On-line Ramsey Numbers of Paths and Cycles

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Abstract

Consider a game played on the edge set of the infinite clique by two players, Builder and Painter. In each round, Builder chooses an edge and Painter colours it red or blue. Builder wins by creating either a red copy of G or a blue copy of H for some fixed graphs G and H. The minimum number of rounds within which Builder can win, assuming both players play perfectly, is the *on-line Ramsey number* $\tilde{r}(G,H)$. In this paper, we consider the case where G is a path P_k. We prove that $\tilde{r}(P_3,P_{\ell+1}) = \lceil 5\ell/4 \rceil = \tilde{r}(P_3,C_\ell)$ for all $\ell \geqslant 5$, and determine $\tilde{r}(P_4,P_{\ell+1})$ up to an additive constant for all $\ell \geqslant 3$. We also prove some general lower bounds for on-line Ramsey numbers of the form $\tilde{r}(P_{k+1},H)$.

Keywords: On-line Ramsey theory; Combinatorial games; Paths; Cycles.

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1 Introduction

Ramsey's theorem [16] states that for all $k \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that any red-blue edge colouring of a clique K_t contains a monochromatic clique of order k. We call the least such t the k^{th} *Ramsey number*, and denote it by $r(k)$. Ramsey numbers and their generalisations have been a fundamentally important area of study in combinatorics for many years. Particularly well-studied are Ramsey numbers for graphs. Here the *Ramsey number* of two graphs G and H, denoted by $r(G,H)$, is the least t such that any red-blue edge colouring of K_t contains a red copy of G or a blue copy of H. See e.g. [15] for a survey of known Ramsey numbers.

An important generalisation of Ramsey numbers, first defined by Erdős, Faudree, Rousseau and Schelp [5], is as follows. Let G and H be two graphs. We say that a graph K has the (G,H)-*Ramsey property* if any red-blue edge colouring of K must contain either a red copy of G or a blue copy of H. Then the *size Ramsey number* $\hat{r}(G,H)$ is given by the minimum number of edges of any graph with the (G,H)-Ramsey property.

In this paper, we consider the following related generalisation defined independently by Beck [1] and Kurek and Ruciński [10]. Let G and H be two graphs. Consider a game played on the edge set of the infinite clique $K_{\mathbb{N}}$ with two players, Builder and Painter. In each round of the game, Builder chooses an edge and Painter colours it red or blue. Builder wins by creating either a red copy of G or a blue copy of H, and wishes to do so in as few rounds as possible. Painter wishes to delay Builder for as many rounds as possible. (Note that Painter may not delay Builder indefinitely – for example, Builder may simply choose every edge of $K_{r(G,H)}$.) The *on-line Ramsey number* $\tilde{r}(G,H)$ is the minimum number of rounds it takes Builder to win, assuming that both Builder and Painter play optimally. We call this game the $\tilde{r}(G,H)$-*game*, and write $\tilde{r}(G) = \tilde{r}(G,G)$. Note that $\tilde{r}(G,H) \geqslant e(G) + e(H) - 1$ for all graphs G and H, as Painter may simply colour the first $e(G) - 1$ edges red and all subsequent edges blue. It is also clear that $\tilde{r}(G,H) \leqslant \hat{r}(G,H)$.

On-line Ramsey theory has been well-studied. The best known bounds for $\tilde{r}(K_t)$ are given by

$$\frac{r(t)-1}{2} \leqslant \tilde{r}(K_t) \leqslant t^{-c\frac{\log t}{\log\log t}} 4^t,$$

where c is a positive constant. The lower bound is due to Alon (and was first published in a paper of Beck [1]), and the upper bound is due to Conlon [3]. Note that these bounds are similar to the best known bounds for classical Ramsey numbers $r(t)$, although Conlon also proves in [3] that

$$\tilde{r}(K_t) \leqslant C^{-t}\binom{r(t)}{2}$$

for some constant $C > 1$ and infinitely many values of t, which gives positive evidences supporting a conjecture of Kurek and Ruciński [10] that $\tilde{r}(K_t) = o(r(t)^2)$. For general graphs G, the best known lower bound for $\tilde{r}(G)$ is given by Grytczuk, Kierstead and Prałat [8].

Theorem 1. *For graphs G, we have $\tilde{r}(G) \geqslant \beta(G)(\Delta(G)-1)/2 + e(G)$, where $\beta(G)$ denotes the vertex cover number of G.*

Various general strategies for Builder and Painter have also been studied. For example, consider the following strategy for Builder in the $\tilde{r}(G,H)$-game. Builder chooses a large but finite set of vertices in $K_{\mathbb{N}}$, say a set of size $n \in \mathbb{N}$, with $n \geqslant r(G,H)$. Then Builder chooses the edges of the induced K_n in a uniformly random order, allowing Painter to colour each edge as they wish, until the game ends. This strategy was analysed for the $\tilde{r}(K_3)$-game by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali [6], and for the more general $\tilde{r}(G)$-game by Marciniszyn, Spöhel and Steger [11, 12].

Finally, let $\tilde{r}_\chi(G)$-game be the $\tilde{r}(G)$-game in which Builder is forbidden to uncovering a graph with chromatics number greater than $\chi(G)$. Grytczuk, Hałuszczak and Kierstead [7] proved that Builder can win the $\tilde{r}_\chi(G)$-game. Kierstead and Konjevod [9] proved the hypergraph generalisation.

Given the known bounds on $\tilde{r}(K_t)$, it is not surprising that determining on-line Ramsey numbers exactly has proved even more difficult than determining classical Ramsey numbers exactly, and very few results are known. A significant amount of effort has been focused on the special case where G and H are paths. Grytczuk, Kierstead and Prałat [8] and Prałat [13, 14] have determined $\tilde{r}(P_{k+1}, P_{\ell+1})$ exactly when $\max\{k,\ell\} \leqslant 8$ (where P_s is a path on s verices). In addition, Beck [2] has proved that the size Ramsey number $\hat{r}(P_k, P_k)$ is linear in k. (The best known upper bound, due to Dudek and Prałat [4], is $\hat{r}(P_k, P_k) \leqslant 137k$.) The best known bounds on $\tilde{r}(P_{k+1}, P_{\ell+1})$ were proved in [8].

Theorem 2. *For all $k, \ell \in \mathbb{N}$, we have $k + \ell - 1 \leqslant \tilde{r}(P_{k+1}, P_{\ell+1}) \leqslant 2k + 2\ell - 3$.*

In general, it seems difficult to bound on-line Ramsey numbers $\tilde{r}(G,H)$ below. One of the major difficulties in doing so is the variety of possible strategies for Builder. We present a strategy for Painter which mitigates this problem somewhat.

Definition 3. Let $\mathcal{F}$ be a family of graphs. We define the *$\mathcal{F}$-blocking strategy* for Painter as follows. Write R_i for the graph consisting of all uncovered red edges immediately before the ith move of the game, and write e_i for the ith edge chosen by Builder. Then Painter colours e_i red if $R_i + e_i$ is $\mathcal{F}$-free, and blue otherwise. (Recall that a graph is *$\mathcal{F}$-free* if it contains no graph in $\mathcal{F}$ as a subgraph.)

In an $\tilde{r}(G,H)$-game, it is natural to consider $\mathcal{F}$-blocking strategies with $G \in \mathcal{F}$. For example, if $\mathcal{F} = \{G\}$, then the $\mathcal{F}$-blocking strategy for Painter consists of colouring every edge red unless doing so would cause Painter to lose the game. If Painter is using an $\mathcal{F}$-blocking strategy, one clear strategy for Builder would be to construct a red $\mathcal{F}$-free graph, then use it to force a blue copy of H in $e(H)$ moves. We will show that this is effectively Builder's only strategy (see Proposition 13), and thus to bound $\tilde{r}(G,H)$ below it suffices to prove that no small red $\mathcal{F}$-free graph can be used to force a blue copy of H. We use this technique to derive some lower bounds for on-line Ramsey numbers of the form $\tilde{r}(P_{k+1}, H)$, taking $\mathcal{F} = \{P_{k+1}\} \cup \{C_i : i \geqslant 3\}$.

Theorem 4. *Let $k, \ell \in \mathbb{N}$ with $k \geqslant 2$. Let H be a graph on $|H|$ vertices with ℓ edges and let $\Delta = \Delta(H)$. Then*

$$\tilde{r}(P_{k+1}, H) \geqslant \begin{cases} (2\Delta+1)\ell/(2\Delta) & \text{if } k = 2, \\ (5\Delta+4)\ell/(5\Delta) & \text{if } k = 3, \\ (\Delta+1)\ell/\Delta & \text{if } k \geqslant 4. \end{cases}$$

Moreover, if H is connected and $k \geqslant 4$, then

$$\tilde{r}(P_{k+1}, H) \geqslant (\Delta+1)\ell/\Delta + \min\{k/2 - 2, |H| - 1\}.$$

For $k = 2$, we show that if $H = P_{\ell+1}$ for $\ell \geqslant 2$ or $H = C_\ell$ for $\ell \geqslant 5$, then the bound on $\tilde{r}(P_3, H)$ given by Theorem 4 is tight.

Theorem 5. *For all $\ell \geqslant 2$, we have $\tilde{r}(P_3, P_{\ell+1}) = \lceil 5\ell/4 \rceil$. Also,*

$$\tilde{r}(P_3, C_\ell) = \begin{cases} \ell + 2 & \text{if } \ell = 3, 4, \\ \lceil 5\ell/4 \rceil & \text{if } \ell \geqslant 5. \end{cases}$$

Furthermore, for $k = 3$, we determine $\tilde{r}(P_4, P_{\ell+1})$ up to an additive constant for all $\ell \geqslant 3$.

Theorem 6. *For all $\ell \geqslant 3$, we have $(7\ell+2)/5 \leqslant \tilde{r}(P_4, P_{\ell+1}) \leqslant (7\ell+52)/5$.*

Our proof of the upper bound for $k = 3$ is complicated, so the proof is included in the Appendix. The lower bound follows from Lemma 18, a simple extension of the proof of Theorem 4, and we believe that it is tight.

Conjecture 7. For all $\ell \geqslant 3$, we have $\tilde{r}(P_4, P_{\ell+1}) = \lceil (7\ell+2)/5 \rceil$.

By Theorems 5 and 6, we have

$$\lim_{\ell\to\infty} \tilde{r}(P_3, P_{\ell+1})/\ell = 5/4,$$
$$\lim_{\ell\to\infty} \tilde{r}(P_4, P_{\ell+1})/\ell = 7/5.$$

On the other hand, for all fixed $k \geqslant 4$, Theorems 2 and 4 imply that

$$3/2 \leqslant \liminf_{\ell\to\infty} \tilde{r}(P_{k+1}, P_{\ell+1})/\ell \leqslant \limsup_{\ell\to\infty} \tilde{r}(P_{k+1}, P_{\ell+1})/\ell \leqslant 2,$$

and we make the following conjecture.

Conjecture 8. For $k \geqslant 4$, $\lim_{\ell\to\infty} \tilde{r}(P_{k+1}, P_{\ell+1})/\ell = 3/2$. Moreover, for all $\ell \geqslant k \geqslant 4$, we have $\tilde{r}(P_{k+1}, P_{\ell+1}) = \lceil 3\ell/2 \rceil + k - 3$. In particular, we have $\tilde{r}(P_{k+1}) = \lceil 5k/2 \rceil - 3$ for $k \geqslant 4$.

Note that Conjecture 8 would imply Conjecture 4.1 of [14]. Conjectures 7 and 8 have been confirmed for $\ell \leqslant 8$ by Prałat [13], using a high-performance computer cluster.

Finally, we give some bounds on $\tilde{r}(C_4, P_{\ell+1})$.

Theorem 9. *For $\ell \geqslant 3$, we have $2\ell \leqslant \tilde{r}(C_4, P_{\ell+1}) \leqslant 4\ell - 4$. Moreover, $\tilde{r}(C_4, P_4) = 8$.*

Many of the lower bounds above follow from Theorem 4, and all of them follow from analysing $\mathcal{F}$-blocking strategies. In particular, we obtain tight lower bounds on $\tilde{r}(P_3, P_{\ell+1})$ and $\tilde{r}(P_3, C_\ell)$ in this way, as well as a lower bound on $\tilde{r}(P_4, P_{\ell+1})$ which matches Conjecture 7. We are therefore motivated to ask the following question.

Question 10. For which graphs G and H does there exist a family $\mathcal{F}$ of graphs such that the $\mathcal{F}$-blocking strategy is optimal for Painter in the $\tilde{r}(G, H)$-game?

The paper is laid out as follows. In Section 3, we prove Theorem 4. We prove Theorem 5 in Sections 4 and 5 (see Theorem 21, Proposition 23 and Theorem 24). Finally, in Section 6 we prove Theorem 9. The proof of Theorem 6 is in the Appendix.

2 Notation and conventions

We write $\mathbb{N}$ for the set $\{1, 2, \dots\}$ of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Suppose $P = v_1 \dots v_k$ and $Q = w_1 \dots w_\ell$ are paths. If $i < j$, we write $v_i P v_j$ (or $v_j P v_i$) for the subpath $v_i v_{i+1} \dots v_j$ of P. We also write PQ for the concatenation of P and Q. For example, if $i < j$ and $i' < j'$ then $uv_iPv_jyw_{i'}Qw_{j'}$ denotes the path $uv_iv_{i+1} \dots v_j y w_{i'} w_{i'+1} \dots w_{j'}$.

If G is a graph, we will write $|G|$ for the number of vertices of G and $e(G)$ for the number of edges of G.

In the context of an $\tilde{r}(G, H)$-game, an *uncovered edge* is an edge of $K_{\mathbb{N}}$ that has previously been chosen by Builder, and a *new vertex* is a vertex in $K_{\mathbb{N}}$ not incident to any uncovered edge.

Many of our lemmas say that in an $\tilde{r}(G, H)$-game, given a finite coloured graph $X \subseteq K_{\mathbb{N}}$, Builder can force Painter to construct a coloured graph $Y \subseteq K_{\mathbb{N}}$ satisfying some desired property. We will often apply such a lemma to a finite coloured graph $X' \supsetneq X$, and in these cases we will implicitly require $V(Y) \cap V(X') \subseteq V(X)$. (Intuitively, when Builder chooses a new vertex while constructing Y, it should be new with respect to X' rather than X.) This is formally valid, since we may apply the lemma to an $\tilde{r}(G, H)$-game on the board $K_{\mathbb{N}} - (V(X') \setminus V(X))$ and have Builder choose the corresponding edges in $K_{\mathbb{N}}$.

For technical convenience, we allow Builder to "waste" a round in the $\tilde{r}(G, H)$-game by choosing an uncovered edge. If he does so, the round contributes to the duration of the game but the edge Builder chooses is not recoloured. Since such a move is never optimal for Builder, the definition of $\tilde{r}(G, H)$ is not affected.

3 General lower bounds

Our aim is to bound $\tilde{r}(G,H)$ below for graphs G and H. In this section, Painter will always use an $\mathcal{F}$-blocking strategy for some family $\mathcal{F}$ of graphs with $G \in \mathcal{F}$. Hence, as we shall demonstrate in Proposition 13 below, Builder's strategy boils down to choosing a red graph with which to force a blue copy of H.

Definition 11. Let $\mathcal{F}$ be a family of graphs and let $R \subseteq K_{\mathbb{N}}$ be an $\mathcal{F}$-free graph. We say that an edge $e \in K_{\mathbb{N}} - R$ is $(R,\mathcal{F})$-*forceable* if $R+e$ is not $\mathcal{F}$-free. We say a graph H is $(R,\mathcal{F})$-*forceable* if there exists $H' \subseteq K_{\mathbb{N}} - R$ with H' isomorphic to H such that every edge $e \in E(H')$ is $(R,\mathcal{F})$-forceable. We call H' an $(R,\mathcal{F})$-*forced copy of* H. If R and $\mathcal{F}$ are clear from context, we will omit '$(R,\mathcal{F})$-'.

Definition 12. Let $\mathcal{F}$ be a family of graphs and let H be a graph. We say a graph $R \subseteq K_{\mathbb{N}}$ is an $\mathcal{F}$-*scaffolding for* H if the following properties hold.

(i) R is $\mathcal{F}$-free.

(ii) H is $(R,\mathcal{F})$-forceable.

(iii) R contains no isolated vertices.

Proposition 13. *Let G and H be graphs. Let $\mathcal{F}$ be a family of graphs with $G \in \mathcal{F}$. Suppose every $\mathcal{F}$-scaffolding for H has at least m edges. Then $\tilde{r}(G,H) \geqslant m + e(H)$.*

Proof. Consider an $\tilde{r}(G,H)$-game in which Painter uses an $\mathcal{F}$-blocking strategy. Further suppose Builder wins by claiming edges $e_1,\dots,e_r$. Since Builder choosing an edge which Painter colours blue has no effect on Painter's subsequent choices, without loss of generality we may assume that there exists i such that Painter colours $e_1,\dots,e_i$ red and $e_{i+1},\dots,e_r$ blue. Let $R \subseteq K_{\mathbb{N}}$ be the subgraph with edge set $\{e_1,\dots,e_i\}$, and let $B \subseteq K_{\mathbb{N}}$ be the subgraph with edge set $\{e_{i+1},\dots,e_r\}$. Thus R is the uncovered red graph and B is the uncovered blue graph.

We will show that R is an $\mathcal{F}$-scaffolding for H. First note that R is $\mathcal{F}$-free by Painter's strategy, and R has no isolated vertices by definition. Moreover, since $G \in \mathcal{F}$ and Builder wins, there exists $H' \subseteq B$ with H' isomorphic to H. So $e(B) \geqslant e(H)$. Moreover, by Painter's strategy all edges in B must be $(R,\mathcal{F})$-forceable, so H is $(R,\mathcal{F})$-forceable. Hence R is an $\mathcal{F}$-scaffolding for H, so $e(R) \geqslant m$. Therefore, Builder wins in $r \geqslant e(R)+e(B) \geqslant m+e(H)$ rounds. □

Therefore, to bound $\tilde{r}(G,H)$ below, it suffices to bound the number of edges in an $\mathcal{F}$-scaffolding for H below for some family $\mathcal{F}$ of graphs with $G \in \mathcal{F}$. We first use Proposition 13 to bound $\tilde{r}(C_k,H)$ for connected graphs H.

Lemma 14. *Let H be a connected graph. Then every $\{C_i : i \geqslant 3\}$-scaffolding for H has at least $|H|-1$ edges. Moreover, $\tilde{r}(C_k,H) \geqslant |H| + e(H) - 1$ for all $k \geqslant 3$.*

Proof. Let R be a $\{C_k\}$-scaffolding for H with $e(R)$ minimal. Note that each $(R, \{C_k\})$-forceable edge must lie entirely in a component of R. Since H is connected, R is connected and $|R| \geqslant |H|$. Hence, $e(R) \geqslant |H| - 1$.

By Proposition 13, $\tilde{r}(C_k, H) \geqslant |H| + e(H) - 1$. □

To prove Theorem 4, we set $G = P_{k+1}$ and $\mathcal{F} = \{P_{k+1}\} \cup \{C_i : i \geqslant 3\}$. Thus an $\mathcal{F}$-free graph is a forest whose components have diameter less than k. Lemma 17 gives a lower bound on the number of edges in an $\mathcal{F}$-scaffolding for H.

Note that replacing $\mathcal{F}$ by $\{P_{k+1}\}$ and attempting a similar proof yields a worse lower bound in some cases. For example, taking $H = P_{2k+1}$ with $k \geqslant 3$, if Painter follows the $\{P_{k+1}\}$-blocking strategy then Builder can win in $3k$ moves by first constructing a red C_k.

We will see in the proof of Lemma 17 that if R is a red $\mathcal{F}$-free graph with no isolated vertices, and $X \subseteq V(R)$ is the set of endpoints of P_k's in R, then Builder may force at most $\Delta(H)(|R| + |X|)$ edges of H using R. It will therefore be very useful to bound $|R|+|X|$ above in terms of $e(R)$, first in the special case where R is a tree (see Lemma 15) and then in general (see Lemma 16).

Lemma 15. *Let $k, m \in \mathbb{N}$ with $k \geqslant 2$. Let R be a P_{k+1}-free tree with m edges. Let X be the set of endpoints of P_k's in R. If $X \neq \emptyset$, then $|R| + |X| \leqslant 2m - k + 4$.*

Proof. We claim that if $x \in X$, then x is a leaf of R. Indeed, let P be a P_k with one endpoint equal to x. Let $y \in V(P)$ be the neighbour of x in P, and suppose $xz \in E(R)$ for some $z \neq y$. Then either $z \in V(P)$ and $xzPx$ is a cycle in R, or $z \notin V(P)$ and Pxz is a P_{k+1} in R – both are contradictions. Hence if $x \in X$, then x is a leaf. But since $X \neq \emptyset$, R contains a P_k and hence at least $k - 2$ vertices of degree greater than 1. Hence

$$|R| + |X| \leqslant |R| + |R| - (k - 2) = 2m - k + 4,$$

and the proposition follows. □

Lemma 16. *Let $k, m \in \mathbb{N}$ with $k \geqslant 2$. Let R be a P_{k+1}-free forest with m edges and no isolated vertices. Let X be the set of all endpoints of P_k's in R. Then*

$$|R| + |X| \leqslant \begin{cases} 4m & \text{if } k = 2, \\ 5m/2 & \text{if } k = 3, \\ 2m & \text{if } k \geqslant 4. \end{cases}$$

Moreover, if $k \geqslant 4$ and there exists an edge e such that $R + e$ contains a P_{k+1}, then $|R| + |X| \leqslant 2m - k + 4$.

Proof. Let $R_1, \ldots, R_r$ be the components of R. Let $m_i = e(R_i)$ and $X_i = X \cap V(R_i)$ for all $1 \leqslant i \leqslant r$. If $k = 2$, then R is a disjoint union of m edges and the result is immediate.

Suppose $k = 3$. Without loss of generality, let $R_1, \ldots, R_{r'}$ be those components of R which consist of a single edge. (Note that we may have $r' = 0$.) Then $m = r' + \sum_{i=r'+1}^{r} m_i$

and $r - r' \leqslant m/2$. Then by Lemma 15 we have

$$\begin{aligned}|R| + |X| &= \sum_{i=1}^{r'} |R_i| + \sum_{i=r'+1}^{r} (|R_i| + |X_i|) \leqslant 2r' + \sum_{i=r'+1}^{r} (2m_i + 1) \\ &= 2m + r - r' \leqslant 5m/2\end{aligned}$$

and so the result follows.

Finally, suppose $k \geqslant 4$. Let q be the number of components of R containing a P_k. Without loss of generality suppose that $R_1, \ldots, R_q$ are the components of R which contain a P_k. For $q < i \leqslant r$, we have $|R_i| + |X_i| = |R_i| = m_i + 1 \leqslant 2m_i$. Then by Lemma 15 we have

$$|R| + |X| = \sum_{i=1}^{r} (|R_i| + |X_i|) \leqslant \sum_{i=1}^{q} (2m_i - k + 4) + \sum_{i=q+1}^{r} (2m_i) = 2m - q(k-4). \quad (1)$$

Suppose that there exists an edge e such that $R+e$ contains a P_{k+1}. If $X \neq \emptyset$, then $q \geqslant 1$ and so $|R| + |X| \leqslant 2m - k + 4$ by (1). Hence we may assume that $X = \emptyset$, and so e is an edge between two vertices of R. It follows that R contains two vertex-disjoint paths of combined length at least $k - 1$, and hence that

$$|R| + |X| = |R| = m + r \leqslant m + (m - k + 3) < 2m - k + 4,$$

as desired. The first inequality follows since all edges in a given path must lie in the same component of R. □

Lemma 17. *Let $k, \ell \in \mathbb{N}$ with $k \geqslant 2$. Let H be a graph with ℓ edges and let $\Delta = \Delta(H)$. Let $\mathcal{F} = \{P_{k+1}\} \cup \{C_i : i \geqslant 3\}$. Suppose R is an $\mathcal{F}$-scaffolding for H. Then, we have*

$$e(R) \geqslant \begin{cases} \ell/(2\Delta) & \text{if } k = 2, \\ 4\ell/(5\Delta) & \text{if } k = 3, \\ \ell/\Delta & \text{if } k \geqslant 4. \end{cases}$$

Moreover, if H is connected and $k \geqslant 4$ then $e(R) \geqslant \min\left\{\frac{\ell}{\Delta} + \frac{k}{2} - 2, |H| - 1\right\}$.

Proof. Let $m = e(R)$. Note that R is a P_{k+1}-free forest with m edges and no isolated vertices. Let X be the set of endpoints of P_k's in R and let $Y = V(R) \setminus X$.

We first claim that any $(R, \mathcal{F})$-forceable edge is either incident to X or internal to Y. Suppose not. Then there exist $y \in Y$ and $z \notin V(R)$ such that yz is a forceable edge. Let $F \in \mathcal{F}$ be such that $F \subseteq R+e$. Note that $e \in E(F)$, since R is $\mathcal{F}$-free. Since $d_{R+e}(z) = 1$, we have $F = P_{k+1}$. But then y is an endpoint of a P_k in R, contradicting $y \in Y$.

Let H' be a forced copy of H. Then H' contains at most $\Delta|X|$ edges incident to X, and at most $\Delta|Y|/2$ edges internal to Y. All edges of H' are forceable, so it follows that

$$\ell = e(H') \leqslant \Delta|X| + \frac{\Delta|Y|}{2} = \frac{\Delta(|R| + |X|)}{2}. \quad (2)$$

Lemma 16 and (2) imply the lemma holds unless $k \geqslant 4$ and H is connected.

Now suppose H is connected and $k \geqslant 4$. If there exists an edge e such that $R + e$ contains a P_{k+1}, then $|R| + |X| \leqslant 2m - k + 4$ by Lemma 16. Hence, (2) implies that $m \geqslant \frac{\ell}{\Delta} + \frac{k}{2} - 2$. Therefore, we may assume that no such edge exists, and in particular that $X = \emptyset$. This implies that R is a $\{C_i : i \geqslant 3\}$-scaffolding for H. Lemma 14 implies that $m \geqslant |H| - 1$ as required.

□

Theorem 4 follows immediately from Proposition 13 and Lemma 17.

We now bound $\tilde{r}(P_4, P_{\ell+1})$ from below.

Lemma 18. *Let $\ell \in \mathbb{N}$ with $\ell \geqslant 3$. Then we have $\tilde{r}(P_4, P_{\ell+1}) \geqslant (7\ell + 2)/5$.*

Proof. Let $\mathcal{F} = \{P_4\} \cup \{C_i : i \geqslant 3\}$. Let R be an $\mathcal{F}$-scaffolding for $P_{\ell+1}$. Let X be the set of endpoints of P_3's in R, and let $Y = V(R) \setminus X$. By Lemma 16 and Proposition 13, to prove the lemma it suffices to show that $|R| + |X| \geqslant \ell + 1$.

Let H be a forced copy of $P_{\ell+1}$. Note that any $(R, \mathcal{F})$-forceable edge is either incident to X or internal to Y. Note also that $Y \neq \emptyset$. Indeed, if $X = \emptyset$ then this is immediate. If $X \neq \emptyset$, then R is a P_4-free forest containing a P_3. The central vertex of this P_3 cannot be an element of X, and is therefore an element of Y.

Since $\Delta(H) = 2$, H contains at most $2|X|$ edges incident to X. Moreover, since H is a path, $H[Y]$ is a forest and so $e_H(Y) \leqslant |Y| - 1$. It follows that

$$\ell \leqslant 2|X| + |Y| - 1 = |R| + |X| - 1,$$

and hence $|R| + |X| \geqslant \ell + 1$ as desired.

□

4 Determining $\tilde{r}(P_3, P_{\ell+1})$ for $\ell \geqslant 2$

Theorem 4 implies that $\tilde{r}(P_3, P_{\ell+1}) \geqslant \lceil 5\ell/4 \rceil$ for $\ell \geqslant 2$. To bound $\tilde{r}(P_3, P_{\ell+1})$ above, we shall present a strategy for Builder. In the discussion that follows, we assume for clarity that Painter will never voluntarily lose the $\tilde{r}(P_3, P_{\ell+1})$-game.

Builder will use the threat of a red P_3 to force a blue $P_{\ell+1}$. First, Builder will use Lemma 19 to construct a blue path P with one endpoint incident to a red edge. Builder will then use a procedure outlined in Lemma 20 to efficiently extend P until it has length between $\ell - 4$ and ℓ. Finally, Builder will carefully extend P into a blue $P_{\ell+1}$, yielding a tight upper bound for $\tilde{r}(P_3, P_{\ell+1})$ (see Theorem 21).

Lemma 19. *Let $q \in \mathbb{N}$ with $q \geqslant 5$. Builder can force one of the following structures independent of Painter's choices:*

(i) *a red P_3 in at most $q - 1$ rounds.*

(ii) *a blue P_q in $q - 1$ rounds.*

(iii) *a blue P_t with one endpoint incident to a red edge in t rounds for some $4 \leqslant t \leqslant q-1$.*

Proof. Builder first chooses an arbitrary vertex x_1, then proceeds as follows. Suppose that Builder has already obtained a blue path $x_1 \dots x_i$ in $i-1$ rounds for some $1 \leqslant i < q$. Builder then chooses the edge $x_i x_{i+1}$, where x_{i+1} is a new vertex. If Painter colours $x_i x_{i+1}$ blue, we have obtained a blue path $x_1 \dots x_{i+1}$ in i rounds, and so if $i+1 < q$ we may repeat the process. If Painter colours all such edges blue, we will obtain a blue path $x_1 \dots x_q$ in $q-1$ rounds and achieve (ii). Suppose instead that for some $1 \leqslant i \leqslant q-1$, within i rounds we obtain a path $x_1 \dots x_{i+1}$ such that $x_1 \dots x_i$ is blue and $x_i x_{i+1}$ is red. If $i \geqslant 4$ then we have achieved (iii), so suppose in addition $i \leqslant 3$.

First suppose $i \in \{1, 2\}$. In this case, Builder chooses the two edges $x_i v$ and $v x_{i+1}$ where v is a new vertex. If $i = 1$, Builder also chooses the edge $x_{i+1} w$ where w is a new vertex. If Painter colours $x_i v$, $v x_{i+1}$ or $x_{i+1} w$ red, then $x_{i+1} x_i v$, $v x_{i+1} x_i$ or $x_i x_{i+1} w$ respectively is a red P_3 and we have achieved (i). Otherwise, we have achieved (iii). Indeed, if $i = 1$ then $x_1 v x_2 w$ is a blue P_4 constructed in 4 rounds with x_1 incident to the red edge $x_1 x_2$, and if $i = 2$ then $x_1 x_2 v x_3$ is a blue P_4 constructed in 4 rounds with x_3 incident to the red edge $x_3 x_2$.

Finally, suppose $i = 3$. Then Builder chooses the edge $x_4 x_1$. If Painter colours the edge red, then $x_3 x_4 x_1$ is a red P_3 and we have achieved (i), so suppose Painter colours the edge blue. Then $x_4 x_1 x_2 x_3$ is a blue P_4 constructed in 4 rounds with x_3 incident to the red edge $x_3 x_4$, so we have achieved (iii). □

Lemma 20. *Let $\ell \in \mathbb{N}$ with $\ell \geqslant 4$. Builder can force one of the following structures independent of Painter's choices:*

(i) *a red P_3 in at most $5\ell/4 - 1$ rounds.*

(ii) *a blue $P_{\ell+1}$ in at most $5\ell/4 - 1$ rounds.*

(iii) *a blue P_t with one endpoint incident to a red edge in at most $5t/4 - 1$ rounds for some $\ell - 3 \leqslant t \leqslant \ell$.*

Proof. Throughout the proof, we assume for clarity that Painter will always avoid (i) and (ii) if possible. By Lemma 19 (taking $q = \ell + 1$) we may assume that Builder has constructed a blue P_t, say $v_1 \dots v_t$, which satisfies

(∗) $v_1 \dots v_t$ has one endpoint incident to a red edge $v_1 u$, and Builder constructed $v_1 \dots v_t$ in at most $5t/4 - 1$ rounds. Moreover, $4 \leqslant t \leqslant \ell$.

Note that $t \leqslant 5t/4 - 1$ since $t \geqslant 4$.

If $t \geqslant \ell - 3$, then we have achieved (iii). Hence, we may assume that $4 \leqslant t < \ell - 3$. Without loss of generality, let $v_1 u$ be a red edge as in (∗). Builder will extend $v_1 \dots v_t$ as follows. We apply Lemma 19 with $q = \ell - t + 1 \geqslant 5$ on a set of new vertices. We split into cases depending on Painter's choice.

Case 1: Builder obtains a red P_3 in at most $\ell - t$ rounds, as in Lemma 19(i).

In this case, Builder has spent at most $5t/4 - 1 + \ell - t \leqslant 5\ell/4 - 2$ rounds in total since $t \leqslant \ell - 4$, and so we have achieved (i).

Case 2: Builder obtains a blue path $w_1 \dots w_{\ell-t+1}$ in $\ell - t$ rounds, as in Lemma 19(ii).

In this case, Builder has again spent at most $5\ell/4 - 2$ rounds in total. Builder now chooses the edge w_1v_1. If Painter colours it red, then w_1v_1u is a red P_3 and we have achieved (i). If Painter colours it blue, then $w_{\ell-t+1} \dots w_1v_1 \dots v_t$ is a blue $P_{\ell+1}$ and we have achieved (ii).

Case 3: Builder obtains a blue path $w_1 \dots w_{t'}$ and a red edge w_1x in at most t' rounds for some $4 \leqslant t' \leqslant \ell - t$, as in Lemma 19(iii).

In this case, Builder has spent at most

$$\frac{5t}{4} - 1 + t' = \frac{5t}{4} + \frac{5t'}{4} - \frac{t'}{4} - 1 \leqslant \frac{5(t+t')}{4} - 2 \leqslant \frac{5\ell}{4} - 2$$

rounds in total. Builder now chooses the edge v_tw_1. If Painter colours it red, then v_tw_1x is a red P_3 and we have achieved (i). If Painter colours it blue, then $v_1 \dots v_tw_1 \dots w_{t'}$ is a blue $P_{t+t'}$ with v_1 incident to the red edge v_1u. Moreover, this $P_{t+t'}$ satisfies $(*)$ with $t + t' > t$. Hence by iterating the argument above, the result follows. □

Theorem 21. *For all $\ell \geqslant 2$, $\tilde{r}(P_3, P_{\ell+1}) = \lceil 5\ell/4 \rceil$.*

Proof. Theorem 4 implies that $\tilde{r}(P_3, P_{\ell+1}) \geqslant \lceil 5\ell/4 \rceil$. It therefore suffices to prove that Builder can win the $\tilde{r}(P_3, P_{\ell+1})$-game within $\lceil 5\ell/4 \rceil$ rounds. First note that $\tilde{r}(P_3, P_3) = 3$ and $\tilde{r}(P_3, P_4) = 4$, as shown by Grytczuk, Kierstead and Prałat [8] and Prałat [13] respectively, so we may assume $\ell \geqslant 4$. Applying Lemma 20, either Builder obtains a blue path $v_1 \dots v_{t+1}$ and a red edge v_1u in at most $5(t+1)/4-1$ rounds for some $\ell-3 \leqslant t+1 \leqslant \ell$ or we are done. Write

$$r(t) = \left\lceil \frac{5\ell}{4} \right\rceil - \left(\left\lfloor \frac{5(t+1)}{4} \right\rfloor - 1 \right) = \left\lceil \frac{\ell}{4} \right\rceil - \left\lfloor \frac{t+1}{4} \right\rfloor + (\ell - t),$$

and note that Builder has at least $r(t)$ rounds left to construct either a red P_3 or a blue $P_{\ell+1}$. We now split into cases depending on the precise value of t.

Case 1: $t = \ell - 1$, so that $r(t) = 1$.

Builder chooses the edge v_0v_1, where v_0 is a new vertex. If Painter colours it red, then v_0v_1u is a red P_3 and we are done. Otherwise, $v_0v_1 \dots v_\ell$ is a blue $P_{\ell+1}$ and we are done.

Case 2: $t = \ell - 2$, so that $r(t) \geqslant 3$.

Builder chooses the edge $v_{\ell-1}x$, where x is a new vertex. If Painter colours it blue, then we are in Case 1 with an extra round to spare. If Painter colours it red, Builder chooses the edges $v_{\ell-1}w$ and wx, where w is a new vertex. If Painter colours either edge red then $xv_{\ell-1}w$ or $wxv_{\ell-1}$ respectively is a red P_3 and we are done. Otherwise, $v_1 \dots v_{\ell-1}wx$ is a blue $P_{\ell+1}$ and we are done.

Case 3: $t = \ell - 3$, so that $r(t) \geqslant 4$.

Builder chooses the edge $v_{\ell-2}x$, where x is a new vertex. If Painter colours it blue, then we are in Case 2. If Painter colours it red, Builder chooses the edges $v_{\ell-2}w$, wx and xy, where w and y are new vertices. If Painter colours any of these edges red then $xv_{\ell-2}w$, $wxv_{\ell-2}$ or $v_{\ell-2}xy$ respectively is a red P_3 and we are done. Otherwise, $v_1 \dots v_{\ell-2}wxy$ is a blue $P_{\ell+1}$ and we are done.

Case 4: $t = \ell - 4$, so that $r(t) \geqslant 5$.

Builder chooses the edge $v_{\ell-3}x$, where x is a new vertex. If Painter colours it blue, then we are in Case 3. If Painter colours it red, Builder chooses the edges v_0v_1, $v_{\ell-3}w$, wx and xy, where v_0, w and y are new vertices. If Painter colours any of these edges red then v_0v_1u, $xv_{\ell-3}w$, $wxv_{\ell-3}$ or $v_{\ell-3}xy$ respectively is a red P_3 and we are done. Otherwise, $v_0v_1 \dots v_{\ell-3}wxy$ is a blue $P_{\ell+1}$ and we are done. □

5 Determining $\tilde{r}(P_3, C_\ell)$ for $\ell \geqslant 3$

Our aim is to determine $\tilde{r}(P_3, C_\ell)$ for all $\ell \geqslant 3$, so proving Theorem 5. As a warmup, we first determine $\tilde{r}(P_3, C_3)$ and $\tilde{r}(P_3, C_4)$. Note that Theorem 4 implies that $\tilde{r}(P_3, C_3) \geqslant 5\ell/4$ for all $\ell \geqslant 3$, but this lower bound is too weak when $\ell \leqslant 4$. Instead, we consider the $\{C_\ell\}$-blocking strategy for Painter in an $\tilde{r}(C_\ell, P_3)$-game.

Proposition 22. *For all $\ell \geqslant 3$, we have $\tilde{r}(P_3, C_\ell) \geqslant \ell + 2$.*

Proof. We consider the $\{C_\ell\}$-blocking strategy for Painter in the $\tilde{r}(C_\ell, P_3)$-game. Let R be an edge-minimal $\{C_\ell\}$-scaffolding for P_3. Then R must contain two distinct P_ℓ's, so $e(R) \geqslant \ell$. The result therefore follows from Proposition 13. □

The upper bounds are both relatively straightforward.

Proposition 23. *We have $\tilde{r}(P_3, C_3) = 5$ and $\tilde{r}(P_3, C_4) = 6$.*

Proof. By Proposition 22, we have $\tilde{r}(P_3, C_3) \geqslant 5$ and $\tilde{r}(P_3, C_4) \geqslant 6$. It is easy to show that $r(P_3, C_4) = 4$ (see e.g. Radziszowski [15]), so we also have $\tilde{r}(P_3, C_4) \leqslant \binom{4}{2} = 6$ as Builder may simply choose the edges of a K_4. It therefore suffices to prove that Builder can win the $\tilde{r}(P_3, C_3)$-game in 5 rounds.

Take new vertices u, v, w, x, y and z. Builder first chooses the edges uv, uw and ux. If Painter colours more than one of these edges red, then we have obtained a red P_3 and we are done.

Suppose Painter colours uv, uw and ux blue. Then Builder chooses the edges vw and wx. If Painter colours either edge blue, then $vwuv$ or $wxuw$ respectively is a blue C_3 and we are done. If Painter colours both edges red, then vwx is a red P_3 and we are done.

Finally, suppose Painter colours (without loss of generality) uv red, but uw and ux blue. Then Builder chooses the edge xy. If Painter colours xy red, Builder chooses the edge wx, yielding either a red P_3 (namely wxy), or a blue C_3, $wxuw$, and we are done. If Painter colours xy blue, Builder chooses the edge yu, yielding either a red P_3 (namely yuv) or a blue C_3 (namely $uxyu$), and we are done. □

We now determine $\tilde{r}(P_3, C_\ell)$ for $\ell \geqslant 5$. As in Section 4, Builder's strategy will be to build up a long blue path using Lemma 20. Builder will then carefully close this path into a blue C_ℓ.

Theorem 24. *For all $\ell \geqslant 5$, $\tilde{r}(P_3, C_\ell) = \lceil 5\ell/4 \rceil$.*

Proof. Theorem 4 implies that $\tilde{r}(P_3, C_\ell) \geqslant \lceil 5\ell/4 \rceil$. It therefore suffices to prove that Builder can win the $\tilde{r}(P_3, C_\ell)$-game within $\lceil 5\ell/4 \rceil$ rounds. By Lemma 20, Builder can force one of the following structures independent of Painter's choices:

(i) a red P_3 in at most $5(\ell-1)/4 - 1$ rounds.

(ii) a blue P_ℓ in at most $5(\ell-1)/4 - 1$ rounds.

(iii) a blue P_t with one endpoint incident to a red edge in at most $5t/4 - 1$ rounds for some $\ell - 4 \leqslant t \leqslant \ell - 1$.

If Painter chooses (i), then we are done. Suppose Painter chooses (ii), so that Builder has at least

$$\left\lceil \frac{5\ell}{4} \right\rceil - \left(\frac{5(\ell-1)}{4} - 1 \right) = \left\lceil \frac{5\ell}{4} \right\rceil - \frac{5\ell}{4} + \frac{9}{4} > 2$$

rounds to construct a red P_3 or a blue C_ℓ, and let $v_1 \dots v_\ell$ be the corresponding blue path. Then Builder chooses the edges $v_\ell v_1$, $v_1 v_3$ and $v_\ell v_2$. If Painter colours $v_\ell v_1$ blue then $v_1 \dots v_\ell v_1$ is a blue C_ℓ and we are done. If Painter colours $v_\ell v_1$ red and $v_1 v_3$ or $v_\ell v_2$ red, then $v_\ell v_1 v_3$ or $v_1 v_\ell v_2$ respectively is a red P_3 and we are done. Finally, if Painter colours both $v_1 v_3$ and $v_\ell v_2$ blue, then $v_1 v_3 v_4 \dots v_\ell v_2 v_1$ is a blue C_ℓ and we are done.

Finally, suppose Painter chooses (iii). Let $v_1 \dots v_t$ be the corresponding blue path and let $v_1 u$ be a red edge. Write

$$r(t) = \left\lceil \frac{5\ell}{4} \right\rceil - \left(\left\lfloor \frac{5t}{4} \right\rfloor - 1 \right) = \left\lceil \frac{\ell}{4} \right\rceil - \left\lfloor \frac{t}{4} \right\rfloor + \ell - t + 1,$$

so that Builder has at least $r(t)$ rounds left to construct either a red P_3 or a blue C_ℓ. We split into cases depending on the precise value of t.

Case 1: $t = \ell - 1$, so that $r(t) \geqslant 3$.

Builder first chooses the edge $v_{\ell-1} w$, where w is a new vertex. If Painter colours $v_{\ell-1} w$ blue, then Builder chooses the edge $w v_1$. If Painter colours $w v_1$ red then $w v_1 u$ is a red P_3, and if Painter colours $w v_1$ blue then $v_1 v_2 \dots v_{\ell-1} w v_1$ is a blue C_ℓ. Now suppose Painter colours $v_{\ell-1} w$ red instead. Then Builder chooses the edges $v_{\ell-1} x$ and $x v_1$, where x is a new vertex. If Painter colours either edge red, then $w v_{\ell-1} x$ or $x v_1 u$ respectively is a red P_3 and we are done. Otherwise, $v_1 \dots v_{\ell-1} x v_1$ is a blue C_ℓ and we are done.

Case 2: $t = \ell - 2$, so that $r(t) \geqslant 4$.

Builder first chooses the edge $v_{\ell-2} w$, where w is a new vertex. If Painter colours $v_{\ell-2} w$ blue then we are in Case 1, so suppose Painter colours $v_{\ell-2} w$ red. Builder then chooses the edges $v_{\ell-2} x$, $x w$ and $w v_1$, where x is a new vertex. If Painter colours any of these edges

red, then $wv_{\ell-2}x$, $xwv_{\ell-2}$ or $v_{\ell-2}wv_1$ respectively is a red P_3 and we are done. Otherwise, $v_1v_2\dots v_{\ell-2}xwv_1$ is a blue C_ℓ and we are done.

Case 3: $t = \ell - 3$, so that $r(t) \geqslant 5$.

Builder first chooses the edge $v_{\ell-3}w$, where w is a new vertex. If Painter colours $v_{\ell-3}w$ blue then we are in Case 2, so suppose Painter colours $v_{\ell-3}w$ red. Builder then chooses the edges $v_{\ell-3}x$, xw, wy and yv_1, where x and y are new vertices. If Painter colours any of these edges red, then $wv_{\ell-3}x$, $xwv_{\ell-3}$, $v_{\ell-3}wy$ or yv_1u respectively is a red P_3 and we are done. Otherwise, $v_1v_2\dots v_{\ell-3}xwyv_1$ is a blue C_ℓ and we are done.

Case 4: $t = \ell - 4$, so that $r(t) \geqslant 6$.

Builder first chooses two edges wx and xy, where w, x and y are new vertices. If Painter colours both edges red, wxy is a red P_3 and we are done. Now suppose that Painter colours one edge blue and one red, say wx red and xy blue. Then Builder chooses the edges $v_{\ell-4}w$, wz, zx and yv_1, where z is a new vertex. If Painter colours any of these edges red, then $v_{\ell-4}wx$, xwz, zxw or yv_1u respectively is a red P_3 and we are done. Otherwise, $v_1v_2\dots v_{\ell-4}wzxyv_1$ is a blue C_ℓ and we are done.

We may therefore assume that Painter colours both wx and xy blue. Builder now chooses the edge $v_{\ell-4}w$. If Painter colours $v_{\ell-4}w$ blue, we are in Case 1 (taking our path to be $v_1v_2\dots v_{\ell-4}wxy$), so suppose Painter colours $v_{\ell-4}w$ red. Then Builder chooses the edges $v_{\ell-4}z$, zw and yv_1, where z is a new vertex. If Painter colours any of these edges red, then $wv_{\ell-4}z$, $zwv_{\ell-4}$ or yv_1u respectively is a red P_3 and we are done. Otherwise, $v_1v_2\dots v_{\ell-4}zwxyv_1$ is a blue C_ℓ and we are done. □

6 Bounding $\tilde{r}(C_4, P_{\ell+1})$ for $\ell \geqslant 3$

Our aim is to prove Theorem 9, i.e. to bound $\tilde{r}(C_4, P_{\ell+1})$ for all $\ell \geqslant 3$. First we prove that $\tilde{r}(C_4, P_4) = 8$.

Proposition 25. $\tilde{r}(C_4, P_4) = 8$.

Proof. First, we consider the $\{C_4\}$-blocking strategy for Painter in the $\tilde{r}(C_4, P_4)$-game. Let R be an edge-minimal $\{C_4\}$-scaffolding for P_4. Then R must contain three distinct P_4's, so $e(R) \geqslant 5$ as R is C_4-free. Proposition 13 implies that $\tilde{r}(C_4, P_4) \geqslant 8$.

It therefore suffices to prove that Builder can win the $\tilde{r}(C_4, P_4)$-game within 8 rounds. Builder first chooses the edges uv_1, …, uv_4 for distinct vertices $u, v_1, \dots, v_4$. Without loss of generality we may assume that there exists an integer j such that Painter colours the edges uv_i blue if $i \leqslant j$, and red otherwise.

Suppose $j \geqslant 2$. Then Builder chooses four edges v_1w, v_2w, v_1w' and v_2w', where w and w' are new vertices. If Painter colours all edges red, then $v_1wv_2w'v_1$ is a red C_4. If Painter colours one of the edges blue say v_2w, then v_1uv_2w is a blue P_4.

Suppose $j \leqslant 1$. Then Builder chooses edges v_1v_2 and v_1v_3. If Painter colours both edges red, then $uv_2v_1v_3u$ is a red C_4. Suppose that Painter colours both edges blue. Builder then chooses the edges v_2v_4 and v_3v_4. If Painter colours both v_2v_4 and v_3v_4 red, then $uv_2v_4v_3u$ is a red C_4. Otherwise, $v_3v_1v_2v_4$ or $v_2v_1v_3v_4$ is a blue P_4. Therefore we may

assume that v_1v_2 is blue and v_1v_3 is red. Further suppose that $j = 1$ and so uv_1 is blue. Then Builder chooses the edges v_2v_3 and v_2v_4. If Painter colours one of them blue, then $uv_1v_2v_3$ or $uv_1v_2v_4$ is a blue P_4. Otherwise $uv_3v_2v_4u$ is a red C_4. Finally, suppose that $j = 0$. Builder chooses the edges v_2v_3 and v_3v_4. If Painter colours one of them red, then $uv_1v_3v_2u$ or $uv_1v_3v_4u$ is a red C_4. Otherwise $v_1v_2v_3v_4$ is a blue P_4. □

We now prove Theorem 9.

Proof of Theorem 9. The lower bound follows from Lemma 14 and $\tilde{r}(C_4, P_4) = 8$ by Proposition 25. To prove the theorem, it is enough to show that $\tilde{r}(C_4, P_{\ell+1}) \leqslant 4\ell - 4$ for all $\ell \geqslant 3$. We proceed by induction on ℓ. By Proposition 25, this is true for $\ell = 3$. Suppose instead that $\ell \geqslant 4$ and Builder first spends at most $4\ell - 8$ rounds forcing Painter to construct a red C_4 or a blue $P_\ell = v_1 \dots v_\ell$. (This is possible by the induction hypothesis.) We may assume that the latter holds or else we are done. Then Builder chooses four edges v_1x, $v_\ell x$, v_1y and $v_\ell y$, where x and y are new vertices. If Painter colours all edges red, then $v_1xv_\ell yv_1$ is a red C_4. If Painter colours one of the edges blue, say $v_\ell x$, then $v_1 \dots v_\ell x$ is a blue $P_{\ell+1}$. In total, Builder has chosen at most $4\ell - 4$ edges and the proposition follows. □

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A Bounding $\tilde{r}(P_4, P_{\ell+1})$ for $\ell \geqslant 3$

Here, we prove Theorem 6. Lemma 18 implies that $\tilde{r}(P_4, P_{\ell+1}) \geqslant (7\ell+2)/5$ for $\ell \geqslant 3$. It therefore suffices to bound $\tilde{r}(P_4, P_{\ell+1})$ above, which we do in Theorem 44. In the following discussion we take on the role of Builder, and we will assume for clarity that Painter will not voluntarily lose the game by creating a red P_4.

We will employ the following strategy to construct a blue $P_{\ell+1}$. We will obtain two (initially trivial) vertex-disjoint blue paths Q and R, repeatedly extend them, and then join them together to form a blue $P_{\ell+1}$ when they are sufficiently long. Here Q is distinct from R in that we require one of Q's endpoints to be incident to a red edge bc disjoint from $V(R)$. Some of our methods for extending a blue path require this property, and others destroy it. Thus at each stage we will extend either Q or R depending on which of our extension methods Painter allows us to use.

We will use the following lemma to join Q and R together (and sometimes to extend Q).

Lemma 26. *Let Q be a (possibly trivial) blue path with endpoints a and b, where b is incident to a red edge bc. Let R be a (possibly trivial) blue path vertex-disjoint from $V(Q) \cup \{c\}$. Then Builder can force Painter to construct one of the following while uncovering at most 2 edges:*

(i) *a blue path Q' of length $e(Q) + e(R) + 1$ with one endpoint incident to a red edge.*

(ii) *a red P_4.*

Proof. First suppose that R is non-trivial, and let x and y be the endpoints of R. Moreover, suppose that either $a = c$ or Q is trivial, so that both endpoints of Q are incident

to bc. Builder chooses the edges bx and cy. If Painter colours both edges red, then $xbcy$ is a red P_4. Hence, without loss of generality, we may assume that Painter colours bx blue. Then $Q' := aQbxRy$ is a blue path of length $e(Q) + e(R) + 1$, where a is incident to the red edge bc.

Now suppose that Q is non-trivial and $a \neq c$. Builder chooses the edge ax. If Painter colours ax blue, then $bQaxRy$ is a blue path of length $e(Q) + e(R) + 1$ with endpoint b incident to the red edge bc. So we may assume that Painter colours ax red. Builder then chooses the edge bx. If Painter colours bx red, then $cbxa$ is a red P_4. Otherwise $Q' := aQbxRy$ is a blue path of length $e(Q)+e(R)+1$ where a is incident to the red edge ax.

Finally, suppose R is trivial with endpoint x. Let y be a new vertex. Then the argument above implies the lemma on replacing xRy with x throughout. □

The arguments that follow are by necessity somewhat technical. The reader may therefore find the following intuition useful.

(i) For every seven edges we uncover, we will extend either Q or R by five blue edges.

(ii) When we join Q and R, $e(Q) + e(R) + 1$ should not be too much greater than ℓ.

It is clear that following the above principles will yield a bound of the form $\tilde{r}(P_4, P_{\ell+1}) \leqslant 7\ell/5 + C$ for some constant C. We will violate (i) in the first and last phases of Builder's strategy, but this introduces only constant overhead.

Before we can apply Lemma 26 to join Q and R and obtain a blue $P_{\ell+1}$, we must extend them until $e(Q) + e(R) + 1 \geqslant \ell$. Each time we extend Q and R, we require two independent edges of the same colour. (Naturally, we can obtain these by choosing three independent edges.) If these edges are blue, we may extend Q efficiently using Lemma 30 (see Section A.1). If they are red, we may extend either Q or R efficiently using Lemma 39 (see Section A.2). Note that the latter case is significantly harder. We then apply Lemmas 30 and 39 repeatedly to prove Theorem 44 (see Section A.3).

In our figures throughout the section, we shall represent blue edges with solid lines and red edges with dotted lines.

A.1 Extending Q using two independent blue edges e and f.

Throughout this subsection, e and f will be two independent blue edges vertex-disjoint from Q and R. We will prove that we can use these two edges to efficiently extend Q – see Lemma 30. We first define a special type of path which will be important to the extension process.

Definition 27. We say that a path $xySz$ is *of type A* if xy is a red edge and S is a non-trivial blue path with endpoints y and z.

Note that the above definition requires $x \notin V(S)$. For the remainder of the section, if we refer to a path $xySz$ of type A, we shall take it as read that x, y, z and S are as in Definition 27.

We now sketch the proof of Lemma 30. By greedily extending the blue edge e into a path, Builder can obtain either a long blue path or a path of type A (see Lemma 28). If Builder obtains a long blue path P, then we can simply join P and Q together using Lemma 26. Suppose instead Builder obtains a path $xySz$ of type A. Then we use Lemma 29 to efficiently join S and Q together. In either case, the resulting blue path Q' also has an endpoint incident to a red edge, so Q' retains the defining property of Q.

We first prove that Builder can obtain either a long blue path or a path of type A by greedily extending e.

Lemma 28. *Let $m \in \mathbb{N}$ and let e be a blue edge. Then Builder can force Painter to construct one of the following:*

(i) *a path $xySz$ of type A with $e(S) = t$ while uncovering t edges for some $1 \leqslant t < m$.*

(ii) *a blue path of length m while uncovering $m - 1$ edges.*

Proof. Let S_1 be the blue path formed by e. Builder proceeds to extend S_1 greedily until either Builder has constructed a blue path of length m or Painter has coloured an edge red.

Indeed, suppose S_i is a blue path of length i for some $1 \leqslant i \leqslant m-1$ with endpoints y and z, and that Builder has uncovered $i - 1$ edges in forming S_i from S_1. Then Builder chooses the edge xy, where x is a new vertex. If Painter colours xy red then xyS_iz is a path of type A with $e(S_i) = i$, where $1 \leqslant i < m$. Moreover, Builder has uncovered i edges in constructing it, and so we have achieved (i). If instead Painter colours xy blue, then $S_{i+1} := xyS_iz$ is a blue path of length $i + 1$ and Builder has uncovered i edges in constructing it.

By repeating this process, Builder must either obtain a path of type A as in (i) or a blue path S_m of length m as in (ii). □

We now prove that Builder can use a path of type A to efficiently extend Q. Recall that we were given two independent blue edges, e and f, and that we have already used e to construct a path of type A.

Lemma 29. *Suppose Q is a non-trivial blue path with endpoints a and b, where b is incident to a red edge bc.Suppose $xySz$ is a path of type A which is vertex-disjoint from $V(Q) \cup \{c\}$. Further suppose that $f = vw$ is a blue edge vertex-disjoint from $V(Q) \cup V(xySz) \cup \{c\}$. Then Builder can force Painter to construct one of the following:*

(i) *a blue path Q' of length $e(Q) + e(S) + 2$ with one endpoint b' incident to a red edge $b'c'$ while uncovering 2 edges. Moreover, f is vertex-disjoint from $V(Q') \cup \{c'\}$.*

(ii) *a blue path Q' of length $e(Q) + e(S) + 4$ with one endpoint incident to a red edge $b'c'$ while uncovering 4 edges. (Note that f need not be vertex-disjoint from $V(Q') \cup \{c'\}$.)*

(iii) *a red P_4 while uncovering at most 4 edges.*

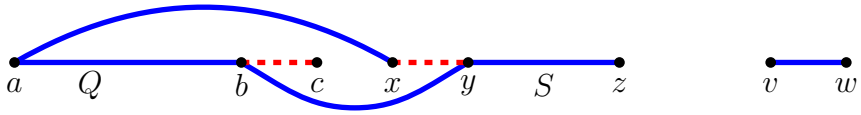

Figure 1: Extending Q using a path of type A as in Lemma 29(i).

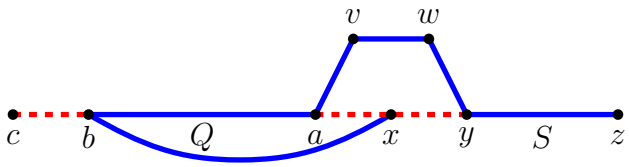

Figure 2: Extending Q using a path of type A and an blue independent edge vw as in Lemma 29(ii).

Proof. Builder chooses the edge ax. First suppose Painter colours ax blue. Builder then chooses the edge by. If Painter colours the edge by red, then $cbyx$ is a red P_3 and we have achieved (iii). Suppose not. Then $Q' := xaQbySz$ (see Figure 1) is a blue path of length $e(Q) + e(S) + 2$, where x is incident to the red edge xy, and we have achieved (i).

Now suppose Painter instead colours ax red. Builder then chooses the edges av, wy and xb. If Painter colours any of these edges red, then $yxav$, $wyxa$ or $yxbc$ respectively is a red P_4 and we have achieved (iii). Suppose not. Then $Q' := xbQavwySz$ (see Figure 2) is a blue path of length $e(Q) + e(S) + 4$, where x is incident to the red edge xy, and we have achieved (ii). □

We now consolidate Lemmas 28 and 29 into a single lemma which says that given two independent blue edges, Builder can efficiently extend Q. In applying Lemma 30, we will take m to be $\ell - e(Q) - e(R) - 1$. Thus if we can extend Q by at least m edges, then we can join Q and R to obtain a blue $P_{\ell+1}$ immediately afterwards.

Lemma 30. *Let $m \in \mathbb{N}$. Suppose Q is a non-trivial blue path with endpoints a and b, where b is incident to a red edge bc. Suppose e and f are two independent blue edges which are vertex-disjoint from $V(Q) \cup \{c\}$. Then Builder can force Painter to construct one of the following:*

(i) *a blue path Q' with $e(Q') = e(Q) + \ell'$ for some $3 \leqslant \ell' \leqslant m + 3$ such that Q' has an endpoint b' incident to a red edge $b'c'$. A total of ℓ' edges are uncovered in the process. Moreover, if $\ell' < 5 \leqslant m$, then f is vertex-disjoint from $V(Q') \cup \{c'\}$.*

(ii) *a red P_4 while uncovering at most $m + 3$ edges.*

Proof. We apply Lemma 28 to e and m, and split into cases depending on Painter's choice.

Case 1: As in Lemma 28(i), we obtain a path $xySz$ of type A with $e(S) = t$ for some $1 \leqslant t < m$ which is vertex-disjoint from $V(f) \cup V(Q) \cup \{c\}$, while uncovering t edges.

We apply Lemma 29 to Q, $xySz$ and f. First suppose that as in Lemma 29(i), we obtain a blue path Q' of length $e(Q)+t+2$ with one endpoint incident to a red edge while

preserving f's independence. In total we have uncovered $t+2$ edges. Hence Q' satisfies (i) on setting $\ell' = t+2$.

Now suppose that as in Lemma 29(ii), we obtain a blue path Q' of length $e(Q)+t+4$ with one endpoint incident to a red edge. We have uncovered $t+4$ edges in total. Hence setting $\ell' = t+4$, we have achieved (i) with $\ell' \geqslant 5$.

Finally, suppose that as in Lemma 29(iii) we obtain a red P_4. Then we have uncovered at most $t+4 \leqslant m+3$ edges in total and so we have achieved (ii).

Case 2: As in Lemma 28(ii), we obtain a blue path S of length m which is vertex-disjoint from $V(Q) \cup \{c\}$ while uncovering $m-1$ edges.

We apply Lemma 26 to Q and S to construct either a blue path Q' of length $e(Q)+m+1$ with one endpoint incident to a red edge or a red P_4 while uncovering at most 2 additional edges. We have uncovered at most $m+1$ edges in total. Hence in the former case we have achieved (i), and in the latter case we have achieved (ii). □

A.2 Extending Q and R using two red edges e and f.

In this subsection, our aim is to extend Q or R efficiently when given two independent red edges e and f – see Lemma 39. As in Section A.1, it will be convenient to define some special paths that we will use in the extension process. These paths can be viewed as analogues of paths of type A.

Definition 31. A path $vwxyz$ is *of type B* if vw and yz are red edges, and wx and xy are blue edges.

Definition 32. A path $T_1 \dots T_k$ is *of type C* if the following statements hold:

(C1) k is odd and $k \geqslant 3$.

(C2) T_1 is either a blue edge or a path of the form $x_1y_1z_1$, where $z_1 \in V(T_2)$ and y_1z_1 is red (and x_1y_1 may be red or blue).

(C3) T_k is either a blue edge or a path of the form $x_ky_kz_k$, where $x_k \in V(T_{k-1})$ and x_ky_k is red (and y_kz_k may be red or blue).

(C4) $T_2, T_4, \dots, T_{k-1}$ are blue paths. Exactly one of these paths has length 1 and the rest have length 2.

(C5) $T_3, T_5, \dots, T_{k-2}$ are all red P_3's.

We say $T_1 \dots T_k$ is *incomplete* if T_1 or T_k is a red P_3. Otherwise, we say $T_1 \dots T_k$ is *complete*.

For the remainder of the section, if we refer to a path $vwxyz$ of type B or a path $T_1 \dots T_k$ of type C, we shall take it as read that v, w, x, y, z and $T_1, \dots, T_k$ are as in Definitions 31 and 32 respectively. Note that paths of type C are well-defined with respect to direction of traversal – if $v_1 \dots v_p$ is a path of type C, then so is $v_p \dots v_1$.

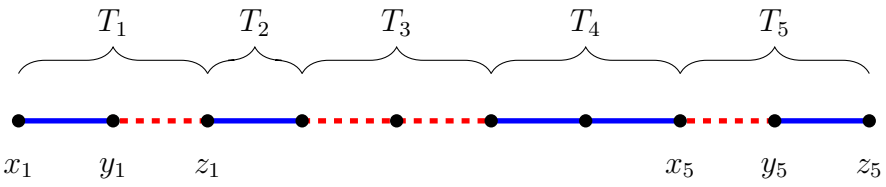

Figure 3: A complete path $T_1 \dots T_5$ of type C.

See Figure 3 for an example of a path of type C.

We now sketch the proof of Lemma 39. Let e and f be two independent red edges. Using these edges, Builder can force either a path of type B or a path of type C using Lemma 33. If Builder obtains a path $vwxyz$ of type B, they will apply Lemma 34 to efficiently extend Q using $vwxyz$.

Suppose instead Builder obtains a path $T_1 \dots T_k$ of type C. Then we run into a problem – $T_1 \dots T_k$ is not complete, and only a complete path of type C may be used to efficiently extend R (see Lemma 38). Builder will therefore use Corollary 37 to extend $T_1 \dots T_k$ into a path $T'_1 \dots T'_{k'}$ of type C which is either complete or arbitrarily long. Builder then uses Lemma 38 to extend R using $T'_1 \dots T'_{k'}$. If $T'_1 \dots T'_{k'}$ is complete, this extension is efficient; otherwise, Builder wins the game immediately afterwards by joining Q and the resulting blue path. Thus an incomplete path of type C is used to extend R at most once over the course of the game, adding only constantly many rounds to the game's length.

We first prove that given two independent red edges Builder can force either a path of type B or a path of type C.

Lemma 33. *Given two independent red edges e and f, Builder can force Painter to construct one of the following:*

(i) *a path of type B while uncovering 2 edges;*

(ii) *an incomplete path $T_1T_2T_3$ of type C and length 5 while uncovering 3 edges;*

(iii) *a red P_4 while uncovering 2 edges.*

Proof. Write $e = uv$ and $f = xy$. Builder chooses the edges vw and wx, where w is a new vertex. If Painter colours both edges red, then $uvwx$ is a red P_4 and we have achieved (iii). Suppose without loss of generality that Painter colours vw blue. If Painter also colours wx blue, then $uvwxy$ is a path of type B and we have achieved (i). If instead Painter colours wx red, then Builder chooses the edge tu. However Painter colours tu, $tuvwxy$ is now a path of type C and length 5, taking $T_1 = tuv$, $T_2 = vw$ and $T_3 = wxy$. Moreover, T_3 is a red P_3, so $T_1T_2T_3$ is incomplete and we have achieved (ii). □

We next prove that Builder can use a path of type B to efficiently extend Q.

Lemma 34. *Suppose Q is a non-trivial blue path with endpoints a and b, where b is incident to a red edge bc. Suppose $vwxyz$ is a path of type B vertex-disjoint from $V(Q) \cup \{c\}$. Then, by uncovering at most 3 edges, Builder can force Painter to construct one of the following:*

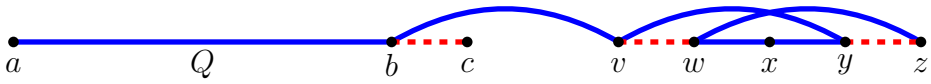

Figure 4: Extending Q using a path of type B as in Lemma 34.

(i) *a blue path Q' of length $e(Q)+5$ with one endpoint b' incident to a red edge $b'c'$.*

(ii) *a red P_4.*

Proof. Builder chooses the edges bv, vy and wz. If Painter colours any of these edges red, then $cbvw$, $wvyz$ or $vwzy$ respectively is a red P_4 and we have achieved (ii). Otherwise, $aQbvyxwz$ is a blue path of length $e(Q)+5$, where z is incident to the red edge zy (see Figure 4), and we have achieved (i). □

We now focus on paths of type C. We first note the following simple property of such paths, which follows immediately from their definition (Definition 32).

Proposition 35. *Suppose $T_1 \dots T_k$ is a path of type C. Then*

$$e(T_1 \dots T_k) = 2k - 5 + e(T_1) + e(T_k).$$

Let $T_1 \dots T_k$ be an incomplete path of type C. We first prove an ancillary lemma, which says that Builder can always extend an incomplete path of type C into a slightly longer path of type C.

Lemma 36. *Suppose $T_1 \dots T_k$ is an incomplete path of type C and length ℓ. Then Builder can force Painter to do one of the following:*

(i) *for some $i \in \{3,4\}$, extend $T_1 \dots T_k$ to a path $T'_1 \dots T'_{k+2}$ of type C and length $\ell+i$ while uncovering i edges.*

(ii) *construct a red P_4 while uncovering at most 4 edges.*

Proof. Suppose without loss of generality that $T_k = x_k y_k z_k$ is a red P_3, where $x_k \in V(T_{k-1})$. Set $T'_i = T_i$ for $i \leqslant k$. Then Builder chooses two edges uv and vw, where u, v and w are new vertices.

First suppose Painter colours both edges blue. Then Builder chooses the edge $z_k u$. If Painter colours $z_k u$ red, then $x_k y_k z_k u$ is a red P_4 and we have achieved (ii). If Painter colours $z_k u$ blue, then set $T'_{k+1} = z_k uv$ and $T'_{k+2} = vw$. Thus, $T'_1 \dots T'_{k+2}$ is a path of type C and length $\ell+3$, and we have achieved (i).

Now suppose that Painter colours both uv and vw red. Then Builder chooses the edges $z_k t$ and tu, where t is a new vertex. If Painter colours one of these edges red, then $x_k y_k z_k t$ or $tuvw$ is a red P_4, respectively, and we have achieved (ii). If Painter colours both $z_k t$ and tu blue, then set $T'_{k+1} = z_k tu$ and $T'_{k+2} = uvw$. Thus, $T'_1 \dots T'_{k+2}$ is a path of type C and length $\ell+4$, and we have achieved (i).

Finally, suppose without loss of generality that Painter colours uv blue and vw red. Then Builder chooses the edges $z_k u$ and wx, where x is a new vertex. If Painter colours $z_k u$ red, then $x_k y_k z_k u$ is a red P_4 and we have achieved (ii). If Painter colours $z_k u$ blue, then set $T'_{k+1} = z_k uv$ and $T'_{k+2} = vwx$. Thus $T'_1 \dots T'_{k+2}$ is a path of type C of length $\ell + 4$, however Painter colours wx, and we have achieved (i). □

By applying Lemma 36 repeatedly, Builder can extend the path $T_1T_2T_3$ of type C given by Lemma 33 into either a complete path of type C or an arbitrarily long incomplete path of type C. Recall from Proposition 35 that a path $T_1 \dots T_k$ of type C has length at most $2k - 1$.

Corollary 37. *Let $k_0 \geqslant 5$ be an odd integer. Suppose $T_1T_2T_3$ is an incomplete path of type C and length 5. Then Builder can force Painter to do one of the following:*

(i) *for some $k, \ell \in \mathbb{N}$, extend $T_1T_2T_3$ to a complete path $T'_1 \dots T'_k$ of type C and length ℓ such that $5 \leqslant k \leqslant k_0$, while uncovering $\ell - 5$ edges.*

(ii) *for some $\ell \in \mathbb{N}$, extend $T_1T_2T_3$ to an incomplete path $T'_1 \dots T'_{k_0}$ of type C and length ℓ while uncovering $\ell - 5$ edges.*

(iii) *construct a red P_4 while uncovering at most $2k_0 - 6$ edges.*

We next prove that Builder can extend R using a path of type C.

Lemma 38. *Suppose $T_1 \dots T_k$ is a path of type C with $k \geqslant 5$ and $e(T_1 \dots T_k) = \ell$. Suppose R is a (possibly trivial) blue path which is vertex-disjoint from $T_1 \dots T_k$. Then Builder can force Painter to construct one of the following:*

(i) *a blue path R' of length $e(R) + (5k - 7)/2$ while uncovering $3(k - 1)/2$ edges. This case can only occur if $T_1 \dots T_k$ is incomplete.*

(ii) *a blue path R' of length $e(R) + \ell'$ while uncovering at most $7\ell'/5 - \ell$ edges for some $1 \leqslant \ell' \leqslant 5(k - 1)/2$. This case can only occur if $T_1 \dots T_k$ is complete.*

(iii) *a red P_4 while uncovering at most $3(k - 1)/2$ edges.*

Proof. Let a and b be the endpoints of R. (If R is trivial, then let $a = b$.)For $i \in \{3, 5, \dots, k - 2\}$, write $T_i = x_i y_i z_i$ where $x_i \in V(T_{i-1})$ and $z_i \in V(T_{i+1})$. Thus $x_i y_i z_i$ is a red P_3 for each $i \in \{3, 5, \dots, k - 2\}$. Builder chooses the set

$$F_1 = \{x_3a, bz_3, x_5c_1, c_1z_5, x_7c_2, c_2z_7, \dots, x_{k-2}c_{\frac{k-5}{2}}, c_{\frac{k-5}{2}}z_{k-2}\}$$

of edges, where $c_1, \dots, c_{\frac{k-5}{2}}$ are new vertices. Note that

$$|F_1| = 2 + 2 \cdot \frac{k-5}{2} = k - 3 < \frac{3(k-1)}{2}. \tag{3}$$

If Painter colours an edge in F_1 red, say $x_i w$ or wz_i for some integer i and some vertex w, then $z_i y_i x_i w$ or $w z_i y_i x_i$ respectively is a red P_4. So in this case we have achieved (iii).

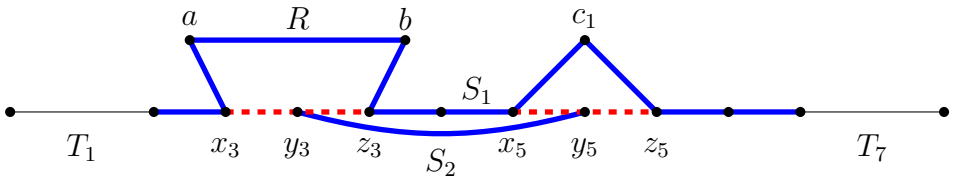

Figure 5: Structure of S_1 and S_2 in Lemma 38 for a path $T_1 \dots T_7$ of type C.

Now suppose Painter colours all edges in F_1 blue. Then we have obtained a blue path

$$S_1 = T_2 x_3 a R b z_3 T_4 x_5 c_1 z_5 T_6 x_7 c_2 z_7 \dots T_{k-3} x_{k-2} c_{\frac{k-5}{2}} z_{k-2} T_{k-1}.$$

Note that S_1 has length

$$\begin{aligned} e(S_1) &= e(T_2) + e(T_4) + \dots + e(T_{k-1}) + |F_1| + e(R) \\ &= \left(2 \cdot \frac{k-3}{2} + 1\right) + (k-3) + e(R) = e(R) + 2k - 5, \end{aligned} \tag{4}$$

where the second equality follows from (3).

Builder now chooses the set

$$F_2 = \{y_3 y_5, y_5 y_7, \dots, y_{k-4} y_{k-2}\}$$

of edges. Note that $|F_2| = (k-5)/2$, so by (3) we have uncovered

$$|F_1| + |F_2| = k - 3 + \frac{k-5}{2} = \frac{3k-11}{2} \tag{5}$$

edges in total so far. If Painter colours an edge in F_2 red, say $y_i y_{i+2}$ for some $i \in \{3, 5, \dots, k-4\}$, then $z_i y_i y_{i+2} x_{i+2}$ is a red P_4. So in this case we have achieved (iii). Suppose Painter colours all edges in F_2 blue. Then we have obtained a blue path

$$S_2 = y_{k-2} y_{k-4} \dots y_5 y_3.$$

Note that S_2 has length $|F_2| = (k-5)/2$. Moreover, S_1 and S_2 are vertex-disjoint (see Figure 5) and by (4) we have

$$e(S_1) + e(S_2) = e(R) + 2k - 5 + \frac{k-5}{2} = e(R) + \frac{5(k-3)}{2}. \tag{6}$$

Our aim is now to join S_1 and S_2 together to form R'. The way in which we do this depends on the structure of T_1 and T_k.

Case 1: $T_1 \dots T_k$ is incomplete.

Without loss of generality we may assume that T_1 is a red P_3, say $x_1 y_1 z_1$ with $z_1 \in V(T_2)$. Builder chooses the edges $y_1 y_{k-2}$, $y_3 x_1$, $x_1 u$ and $u z_1$, where u is a new vertex. In total, Builder has uncovered $|F_1| + |F_2| + 4 = 3(k-1)/2$ edges by (5). If Painter

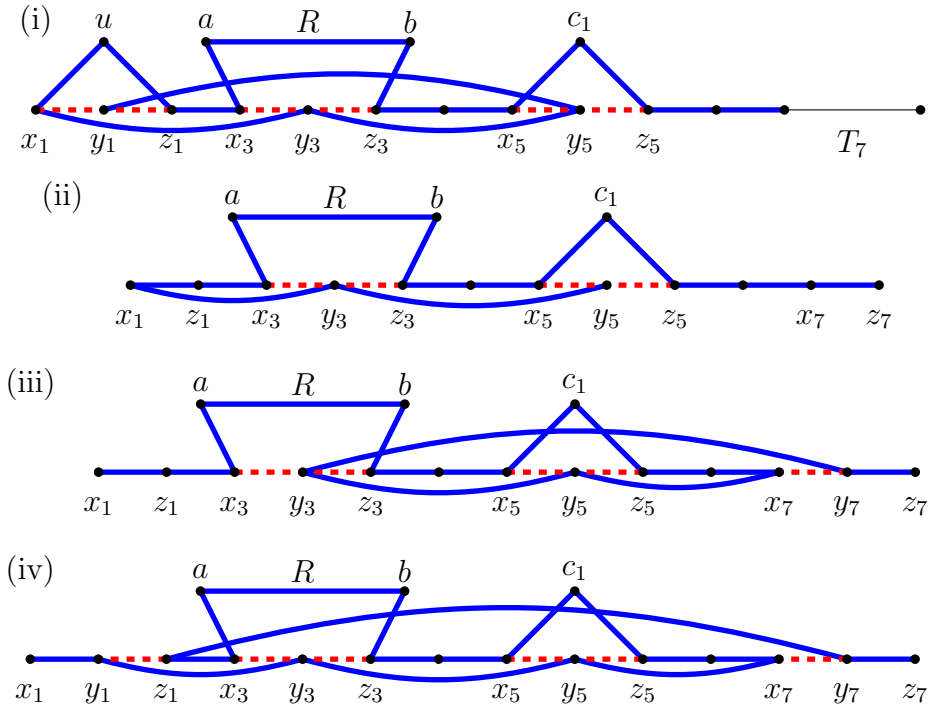

Figure 6: Extending a blue path R with a path $T_1 \dots T_7$ as in cases 1 through 4 (respectively) of Lemma 38.

colours any of the edges red, then $x_1y_1y_{k-2}z_{k-2}$, $y_3x_1y_1z_1$, $z_1y_1x_1u$ or $uz_1y_1x_1$ is a red P_4, respectively, and we have achieved (iii). Suppose Painter colours them all blue. Then $R' := y_1y_{k-2}S_2y_3x_1uz_1S_1$ is a blue path of length $e(S_1) + e(S_2) + 4 = e(R) + (5k-7)/2$ by (6) (see Figure 6(i)) and hence we have achieved (i).

Case 2: $T_1 \dots T_k$ is complete and each of T_1 and T_k is a blue edge.

Write $T_1 = x_1z_1$ and $T_k = x_kz_k$ with $z_1 \in V(T_2)$ and $x_k \in V(T_{k-1})$. First suppose that $k \geqslant 7$. Builder chooses the edges y_3x_1 and $y_{k-2}x_1$. In total, Builder has uncovered $|F_1|+|F_2|+2 = (3k-7)/2$ edges by (5). If Painter colours both edges red, then $x_3y_3x_1y_{k-2}$ is a red P_4 and we have achieved (iii). Suppose Painter colours x_1y_3 blue. Then $R' := S_2y_3x_1z_1S_1x_kz_k$ is a blue path of length $e(S_1) + e(S_2) + 3 = e(R) + (5k-9)/2$ by (6) (see Figure 6(ii)). Writing $\ell' := e(R') - e(R) = (5k-9)/2$, Builder has uncovered

$$\frac{3k-7}{2} < \frac{7}{5} \cdot \frac{5k-9}{2} - (2k-3) = \frac{7\ell'}{5} - \ell$$

edges in total, where the last equality follows from Proposition 35. Hence we have achieved (ii). If instead Painter colours x_1y_{k-2} blue, the same argument shows we have achieved (ii) on replacing S_2y_3 by S_2y_{k-2}. So if $k \geqslant 7$, we are done.

If instead $k = 5$, Builder chooses the edges y_3x_1 and ux_1, where u is a new vertex. If Painter colours both edges red, then $ux_1y_3z_3$ is a red P_4 and we have achieved (iii). Suppose instead Painter colours wx_1 blue for some $w \in \{u, y_3\}$. Then $R' := wx_1z_1S_1x_5z_5$

is a blue path of length $e(S_1) + e(S_2) + 3$ (as $e(S_2) = 0$) and Builder has uncovered $|F_1| + |F_2| + 2$ edges. Thus we have achieved (ii) as above.

Case 3: $T_1 \dots T_k$ is complete and exactly one of T_1 and T_k is a blue edge.

Without loss of generality we may assume that T_1 is a blue edge. Let $T_1 = x_1z_1$ with $z_1 \in V(T_2)$, and let $T_k = x_ky_kz_k$ with $x_k \in V(T_{k-1})$. Note that x_ky_k is red and y_kz_k is blue. Builder chooses the edges x_ky_{k-2} and y_3y_k. In total, Builder has uncovered $|F_1|+|F_2|+2 = (3k-7)/2$ edges by (5). If Painter colours either x_ky_{k-2} or y_3y_k red, then $y_kx_ky_{k-2}x_{k-2}$ or $x_3y_3y_kx_k$ is a red P_4 respectively, and we have achieved (iii). Suppose Painter instead colours both edges blue. Then $R' := x_1z_1S_1x_ky_{k-2}S_2y_3y_kz_k$ is a blue path of length $e(S_1) + e(S_2) + 4 = e(R) + (5k-7)/2$ by (6) (see Figure 6(iii)). Writing $\ell' := e(R') - e(R) = (5k-7)/2$, Builder has uncovered

$$\frac{3k-7}{2} < \frac{7}{5} \cdot \frac{5k-7}{2} - (2k-2) = \frac{7\ell'}{5} - \ell$$

edges in total, where the last equality follows from Proposition 35. Hence we have achieved (ii).

Case 4: $T_1 \dots T_k$ is complete and neither T_1 nor T_k is a blue edge.

Let $T_1 = x_1y_1z_1$ and $T_k = x_ky_kz_k$ where $z_1 \in V(T_2)$ and $x_k \in V(T_{k-1})$. Thus x_1y_1 and y_kz_k are blue, and y_1z_1 and x_ky_k are red. Then Builder chooses the edges y_kz_1, x_ky_{k-2}, and y_3y_1. In total, Builder has uncovered $|F_1| + |F_2| + 3 = (3k-5)/2$ edges by (5). If Painter colours one of these edges red, then $x_ky_kz_1y_1$, $y_kx_ky_{k-2}x_{k-2}$ or $z_3y_3y_1z_1$ respectively is a red P_4 and we have achieved (iii). Suppose Painter colours them all blue. Then $R' := z_ky_kz_1S_1x_ky_{k-2}S_2y_3y_1x_1$ is a blue path (see Figure 6(iv)) of length $e(S_1) + e(S_2) + 5 = e(R) + 5(k-1)/2$ by (6). Writing $\ell' := e(R') - e(R) = (5k-5)/2$, Builder has uncovered

$$\frac{3k-5}{2} = \frac{7}{5} \cdot \frac{5k-5}{2} - (2k-1) = \frac{7\ell'}{5} - \ell$$

edges in total, where the last equality follows from Proposition 35. We have achieved case (ii). □

Finally, we consolidate Lemmas 33, 34 and 38 and Corollary 37 into a single lemma which says that given two independent red edges, Builder can extend either Q or R. As with Lemma 30, in applying Lemma 39 we will take m to be $\ell - e(Q) - e(R) - 1$.

Lemma 39. *Let $m \geqslant 9$ be an integer. Let Q and R be blue paths and let e and f be two red edges. Suppose that Q is non-trivial and has an endpoint b incident to a red edge bc. Further suppose that $V(Q) \cup \{c\}$, R, e and f are pairwise vertex-disjoint. Then Builder can force Painter to construct one of the following:*

(i) *a blue path Q' with one endpoint b' incident to a red edge $b'c'$ such that $e(Q') = e(Q)+5$, while uncovering 5 edges. Moreover, R is vertex-disjoint from $V(Q') \cup \{c'\}$.*

(ii) *a blue path R' such that $e(R') = e(R) + \ell'$ for some $1 \leqslant \ell' \leqslant m+5$ while uncovering at most $7\ell'/5 - 2$ edges. Moreover, R' is vertex-disjoint from $V(Q) \cup \{c\}$.*

(iii) *a blue path R' such that $e(R') \geqslant e(R)+m$ while uncovering at most $7m/5+6$ edges. Moreover, R' is vertex-disjoint from $V(Q) \cup \{c\}$.*

(iv) *a red P_4 while uncovering at most $7m/5+6$ edges.*

Proof. We first apply Lemma 33 to e and f. If as in Lemma 33(iii) we obtain a red P_4 while uncovering 2 edges, then we have achieved (iv). Suppose we do not. Then we split into cases depending on Painter's choice.

Case 1: We obtain a path $vwxyz$ of type B while uncovering 2 edges, as in Lemma 33(i). Moreover, $vwxyz$ is vertex-disjoint from $V(Q) \cup \{c\}$ and R.

We apply Lemma 34 to Q and $vwxyz$. Hence we have uncovered at most 5 edges in total. If we obtain a red P_4, then we have achieved (iv). Suppose instead we obtain a blue path Q' of length $q+5$ with one endpoint b' incident to a red edge $b'c'$, where $V(Q') \cup \{c'\}$ is vertex-disjoint from R. Then we have achieved (i).

Case 2: We obtain an incomplete path $T_1T_2T_3$ of type C and length 5 while uncovering 3 edges, as in Lemma 33(ii). Moreover, $T_1T_2T_3$ is vertex-disjoint from $V(Q) \cup \{c\}$ and R.

Let k_0 be the least odd number such that $k_0 \geqslant (2m+7)/5$. Since $5k_0 < (2m+7)+5 \cdot 2$, and both $5k_0$ and $2m+17$ are odd integers, we have $k_0 \leqslant 2m/5+3$. Moreover, $k_0 \geqslant (2m+7)/5 \geqslant 5$ since $m \geqslant 9$. We apply Corollary 37 to $T_1T_2T_3$ and k_0. If we obtain a red P_4 while uncovering at most $2k_0-6$ additional edges, then we have achieved (iv). Suppose we do not. Then we split into subcases depending on Painter's choice.

Case 2a: For some $k, \ell \in \mathbb{N}$, we obtain a complete path $T'_1 \dots T'_k$ of type C and length ℓ such that $5 \leqslant k \leqslant k_0$ while uncovering $\ell - 5$ additional edges, as in Corollary 37(i). Moreover, $T'_1 \dots T'_k$ is vertex-disjoint from $V(Q) \cup \{c\}$ and R.

We now apply Lemma 38 to $T'_1 \dots T'_k$ and R. Suppose we obtain a blue path R' with length $e(R) + \ell'$, where

$$\ell' \leqslant \frac{5(k-1)}{2} \leqslant \frac{5(k_0-1)}{2} \leqslant \frac{5}{2} \cdot \left(\frac{2m}{5} + 2 \right) = m + 5,$$

while uncovering at most $7\ell'/5 - \ell$ edges as in Lemma 38(ii). Note that R' is vertex-disjoint from $V(Q) \cup \{c\}$. In total we have uncovered at most $3 + (\ell - 5) + (7\ell'/5 - \ell) = 7\ell'/5 - 2$ edges, so we have achieved (i).

Suppose instead we obtain a red P_4 while uncovering at most $3(k-1)/2$ edges as in Lemma 38(iii). Note that $\ell \leqslant 2k_0 - 1$ by Proposition 35. In total we have therefore uncovered at most

$$3 + (\ell - 5) + \frac{3(k_0-1)}{2} \leqslant \frac{7k_0-9}{2} \leqslant \frac{7}{2} \cdot \left(\frac{2m}{5} + 3 \right) - \frac{9}{2} = \frac{7m}{5} + 6 \tag{7}$$

edges, and thus we have achieved (iv).

Case 2b: For some $\ell \in \mathbb{N}$, we obtain an incomplete path $T'_1 \dots T'_{k_0}$ of type C and length ℓ while uncovering $\ell - 5$ additional edges, as in Corollary 37(ii). Moreover, $T'_1 \dots T'_{k_0}$ is vertex-disjoint from $V(Q) \cup \{c\}$ and R.

We apply Lemma 38 to $T'_1 \dots T'_{k_0}$ and R. Whatever the outcome, we uncover at most $3(k_0-1)/2$ edges. We have therefore uncovered at most $7m/5+6$ edges in total, as in (7). If we obtain a red P_4 as in Lemma 38(iii), then we have achieved (iv). Hence we may assume that we obtain a blue path R' of length

$$e(R) + \frac{5k_0 - 7}{2} \geqslant e(R) + \frac{5}{2} \cdot \frac{2m+7}{5} - \frac{7}{2} = e(R) + m,$$

as in Lemma 38(i). (The inequality follows from the definition of k_0.) We have therefore achieved (iii). □

A.3 An upper bound on $\tilde{r}(P_4, P_{\ell+1})$ for $\ell \geqslant 3$

We now use Lemmas 26, 30 and 39 to bound $\tilde{r}(P_4, P_{\ell+1})$ above in Theorem 44. Together with Theorem 4, this will imply Theorem 6.

Recall that Builder's strategy is to extend blue paths Q and R using independent edges. For the remainder of the section, we denote the graph Builder has uncovered by G. In order to keep track of the lengths of Q and R and the number of independent edges available, we introduce the following notation.

Definition 40. Let G be a graph. Given $q, r, n_{\text{blue}}, n_{\text{red}} \in \mathbb{N}_0$, we say that G contains a $(q, r, n_{\text{blue}}, n_{\text{red}})$-*structure* if it satisfies the following properties:

(P1) G contains a (possibly trivial) blue path Q of length q with one endpoint b incident to a red edge bc.

(P2) G contains a (possibly trivial) blue path R of length r.

(P3) G contains a set F of independent edges containing n_{blue} blue edges and n_{red} red edges.

(P4) $V(Q) \cup \{c\}$, R and F are pairwise vertex-disjoint.

This notation substantially simplifies the statements of Lemmas 26, 30 and 39. The corresponding statements are as follows.

Corollary 41. *Let $q, r, n_{\text{red}}, n_{\text{blue}} \in \mathbb{N}_0$ and let G be a graph. Suppose G contains a $(q, r, n_{\text{blue}}, n_{\text{red}})$-structure. Then Builder can force Painter to construct a graph $G' \supseteq G$ with $e(G') \leqslant e(G)+2$ such that G' contains a $(q+r+1, 0, n_{\text{blue}}, n_{\text{red}})$-structure or a red P_4.*

Corollary 42. *Let $m, q, r, n_{\text{red}} \in \mathbb{N}_0$ with $q, m \geqslant 1$. Suppose G is a graph containing a $(q, r, 2, n_{\text{red}})$-structure. Then Builder can force Painter to construct a graph $G' \supseteq G$ such that one of the following holds:*

(i) *G' contains a $(q + \ell', r, n_{\text{blue}}, n_{\text{red}})$-structure and $e(G') = e(G) + \ell'$ for some $3 \leqslant \ell' \leqslant m + 3$ and some $n_{\text{blue}} \in \mathbb{N}_0$. Moreover, if $3 \leqslant \ell' < 5 \leqslant m$, then we may take $n_{\text{blue}} = 1$.*

(ii) *G' contains a red P_4 and $e(G') \leqslant e(G) + m + 3$.*

Corollary 43. *Let $m, q, r, n_{\text{blue}} \in \mathbb{N}_0$ with $q \geqslant 1$ and $m \geqslant 9$. Suppose G is a graph containing a $(q, r, n_{\text{blue}}, 2)$-structure. Then Builder can force Painter to construct a graph $G' \supseteq G$ such that one of the following holds:*

(i) *$e(G') = e(G) + 5$ and G' contains a $(q + 5, r, n_{\text{blue}}, 0)$-structure.*

(ii) *There exists $1 \leqslant \ell' \leqslant m + 5$ such that $e(G') \leqslant e(G) + 7\ell'/5 - 2$ and G' contains a $(q, r + \ell', n_{\text{blue}}, 0)$-structure.*

(iii) *$e(G') \leqslant e(G) + 7m/5 + 6$ and G' contains a $(q, r + m, n_{\text{blue}}, 0)$-structure.*

(iv) *$e(G') \leqslant e(G) + 7m/5 + 6$ and G' contains a red P_4.*

Theorem 44. *For all $\ell \in \mathbb{N}$, we have $\tilde{r}(P_4, P_{\ell+1}) \leqslant (7\ell + 52)/5$.*

Proof. Our aim is to show that Builder can construct a graph G with $e(G) \leqslant (7\ell + 52)/5$ containing a red P_4 or a blue $P_{\ell+1}$.

We first obtain an initial blue path Q with one endpoint incident to a red edge. We claim that either Builder can construct a path $xySz$ of type A with $e(S) < \ell$, while uncovering at most $(7e(S) + 4)/5$ edges, or we are done. We proceed as follows. Builder chooses an edge $e = uv$. First suppose Painter colours uv blue. Then apply Lemma 28 to uv, taking $m = \ell$. If we find a blue $P_{\ell+1}$ while uncovering $\ell - 1$ additional edges, then since we have uncovered ℓ edges in total we are done. Suppose instead we find a path $xySz$ of type A with $e(S) < \ell$, while uncovering $e(S)$ additional edges in the process. Then in total Builder has uncovered $e(S) + 1 < (7e(S) + 4)/5$ edges, as desired.

Suppose instead Painter colours uv red. Then Builder chooses the edge vx, where x is a new vertex. If Painter colours vx blue, then uvx is a path of type A constructed while uncovering $2 < (7 + 4)/5$ edges in total. If Painter colours vx red, then Builder chooses the edges tu, uw and wx, where t and w are new vertices. If Painter colours any of these edges red, then $tuvx$, $xvuw$ or $wxvu$ respectively is a red P_4 and we are done. Otherwise, $tuwxv$ is a path of type A (taking $S = tuwx$), constructed while uncovering $5 = (7 \cdot 3 + 4)/5$ edges in total. Therefore, we may assume that Builder has constructed a path $xySz$ of type A with $e(S) < \ell$ while uncovering at most $(7e(S) + 4)/5$ edges as claimed.

Let G_0 be the graph consisting of all edges uncovered so far. Thus G_0 contains a $(q_0, 0, 0, 0)$-structure for some $1 \leqslant q_0 < \ell$, and $e(G_0) \leqslant (7q_0 + 4)/5$. Suppose that for some $i \geqslant 0$, Builder has already constructed a graph G_i such that there exist $q_i, r_i, n_{\text{blue},i}, n_{\text{red},i} \in \mathbb{N}_0$ satisfying the following properties:

(G1) $G_i \subseteq K_{\mathbb{N}}$ is the graph of all uncovered edges.

(G2) G_i contains a $(q_i, r_i, n_{\text{blue},i}, n_{\text{red},i})$-structure, and $q_i > 0$.

(G3) $q_i + r_i \leqslant \ell + 4$.

(G4) $n_{\mathrm{red},i}, n_{\mathrm{blue},i} \leqslant 1$.

(G5) $e(G_i) \leqslant (7(q_i + r_i) + 4)/5 + n_{\mathrm{blue},i} + n_{\mathrm{red},i}$.

Note that (G1)–(G5) hold for $i = 0$. We are going to show that Builder can force a graph $G_{i+1} \supseteq G_i$ such that one of the following holds:

(a) G_{i+1} contains a red P_4 or a blue $P_{\ell+1}$ and $e(G_{i+1}) \leqslant (7\ell + 52)/5$.

(b) there exist $q_{i+1}, r_{i+1}, n_{\mathrm{blue},i+1}, n_{\mathrm{red},i+1} \in \mathbb{N}_0$ such that $q_{i+1} + r_{i+1} > q_i + r_i$ and G_{i+1}, $q_{i+1}, r_{i+1}, n_{\mathrm{blue},i+1}$ and $n_{\mathrm{red},i+1}$ together satisfy (G1)–(G5).

If (a) holds, we are done. If (b) holds, then Builder can repeat the algorithm to obtain G_{i+2}. We then simply repeat the process until it terminates, which must happen by (G3) (since $q_{i+1} + r_{i+1} > q_i + r_i$ whenever these quantities are defined). It therefore remains only to prove that forcing such a graph is possible.

Let $m = \ell - q_i - r_i - 1$. We split into cases depending on the values of $q_i, r_i, n_{\mathrm{blue},i}$ and $n_{\mathrm{red},i}$.

Case 1: $q_i + r_i \geqslant \ell - 1$.

In this case, we may simply join our two blue paths together to achieve (a). Apply Corollary 41 to G_i. Builder obtains a graph $G_{i+1} \supseteq G_i$ with

$$e(G_{i+1}) = e(G_i) + 2 \overset{\text{(G5)}}{\leqslant} \frac{7(q_i + r_i) + 4}{5} + n_{\mathrm{blue},i} + n_{\mathrm{red},i} + 2 \overset{\text{(G3),(G4)}}{\leqslant} \frac{7\ell + 52}{5}.$$

Moreover, G' contains a red P_4 or a blue $P_{\ell+1}$, so we have achieved (a).

Case 2: $\ell - 9 \leqslant q_i + r_i \leqslant \ell - 2$, so that $1 \leqslant m \leqslant 8$.

In this case, it is more efficient to naively extend our paths to the right combined length and join them than it is to apply our normal extension methods and potentially end up with paths longer than we need. Builder will force a red P_4 or a blue $P_{\ell+1}$ as follows. Apply Corollary 41 to G_i to obtain a graph $G' \supseteq G_i$ with $e(G') = e(G_i) + 2$. Note that G' contains a red P_4 or a $(q_i + r_i + 1, 0, n_{\mathrm{blue},i}, n_{\mathrm{red},i})$-structure. By repeating the process at most m additional times, Builder obtains a graph $G'' \supseteq G' \supseteq G_i$, where

$$\begin{aligned} e(G'') &\leqslant e(G) + 2m + 2 \overset{\text{(G5)}}{\leqslant} \frac{7(q_i + r_i) + 4}{5} + n_{\mathrm{blue},i} + n_{\mathrm{red},i} + 2m + 2 \\ &\overset{\text{(G4)}}{\leqslant} \frac{7(\ell - m - 1) + 4}{5} + 2 + 2m + 2 = \frac{7\ell}{5} + \frac{3m + 17}{5} \leqslant \frac{7\ell + 41}{5}, \end{aligned}$$

such that G'' contains a red P_4 or a $(q_i + r_i + m + 1, 0, n_{\mathrm{blue},i}, n_{\mathrm{red},i})$-structure (which contains a blue $P_{\ell+1}$). Thus we have achieved (a).

Case 3: $q_i + r_i \leqslant \ell - 10$, so that $m \geqslant 9$.

In this case, we will extend our paths efficiently using Corollaries 42 and 43. By choosing at most $3 - n_{\mathrm{blue},i} - n_{\mathrm{red},i}$ additional independent edges (on new vertices), Builder

obtains a graph $G_i' \supseteq G_i$ containing a $(q_i, r_i, n'_{\text{blue}}, n'_{\text{red}})$-structure such that $n'_{\text{blue}}+n'_{\text{red}} \leqslant 3$, either $n'_{\text{blue}} = 2$ or $n'_{\text{red}} = 2$, and

$$e(G_i') \overset{\text{(G5)}}{\leqslant} \frac{7(q_i + r_i) + 4}{5} + n'_{\text{blue}} + n'_{\text{red}}. \tag{8}$$

We split into subcases depending on the values of n'_{blue} and n'_{red}.

Case 3a: $n'_{\text{blue}} = 2$ and $n'_{\text{red}} \leqslant 1$.

We apply Corollary 42 to G_i', obtaining a graph $G' \supseteq G_i'$. First suppose Corollary 42(i) holds, so that there exists some $3 \leqslant \ell' \leqslant m+3$ such that G' contains a $(q_i+\ell', r_i, n''_{\text{blue}}, n'_{\text{red}})$-structure and $e(G') = e(G_i')+\ell'$. Set $G_{i+1} = G'$, $q_{i+1} = q_i+\ell'$, $r_{i+1} = r_i$ and $n_{\text{red},i+1} = n'_{\text{red}}$. Set $n_{\text{blue},i+1} = 0$ if $\ell' \geqslant 5$ and $n_{\text{blue},i+1} = 1$ otherwise. Clearly $q_{i+1} + r_{i+1} > q_i + r_i$, and (G1) and (G4) are satisfied. Recall from Corollary 42(i) that if $\ell' < 5 \leqslant m$ then we may take $n''_{\text{blue}} = 1$, so (G2) is satisfied. We have $q_{i+1} + r_{i+1} \leqslant q_i + m + 3 + r_i = \ell + 2$, so (G3) is satisfied. If $3 \leqslant \ell' \leqslant 4$, we have

$$\begin{aligned} e(G') = e(G_i') + \ell' &\overset{(8)}{\leqslant} \frac{7(q_i + r_i) + 4}{5} + 2 + n'_{\text{red}} + \ell' \\ &= \frac{7(q_i + r_i + \ell') + 4}{5} - \frac{2\ell'}{5} + 2 + n'_{\text{red}} \leqslant \frac{7(q_{i+1} + r_{i+1}) + 4}{5} + 1 + n'_{\text{red}} \\ &= \frac{7(q_{i+1} + r_{i+1}) + 4}{5} + n_{\text{blue},i+1} + n_{\text{red},i+1}. \end{aligned}$$

So (G5) is satisfied and we have therefore achieved (b). A similar argument holds for the case when $\ell' \geqslant 5$.

Suppose instead that Corollary 42(ii) holds, so that G' contains a red P_4 and $e(G') \leqslant e(G_i') + m + 3$. Then we have

$$e(G') \overset{(8)}{\leqslant} \frac{7(q_i + r_i) + 4}{5} + 2 + n'_{\text{red}} + m + 3 \leqslant \frac{2(q_i + r_i) + 4}{5} + \ell + 5 \leqslant \frac{7\ell + 9}{5},$$

where the final inequality follows since $q_i + r_i \leqslant \ell - 10$. We have therefore achieved (a).

Case 3b: $n'_{\text{red}} = 2$ and $n'_{\text{blue}} \leqslant 1$.

We apply Corollary 43 to G_i', obtaining a graph $G' \supseteq G_i'$. Suppose Corollary 43(i) or (ii) holds. In either case, it follows that there exist q' and r' such that G' contains a $(q', r', n'_{\text{blue}}, 0)$-structure and

$$1 \leqslant q' + r' - (q_i + r_i) \leqslant m + 5.$$

Write $\ell' = q' + r' - (q_i + r_i)$. Set $G_{i+1} = G'$, $q_{i+1} = q'$, $r_{i+1} = r'$, $n_{\text{blue},i+1} = n'_{\text{blue}}$ and $n_{\text{red},i+1} = 0$. Clearly (G1)–(G4) are satisfied, and $q_{i+1} + r_{i+1} > q_i + r_i$. Moreover, we have

$$\begin{aligned} e(G_{i+1}) \leqslant e(G_i') + \frac{7\ell'}{5} - 2 &\overset{(8)}{\leqslant} \frac{7(q_i + r_i + \ell') + 4}{5} + n'_{\text{blue}} \\ &= \frac{7(q_{i+1} + r_{i+1}) + 4}{5} + n_{\text{blue},i+1} + n_{\text{red},i+1}, \end{aligned}$$

so (G5) is satisfied. We have therefore achieved (b).

Now suppose Corollary 43(iii) holds, so that G' contains a $(q_i, r_i+m, n'_{\text{blue}}, 0)$-structure and $e(G') \leqslant e(G'_i) + 7m/5 + 6$. We apply Corollary 41 to G', obtaining a graph G'' such that

$$\begin{aligned} e(G'') &= e(G') + 2 \leqslant e(G'_i) + \frac{7m}{5} + 8 \\ &\overset{(8)}{\leqslant} \frac{7(q_i + r_i + m) + 4}{5} + n'_{\text{blue}} + 10 \leqslant \frac{7\ell + 52}{5}. \end{aligned}$$

Moreover, G'' contains a red P_4 or an $(\ell, 0, n'_{\text{blue}}, 0)$-structure (which contains a blue $P_{\ell+1}$). We have therefore achieved (a).

Finally suppose Corollary 43(iv) holds, so that G' contains a red P_4 and $e(G') \leqslant e(G'_i) + 7m/5 + 6$. Then we have

$$e(G') \leqslant e(G'_i) + \frac{7m}{5} + 6 \overset{(8)}{\leqslant} \frac{7(q_i + r_i + m) + 4}{5} + n'_{\text{blue}} + 8 \leqslant \frac{7\ell + 42}{5}.$$

We have therefore achieved (a). This completes the proof of the theorem. □