m} |N_G(u_i) \cap X| + \sum_{m+1 \leqslant i \leqslant k} |N_G(u_i) \cap Y| \\
&\quad + \sum_{1 \leqslant i \leqslant k} |N_G(u_i) \cap Z| + \sum_{m+1 \leqslant i \leqslant k} |N_G(u_i) \cap \{w\}| + 1 \\
&= \sum_{1 \leqslant i \leqslant k} \deg_G(u_i) + 1 \\
&\geqslant \sigma_k(G) + 1.
\end{aligned}$$

Hence $\sigma_k(G) \leqslant |A| - 1 \leqslant \max\{|A|, |B|\} - 1$. By (2), this is a contradiction.

Next assume $m = k$. By (U1), we have $Leaf(T) \subseteq A$. Since $|X| \geqslant |Y| + 1$, it follows that $X_p \neq \emptyset$ for some $1 \leqslant p \leqslant l$. Let $x \in X_p$. Then, there exists an integer q with $1 \leqslant q \leqslant k$ and $q \neq p$ such that $x \in N_G(u_q)$. Since $T_1 = T - wv_p + xu_q$ is a spanning tree of G with $\mathrm{te}(T;k) \geqslant \mathrm{te}(T_1;k)$, it follows from (T2) that $|Leaf(T)| = |Leaf(T_1)|$, that is, v_p is a leaf of T_1. Therefore $Leaf(T_1) \cap A \neq \emptyset$ and $Leaf(T_1) \cap B \neq \emptyset$. This contradicts (T3).

Case 2. $|X| \leqslant |Y|$ and $m \geqslant 1$.

By Claim 6 (ii), we have $|O \cap B| \geqslant k - m + 1$. By the assumption of Case 2 and Claim 7 (ii), we obtain

$$\begin{aligned}
|B| &= |X| + (k-1)|Y| + |Z| + |O \cap B| \\
&= |X| + (k-m)|Y| + (m-1)|Y| + |Z| + |O \cap B| \\
&\geqslant |X| + (k-m)|Y| + (m-1)|X| + |Z| + |O \cap B| \\
&\geqslant m|X| + (k-m)|Y| + |Z| + k - m + 1 \\
&\geqslant \sum_{1 \leqslant i \leqslant k} |N_G(u_i) \cap \bigcup_{1 \leqslant j \leqslant l} V(D_j)| + \sum_{m+1 \leqslant i \leqslant k} |N_G(u_i) \cap \{w\}| + 1 \\
&\geqslant \sigma_k(G) + 1.
\end{aligned}$$

Hence $\sigma_k(G) \leqslant |B| - 1 \leqslant \max\{|A|, |B|\} - 1$, a contradiction.

Case 3. $m = 0$.

By (U1), $Leaf(D_i) \subseteq B$ for all $1 \leqslant i \leqslant l$. Note that $X = \emptyset$ since $m = 0$, and hence we can ignore condition (U2) and use a symmetry among X_i's. Since $\deg_T(w) \geqslant k+1$, if $\deg_T(a) \geqslant k$ for all $a \in A - \{w\}$ then $|B| \geqslant (k-1)|A| + 2$, a contradiction. Hence we may assume that there exists a vertex $a_h \in V(D_h) \cap A$ such that $\deg_T(a_h) \leqslant k-1$ for some $1 \leqslant h \leqslant l$. Without loss of generality, we may assume that $h = 1$. By Claim 1 (i), $\{a_1, u_2, \ldots, u_k\}$ is an independent set of G.

We regard D_1 as a rooted tree with root a_1, and change the definitions of Y_i $(1 \leqslant i \leqslant l)$ and O_i $(2 \leqslant i \leqslant l)$ as follows;

$$Y_i := \bigcup_{2 \leqslant j \leqslant k, j \neq i} (N_G(u_j) \cap V(D_i)), \quad O_i := V(D_i) - (N_G(a_1) \cup R_i \cup S_i).$$

Following the above change of Y_i, we also change the definition of R_i $(2 \leqslant i \leqslant l)$.

We first consider D_1. By Claim 1 (ii), $(N_G(a_1) \cap V(D_1)) \cap (R_1 \cap B) = \emptyset$. Therefore

$$\begin{aligned}
& |N_G(a_1) \cap V(D_1)| + \sum_{2 \leqslant i \leqslant k} |N_G(u_i) \cap V(D_1)| \\
\leqslant\ & |N_G(a_1) \cap V(D_1)| + (k-1)|Y_1| \\
=\ & |N_G(a_1) \cap V(D_1)| + |R_1 \cap B| \\
\leqslant\ & |V(D_1) \cap B|.
\end{aligned}$$

We next consider $D_2, \ldots, D_l$. By Claim 1 (iii), we have $(N_G(a_1) \cap V(D_i)) \cap (R_i \cap B) = \emptyset$. In the same proof as Claim 5 (ii), we can prove that $(N_G(a_1) \cap V(D_i)) \cap (S_i \cap B) = \emptyset$ for each $2 \leqslant i \leqslant l$.

Using the same argument in the proof of Claim 6 (ii), we can prove that $O_i \cap B \neq \emptyset$ for each $2 \leqslant i \leqslant l$, and hence,

$$\begin{aligned}
& |N_G(a_1) \cap V(D_i)| + \sum_{2 \leqslant j \leqslant k} |N_G(u_j) \cap V(D_i)| \\
\leqslant\ & |N_G(a_1) \cap V(D_i)| + (k-1)|Y_i| + |Z_i| \\
=\ & |N_G(a_1) \cap V(D_i)| + |R_i \cap B| + |S_i \cap B| \\
=\ & |V(D_i) \cap B| - |O_i \cap B| \\
\leqslant\ & |V(D_i) \cap B| - 1.
\end{aligned}$$

By summing above two inequalities, we deduce

$$\begin{aligned}
\sigma_k(G) \leqslant\ & \sum_{1 \leqslant i \leqslant l} \Big(|N_G(a_1) \cap V(D_i)| + \sum_{2 \leqslant j \leqslant k} |N_G(u_j) \cap V(D_i)| \Big) \\
& + \sum_{2 \leqslant j \leqslant k} |N_G(u_j) \cap \{w\}| \\
\leqslant\ & \sum_{1 \leqslant i \leqslant l} |V(D_i) \cap B| - (l-1) + (k-1) \\
\leqslant\ & |B| - 1,
\end{aligned}$$

a contradiction. Consequently the proof is complete.

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