On a refinement of Wilf-equivalence for permutations

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Abstract

Recently, Dokos et al. conjectured that for all $k, m \geqslant 1$, the patterns $12\ldots k(k+m+1)\ldots(k+2)(k+1)$ and $(m+1)(m+2)\ldots(k+m+1)m\ldots 21$ are maj-Wilf-equivalent. In this paper, we confirm this conjecture for all $k \geqslant 1$ and $m = 1$. In fact, we construct a descent set preserving bijection between $12\ldots k(k-1)$-avoiding permutations and $23\ldots k1$-avoiding permutations for all $k \geqslant 3$. As a corollary, our bijection enables us to settle a conjecture of Gowravaram and Jagadeesan concerning the Wilf-equivalence for permutations with given descent sets.

Keywords: maj-Wilf-equivalent; pattern avoiding permutation; bijection.

1 Introduction

Denote by $\mathcal{S}_n$ the set of all permutations on $[n]$. Given a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{S}_n$ and a permutation $\tau = \tau_1\tau_2\ldots\tau_k \in \mathcal{S}_k$, we say that π contains the *pattern* τ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ of π that is order-isomorphic to τ. Otherwise, π is said to *avoid* the pattern τ or be τ-*avoiding*. Denote by $\mathcal{S}_n(\tau)$ the set of all τ-avoiding permutations in $\mathcal{S}_n$. Pattern avoiding permutations have been extensively studied over last decade. For a thorough summary of the current status of research, see Bóna's book [5] and Kitaev's book [12].

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If two patterns $\sigma, \tau \in \mathcal{S}_m$ are said to be *Wilf-equivalent* if and only if $|\mathcal{S}_n(\sigma)| = |\mathcal{S}_n(\tau)|$. A permutation statistic is defined to be a function $s : \mathcal{S}_n \to T$, where T is any fixed set. The most studied statistics include the inversion number and the major index. Let $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{S}_n$. The set of *inversions* of π is

$$\mathcal{I}(\pi) = \{(i,j)|i < j \text{ and } \pi_i > \pi_j\}.$$

The inversion number of π, denoted by $inv(\pi)$, is the cardinality of $\mathcal{I}(\pi)$. The *decent set* of π is

$$\mathcal{D}(\pi) = \{i|\pi_i > \pi_{i+1}\}.$$

The *ascent set* of π is

$$\mathcal{A}(\pi) = \{i|\pi_i < \pi_{i+1}\}.$$

The *major index* of π, denoted by $maj(\pi)$, is given by $maj(\pi) = \sum_{i\in\mathcal{D}(\pi)} i$.

Given a permutation statistic s, we say that σ and τ are s-wilf-equivalent if there exists a bijection $\Theta : \mathcal{S}_n(\sigma) \to \mathcal{S}_n(\tau)$ such that $s(\pi) = s(\Theta(\pi))$ for all $\pi \in \mathcal{S}_n(\sigma)$. In other words, the statistic s is equally distributed on the sets $\mathcal{S}_n(\sigma)$ and $\mathcal{S}_n(\tau)$. This refinement of Wilf-equivalence for patterns of length 3 has been extensively studied, see [2, 3, 7, 8, 10, 14]. However, little is known about permutation statistics and patterns of length 4 or greater. Recently, Dokos et al. [9] posed the following two conjectures on the maj-Wilf-equivalence for patterns of length 4 or greater.

Conjecture 1.1. *([9], Conjecture 2.7) For all $k, m \geqslant 1$, the patterns $12\ldots k(k+m+1)\ldots(k+2)(k+1)$ and $(m+1)(m+2)\ldots(k+m+1)m\ldots21$ are maj-Wilf-equivalent.*

Conjecture 1.2. *([9], Conjecture 2.8) The major index is equally distributed on the sets $\mathcal{S}_n(2413)$, $\mathcal{S}_n(1423)$ and $\mathcal{S}_n(2314)$*

Recently, Bloom [4] confirmed Conjecture 1.2 by providing descent set preserving bijections between the set $\mathcal{S}_n(2413)$ and the set $\mathcal{S}_n(1423)$, and between the set $\mathcal{S}_n(2413)$ and the set $\mathcal{S}_n(2314)$. In their paper [9], Dokos et al. showed that Conjecture 1.1 is true for $m = k = 1$. The main purpose of this paper is to confirm Conjecture 1.1 for all $k \geqslant 1$ and $m = 1$. Actually, we obtain the following stronger result.

Theorem 1.3. *For $k \geqslant 3$, there exists a descent set preserving bijection between the set $\mathcal{S}_n(12\ldots k(k-1))$ and the set $\mathcal{S}_n(23\ldots k1)$.*

Denote by $J_k = 12\ldots k$, $F_k = 23\ldots k1$ and $G_k = 12\ldots k(k-1)$, respectively. Give a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, suppose that $\mathcal{D}(\pi) = \{i_1, i_2, \ldots, i_s\}$. Then we call the subsequence $\pi_1\pi_2\ldots\pi_{i_1}$ the first *block* of π, the subsequence $\pi_{i_1+1}\pi_{i_1+2}\ldots\pi_{i_2}$ the second block of π, and so on. We say that a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ contains an occurrence of H_k if there exists indices $i_1 < i_2 < \ldots < i_k$ such that the subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ is isomorphic to J_k and entries $\pi_{i_{k-1}}$ and π_{i_k} belong to two different blocks. That is, there exists a $j \in \mathcal{D}(\pi)$ with $i_{k-1} \leqslant j < i_k$. Otherwise, we say that π avoids H_k. For example, the subsequence 13579 of the permutation $\pi = 13576894(10)2(11) \in \mathcal{S}_{11}$ is an occurrence of H_5, while the subsequence 13569 is not an occurrence of H_5. We say that a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ contains an occurrence of Q_k if there exists indices $i_1 < i_2 < \ldots < i_k$ such that the subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ is isomorphic to G_k and $\pi_{i_{k-1}} < \pi_{i_{k-1}+1} < \ldots < \pi_{i_k-1} > \pi_{i_k}$. Otherwise, we say that π avoids Q_k. For example, the subsequence 13586 of the permutation $\pi = 1358(10)67492(11) \in \mathcal{S}_{11}$ is an occurrence of Q_5, while the subsequence 13587 is not an occurrence of Q_5.

In order to prove Theorem 1.3, we obtain the following two theorems.

Theorem 1.4. *For $k \geqslant 3$, there is a bijection f between the set $\mathcal{S}_n(G_k)$ and the set $\mathcal{S}_n(H_k, Q_k)$ such that for any $\pi \in \mathcal{S}_n(G_k)$, we have $\mathcal{D}(\pi) = \mathcal{D}(f(\pi))$.*

Theorem 1.5. *For $k \geqslant 3$, there is a bijection Φ between the set $\mathcal{S}_n(F_k)$ and the set $\mathcal{S}_n(H_k, Q_k)$ such that for any $\pi \in \mathcal{S}_n(F_k)$, we have $\mathcal{D}(\pi) = \mathcal{D}(\Phi(\pi))$.*

Combining Theorems 1.4 and 1.5, we are led to Theorem 1.3.

Given a positive integer t, Let $D_n^t = \{i | 1 \leqslant i \leqslant n-1 \,\text{and}\, i \equiv 0 \bmod t\}$. Denote by $\mathcal{S}_n^t(12\ldots k(k-1))$ (resp. $\mathcal{S}_n^t(23\ldots k1)$) the set of permutations $\pi \in \mathcal{S}_n(12\ldots k(k-1))$ (resp. $\pi \in \mathcal{S}_n^t(23\ldots k1)$) with $\mathcal{D}(\pi) = D_n^t$. From Theorem 1.3, we obtain the following result as conjectured by Gowravaram and Jagadeesan [11].

Corollary 1.6. *([11], Conjecture 6.2) For $t \geqslant 1$ and $k \geqslant 3$, we have $|\mathcal{S}_n^t(12\ldots k(k-1))| = |\mathcal{S}_n^t(23\ldots k1)|$.*

2 Proof of Theorem 1.4

We begin with some definitions and notations. An entry of a permutation is said to have *rank* k if the length of the longest increasing subsequence that ends in that entry is k. We now construct a map f from the set $\mathcal{S}_n(G_k)$ to the set $\mathcal{S}_n(H_k, Q_k)$. The map f is a slight modification of a classic bijection, which is given by West [15] to prove the equality $|\mathcal{S}_n(J_k)| = |\mathcal{S}_n(G_k)|$ for all $k \geqslant 3$. Recently, Bona [6] proved that West's bijection also induces a bijection between $12\ldots k$-avoiding alternating permutations and $12\ldots k(k-1)$-avoiding alternating permutations, thereby proving generalized versions of some conjectures of Lewis [13].

Let $\pi \in \mathcal{S}_n(G_k)$. In order to obtain $f(\pi)$, we leave all entries of π that are of rank $k-2$ or less in their place and rearrange the entries of rank $k-1$ or higher. Let $B_1, B_2, \ldots, B_s$ be the blocks of π that are listed from left to right. Let P_i be the set of positions of π in which, an entry that has rank $k-1$ or higher and belongs to the block B_i, is located. Let R be the set of entries of π that are of rank $k-1$ or higher. Now we fill the positions of $P_i's$ as follows.

Step 1. Choose $|P_1|$ largest entries from R and fill the positions of P_1 with the selected entries from left to right in increasing order.

Step 2. Choose $|P_2|$ largest entries from R that have not been placed yet. Then fill the positions of P_2 with the selected entries from left to right in increasing order.

Step 3. Fill the positions of P_3, P_4, …, P_s as in Step 2.

Let $f(\pi)$ be the obtained permutation.

Example 2.1. *Consider* $\pi = 13576894(10)2(11) \in \mathcal{S}_{11}(G_6)$*. Then we have* $B_1 = 1357$*,* $B_2 = 689$*,* $B_3 = 4(10)$ *and* $B_4 = 2(11)$*. Moreover, we have* $P_1 = \emptyset$*,* $P_2 = \{6, 7\}$*,* $P_3 = \{9\}$*,* $P_4 = \{11\}$ *and* $R = \{8, 9, 10, 11\}$*. According to the definition of* f*, we have* $f(\pi) = 13576(10)(11)4928$*.*

Since the existence of π shows that there is at least one way to assign the entries of R to the positions of P_i, the definition of f always enables us to create $f(\pi)$.

Notice that if entry π_i of π has rank $k-2$ or less, then π_i do not move in the above procedure, and the rank of π_i do not change. If entry π_i of π has rank $k-1$ or higher, then π_i may have moved and the rank of π_i in $f(\pi)$ is $k-1$ or higher. We claim that if $\pi_{i-1} > \pi_i$, then the rank of π_i is $k-2$ or less. If not, the longest increasing subsequence that ends in π_i combining with π_{i-1} would form a G_k in π. This contradicts the fact that π avoids G_k.

Now we proceed to show that $\mathcal{D}(\pi) = \mathcal{D}(f(\pi))$. Let $f(\pi) = \sigma_1\sigma_2 \ldots \sigma_n$. If $\pi_{i-1} > \pi_i$ then the rank of π_i is $k-2$ or less and do not move. This implies that $\pi_i = \sigma_i$ and σ_i has rank $k-2$ or less. If π_{i-1} is of rank $k-2$ or less, then we have $\sigma_{i-1} = \pi_{i-1}$. In this case, we have $\sigma_{i-1} = \pi_{i-1} > \pi_i = \sigma_i$. If π_{i-1} is of rank $k-1$ or higher, then σ_{i-1} is of rank $k-1$ or higher. Since σ_i is of rank $k-2$ or less, we have $\sigma_{i-1} > \sigma_i$. Thus, we have concluded that if $\pi_{i-1} > \pi_i$, then $\sigma_{i-1} > \sigma_i$.

Next we aim to show that if $\pi_{i-1} < \pi_i$, then we have $\sigma_{i-1} < \sigma_i$. We have three cases. If π_i is of rank $k-2$ or less in π, then the rank of π_{i-1} is also $k-2$ or less. In this case, we have $\sigma_{i-1} = \pi_{i-1} < \pi_i = \sigma_i$. If both π_i and π_{i-1} are of rank $k-1$ or higher, then according to the definition of f, we have $\sigma_{i-1} < \sigma_i$. If π_i has rank $k-1$ or higher and π_{i-1} is of rank $k-2$ or less, then the rank of σ_{i-1} is $k-2$ or less and σ_i is of rank $k-1$ or higher. This implies that $\sigma_{i-1} < \sigma_i$. Thus, we have concluded that if $\pi_{i-1} < \pi_i$, then $\sigma_{i-1} < \sigma_i$. Therefore, we have $\mathcal{D}(\pi) = \mathcal{D}(f(\pi))$.

Notice that $f(\pi)$ avoids H_k since the existence of such a pattern in $f(\pi)$ would mean that the last two entries of that pattern were not placed according to the rule specified above. Moreover, we have that $f(\pi)$ avoids Q_k. If not, suppose that $\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_k}$ is such a Q_k. Then we have $\sigma_{i_k-1} > \sigma_{i_k}$ and σ_{i_k} has rank $k-1$ or higher. Since $\mathcal{D}(\pi) = \mathcal{D}(f(\pi))$, we have $\pi_{i_k-1} > \pi_{i_k}$. Recall that if $\pi_{i-1} > \pi_i$, then both π_i and σ_i have rank $k-2$ or less. This implies that σ_{i_k} has rank $k-2$ or less, which contradicts the fact that σ_{i_k} has rank $k-1$ or higher. Thus, we deduce that $f(\pi)$ avoids Q_k.

In order to show that the map f is a bijection, we construct a map g from the set $\mathcal{S}_n(H_k, Q_k)$ to the set $\mathcal{S}_n(G_k)$. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in \mathcal{S}_n(H_k, Q_k)$. We aim to obtain $g(\sigma)$ by leaving all entries of σ that are of rank $k-2$ or less in their place and rearranging the entries of rank $k-1$ or higher. Label the blocks of σ from left to right by $B_1, B_2, \ldots, B_s$. Let P_i be the set of positions of π in which an entry, that has rank $k-1$ or higher and belongs to the block B_i, is located, and let R be the set of entries of π that are of rank $k-1$ or higher. Now we fill the positions of P_i as follows.

Step 1. Choose $|P_1|$ smallest entries from R that are larger than the closest entry of rank

$k-2$ on the left of the positions of P_1, and fill the positions of P_1 with the selected entries from left to right in increasing order.

Step 2. Choose $|P_2|$ smallest entries from R that have not been placed yet and are larger than the closest entry of rank $k-2$ on the left of the positions of P_2. Fill the positions of P_2 with the selected entries from left to right in increasing order.

Step 3. Fill the positions of P_3, P_4, …, P_s as in Step 2.

Let $g(\sigma)$ be the obtained permutation.

Example 2.2. *Consider* $\sigma = 13487(10)(11)5926 \in \mathcal{S}_{11}(H_6, Q_6)$. *Then we have* $B_1 = 1348$, $B_2 = 7(10)(11)$, $B_3 = 59$ *and* $B_4 = 26$. *Moreover, we have* $P_1 = \emptyset$, $P_2 = \{6, 7\}$, $P_3 = \{9\}$, $P_4 = \{11\}$ *and* $R = \{6, 9, 10, 11\}$. *According to the definition of* g*, we have* $g(\sigma) = 134879(10)562(11)$.

Since the existence of σ shows that there is at least one way to assign the entries of R to the positions of P_i, the definition of g always enables us to create $g(\sigma)$.

Notice that if entry σ_i has rank $k-2$ or less, then σ_i does not move in the above procedure, and the rank of σ_i do not change. If entry σ_i has rank $k-1$ or higher, then σ_i may have moved and the rank of σ_i in $g(\sigma)$ is $k-1$ or higher. We claim that if $\sigma_{i-1} > \sigma_i$, then the rank of σ_i is $k-2$ or less. If not, there is an increasing subsequence of length $k-1$ that ends in σ_i. Such an increasing subsequence combining with σ_{i-1} would form a Q_k in σ.

By similar reasoning as in the proof of the equality $\mathcal{D}(\pi) = \mathcal{D}(f(\pi))$, one can verify that $\mathcal{D}(\sigma) = \mathcal{D}(g(\sigma))$. Now we proceed to show that $g(\sigma)$ avoids G_k. Let $g(\sigma) = \pi_1\pi_2 \ldots \pi_n$. Suppose that the the subsequence $\pi_{i_1}\pi_{i_2} \ldots \pi_{i_k}$ is a pattern G_k in π with $i_1 < i_2 < \ldots < i_k$. Without loss of generality, assume that $\pi_{i_{k-2}}$ has rank $k-2$. Clearly, both $\pi_{i_{k-1}}$ and π_{i_k} have rank $k-1$ or higher in π. Suppose that $i_{k-1} \in P_j$ for some j, and σ_s is the closest entry of rank $k-2$ on the left of the positions of P_j in σ. Recall that the map g does not change the position of entry σ_i that has rank $k-2$ or less, and the rank of σ_i does not change in π. So we have $\pi_s = \sigma_s$ and $\pi_{i_{k-2}} = \sigma_{i_{k-2}}$, and the rank of π_s (resp. $\sigma_{i_{k-2}}$) is $k-2$ in π (resp. σ). Moreover, since σ_s is the closest entry of rank $k-2$ on the left of the positions of P_j in σ, we have $i_{k-2} \leqslant s$. This implies that $\sigma_s \leqslant \sigma_{i_{k-2}} = \pi_{i_{k-2}}$. Then, we have $\pi_{i_k} > \sigma_s$, which contradicts the selection of $\pi_{i_{k-1}}$ when filling the positions of P_j. Hence, we have $g(\sigma) \in \mathcal{S}_n(G_k)$.

In order to show that the map f is a bijection, it suffices to show that the maps f and g are inverses of each other. First, we wish to prove that for any $\pi \in \mathcal{S}_n(G_k)$, we have $g(f(\pi)) = \pi$. Since $\mathcal{D}(f(\pi)) = \mathcal{D}(\pi)$, π and $f(\pi)$ have the same number of blocks. Suppose that $B'_1, B'_2, \ldots, B'_s$ are the blocks of $f(\pi)$, that are listed from left to right. Let P'_i be the set of positions of $f(\pi)$ in which an entry that has rank $k-1$ or higher and belongs to the block B'_i, is located, and let R' be the set of entries of $f(\pi)$ that are of rank $k-1$ or higher. Recall that if the entry π_i of π has rank $k-2$ or less, then the map f does not change the position of π_i, and the rank of π_i do not change. If entry π_i of π has rank $k-1$ or higher, then π_i may have moved and the rank of π_i in $f(\pi)$ is $k-1$ or higher. So we have $P_i = P'_i$ and $R = R'$. Since π avoids G_k, the positions of P_1 in π are filled with $|P_1|$ smallest elements of R in increasing order which are larger than the closet entry of rank $k-2$ on the left of the positions of P_1. The positions of P_2 are filled with the next $|P_2|$ smallest entries of R in increasing order that have not been placed and larger than the closet entry of rank $k-2$ on the left of the positions of P_2. And the positions of $P_3, \ldots, P_s$ are filled in the same manner as the positions of P_2. Thus, according to the definition of g, it is easy to check that $g(f(\pi)) = \pi$.

Our next goal is to show that for any $\sigma \in \mathcal{S}_n(H_k, Q_k)$, we have $f(g(\sigma)) = \sigma$. Since $\mathcal{D}(g(\sigma)) = \mathcal{D}(\sigma)$, σ and $f(\sigma)$ have the same number of blocks. Suppose that $B'_1, B'_2, \ldots, B'_s$ are the blocks of $f(\sigma)$, that are listed from left to right. Let P'_i be the set of positions of $f(\pi)$ in which an entry that has rank $k-1$ or higher and belongs to the block B'_i, is located, and let R' be the set of entries of $f(\pi)$ that are of rank $k-1$ or higher. Recall that if entry σ_i of σ has rank $k-2$ or less, then the map g does not change the position of σ_i, and the rank of σ_i do not change. If entry σ_i of σ has rank $k-1$ or higher, then σ_i may have moved and the rank of σ_i in $g(\sigma)$ is $k-1$ or higher. So we have $P_i = P'_i$ and $R = R'$. Since σ avoids H_k, the positions of P_1 in σ are filled with $|P_1|$ largest elements of R in increasing order which are larger than the closet entry of rank $k-2$ on the left of the positions of P_1. The positions of P_2 are filled with the next $|P_2|$ largest entries of R in increasing order that have not been placed and larger than the closet entry of rank $k-2$ on the left of the positions of P_2. And the positions of $P_3, \ldots, P_s$ are filled in the same manner as the positions of P_2. Thus, according to the definition of f, it is easy to check that $f(g(\sigma)) = \sigma$.

3 Proof of Theorem 1.5

Let us begin with some necessary definitions and notations. We draw Young diagrams in English notation, and number columns from left to right and rows from bottom to up. For example, the square $(1, 2)$ is the second square in the bottom row of a Young diagram.

A *transversal* of a Young diagram $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n)$ is a filling of the squares of λ with $1's$ and $0's$ such that every row and column contains exactly one 1. Denote by $T = \{(t_i, i)\}_{i=1}^n$ the transversal in which the square (t_i, i) is filled with a 1 for all $i \leqslant n$. For example, the transversal $T = \{(1, 1), (2, 3), (3, 4), (4, 2), (5, 5)\}$ of a Young diagram $(5, 4, 4, 3, 1)$ is illustrated as Figure 1.

In this section, we will consider permutations as permutation matrices. Given a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{S}_n$, its corresponding *permutation matrix* is a transversal of the square shape $\lambda_1 = \lambda_2 = \ldots = \lambda_n = n$ in which the square (π_i, i) is filled with a 1 for all $1 \leqslant i \leqslant n$ and all the other squares are filled with $0's$.

The notion of pattern avoidance is extended to transversal of a Young diagram in [1]. Given a permutation α of $\mathcal{S}_m$, let M be its permutation matrix. A transversal L of a Young diagram λ will be said to contain α if there exists two subsets of the index set $[n]$, namely, $R = \{r_1 < r_2 < \ldots < r_m\}$ and $C = \{c_1 < c_2 < \ldots < c_m\}$, such that the matrix on R and C is a copy of M and each of the squares (r_j, c_j) falls within the Young diagram.

The remaining part of this section is organized as follows. In Subsection 3.1, we

0	0	0	0	1
0	1	0	0	
0	0	0	1	
0	0	1		
1				

Figure 1: The transversal $T = \{(1,1),(2,3),(3,4),(4,2),(5,5)\}$.

describe a transformation ϕ that changes every occurrence of H_k (or Q_k) to an occurrence of F_k. Based on the transformation ϕ, we establish a map Φ from the set $\mathcal{S}_n(F_k)$ to the set $\mathcal{S}_n(H_k, Q_k)$, that recursively transforms every occurrence of H_k (or Q_k) into F_k. In Subsection 3.2, we define a transformation ψ that changes every occurrence of F_k to an occurrence of H_k (or Q_k). Relying on the transformation ψ, we establish a map Ψ from the set $\mathcal{S}_n(H_k, Q_k)$ to the set $\mathcal{S}_n(F_k)$, that recursively transforms every occurrence of F_k into H_k (or Q_k). For the purpose of establishing Theorem 1.5, we investigate certain properties of ϕ and ψ in Subsections 3.1 and 3.2, respectively. In Subsection 3.3, we show that the maps Φ and Ψ are well defined and preserve the descent set. Moreover, they are inverses of each other, thereby establishing Theorem 1.5.

3.1 The map Φ from the set $\mathcal{S}_n(F_k)$ to the set $\mathcal{S}_n(H_k, Q_k)$

Before we describe the map Φ, let us review a transformation θ introduced in [1]

Let $\pi = \{(\pi_1, 1), (\pi_2, 2), \ldots, (\pi_n, n)\}$. Suppose that G is the submatrix of π at columns $c_1 < c_2 < \ldots < c_{k-1} < c_k$ and rows $r_1 < r_2 < \ldots < r_{k-1} < r_k$, which is isomorphic to J_k. In other words, the square (r_i, c_i) is filled with 1 for all $i = 1, 2, \ldots, k$. Let $\theta(G)$ be the submatrix at the same rows and columns as G, such that the squares (r_2, c_1), (r_3, c_2), …, (r_k, c_{k-1}), (r_1, c_k) are filled with $1's$ and all the other squares are filled with $0's$. Clearly, $\theta(G)$ is isomorphic to F_k.

Based on the transformation θ, we define the following three transformations, which will play an essential role in the construction of the map Φ.

Suppose that G is the submatrix of π at columns $c_1 < c_2 < \ldots < c_{k-1} < s < s+1 < \ldots < c_k - 1 < c_k$ and rows $r_1 < r_2 < \ldots < r_{k-1} < r_k > \pi_{c_k - 1} > \ldots > \pi_{s+1} > \pi_s$, in which the squares (r_i, c_i) are filled with $1's$ for all $i = 1, 2, \ldots, k$. Let $\alpha(G)$ be the submatrix at the same rows and columns as G, such that the squares (r_2, c_1), (r_3, c_2), …, (r_k, c_{k-1}), (r_1, s), $(\pi_s, s+1)$, …, (π_{c_k-1}, c_k) are filled with $1's$ and all the other squares are filled with $0's$. Clearly, the submatrix at columns $c_1 < c_2 < \ldots < c_{k-1} < s$ and rows $r_1 < r_2 < \ldots < r_{k-1} < r_k$ is isomorphic to F_k.

Suppose that G is the submatrix of π at columns $c_1 < c_2 < \ldots < c_{k-1} < c_k < c_k + 1 < \ldots < t - 1 < t$ and rows $r_1 < r_2 < \ldots < r_k > \pi_{c_k+1} > \ldots > \pi_{t-1} > \pi_t$, in which the squares (r_i, c_i) are filled with $1's$ for all $i = 1, 2, \ldots, k$. Define $\beta(G)$ to be the submatrix at the same columns and rows as G, such that the squares (r_2, c_1), (r_3, c_2), $\ldots$, (r_k, c_{k-1}), (π_{c_k+1}, c_k), $\ldots$, $(\pi_t, t-1)$, (r_1, t) are filled with $1's$ and all the other squares are filled with $0's$. Clearly, the submatrix at columns $c_1 < c_2 < \ldots < c_{k-1} < t$ and rows $r_1 < r_2 < \ldots < r_{k-1} < r_k$ is isomorphic to F_k.

The transformation ϕ: Suppose that π is a permutation in $\mathcal{S}_n$. First, find the highest square (p_1, q_1) containing a 1, such that there is an H_k or Q_k in π in which the 1 positioned at the square(p_1, q_1) is the leftmost entry. Then, find the leftmost square (p_2, q_2) containing a 1, such that there is an H_k or Q_k in π in which the $1's$ positioned at the squares (p_1, q_1) and (p_2, q_2) are the leftmost two $1's$. Find $(p_3, q_3), (p_4, q_4), \ldots, (p_{k-1}, q_{k-1})$ one by one as (p_2, q_2).

If there is an H_k in which the $1's$ positioned at the squares $(p_1, q_1), (p_2, q_2), \ldots$ (p_{k-1}, q_{k-1}) are the leftmost $k-1$ $1's$, then find the highest square (p_k, q_k) containing a 1, such that the $1's$ positioned at the squares $(p_1, q_1), (p_2, q_2), \ldots$ (p_k, q_k) form an H_k. Find the largest s such that $s-1 \in \mathcal{D}(\pi)$ and $q_{k-1} < s < q_k$. Now we proceed to construct a permutation $\phi(\pi)$ by the following procedure.

Case 1. $q_k = n$ or $\pi_{q_k-1} > \pi_{q_k+1}$. Let G be the submatrix of π at columns $q_1 < q_2 < \ldots < q_{k-1} < s < s+1 < \ldots < q_k - 1 < q_k$ and rows $p_1 < p_2 < \ldots < p_k > \pi_{q_k-1} > \ldots > \pi_{s+1} > \pi_s$. Replace G by $\alpha(G)$ and leave all the other squares fixed.

Case 2. $\pi_{q_k-1} < \pi_{q_k+1}$. Find the least t such that $t > q_k$ and $t \in \mathcal{A}(\pi)$. If such t does not exist, set $t = n$. Let G be the submatrix of π at columns $q_1 < q_2 < \ldots < q_{k-1} < q_k < q_k+1 < \ldots < t-1 < t$ and rows $p_1 < p_2 < \ldots < p_k > \pi_{q_k+1} > \ldots > \pi_{t-1} > \pi_t$. Replace G by $\beta(G)$ and leave all the other squares fixed.

If such an H_k does not exist, then find the leftmost square (p_k, q_k) containing a 1, such that the $1's$ positioned at the squares $(p_1, q_1), (p_2, q_2) \ldots, (p_k, q_k)$ form a Q_k. Construct a permutation $\phi(\pi)$ by the following procedure.

Case 3. $q_k \in \mathcal{A}(\pi)$. Let G be the submatrix of π at columns $q_1 < q_2 < \ldots < q_{k-2} < q_k$ and rows $p_1 < p_2 < \ldots < p_{k-2} < p_k$. Replace G by $\theta(G)$ and leave all the other squares fixed.

Case 4. Otherwise, find the least t such that $t > q_k$ and $t \in \mathcal{A}(\pi)$. If such t does not exist, set $t = n$. Let G be the submatrix of π at columns $q_1 < q_2 < \ldots < q_{k-2} < q_k < q_k+1 < q_k+2 < \ldots < t-1 < t$ and rows $p_1 < p_2 < \ldots < p_{k-2} < p_k > \pi_{q_k+1} > \pi_{q_k+2} > \ldots > \pi_{t-1} > \pi_t$. Replace G by $\beta(G)$ and leave all the other squares fixed.

Remark 3.1. *Notice that the definition of H_k ensures that there exists an s such that $s-1 \in \mathcal{D}(\pi)$ and $q_{k-1} < s \leqslant q_k$. In fact, we have $q_{k-1} < s < q_k$. If not, then the $1's$ positioned at the squares $(p_2, q_2), (p_3, q_3) \ldots (p_{k-1}, q_{k-1}), (\pi_{q_k-1}, q_k-1), (p_k, q_k)$ would form a Q_k, which contradicts the selection of (p_1, q_1).*

Remark 3.2. *We denote the resulting permutation in Case 1, Case 2, Case 3 and Case 4 by $\phi_1(\pi), \phi_2(\pi)$, $\phi_3(\pi)$ and $\phi_4(\pi)$, respectively.*

It is obvious that the transformation ϕ changes every occurrence of H_k (or Q_k) to an occurrence of F_k. Denote by Φ the iterated transformation, that recursively transforms every occurrence of H_k (or Q_k) into F_k.

Using the notation of the algorithm for ϕ_1, we label the squares containing $1's$ in G by $a_1, a_2, \ldots, a_{k-1}, c_1, c_2, \ldots, c_{q_k-s}, a_k$, and the squares containing $1's$ in $\alpha(G)$ by $b_1, b_2, \ldots, b_k, d_1, d_2, \ldots, d_{q_k-s}$, from left to right, see Figure 2 for example.

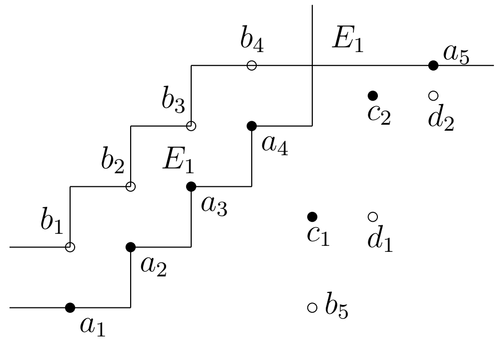

Figure 2: The labelling of squares in $\phi_1(\pi)$ for $k = 5$.

Using the notation of the algorithm for ϕ_2, we label the squares containing $1's$ in G by $a_1, a_2, \ldots, a_{k-1}, a_k, e_1, e_2, \ldots, e_{t-q_k}$, and the squares containing $1's$ in $\beta(G)$ by $b_1, b_2, \ldots, b_{k-1}, f_1, f_2, \ldots, f_{t-q_k}, b_k$, from left to right. We also label the square (π_s, s) by c_1, see Figure 3 for example.

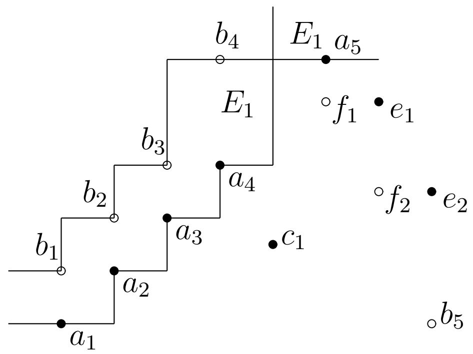

Figure 3: The labelling of squares in $\phi_2(\pi)$ for $k = 5$.

Using the notation of the algorithm for ϕ_3, we label the squares containing $1's$ in G by $a_1, a_2, \ldots, a_{k-2}, a_k$, and the squares containing $1's$ in $\theta(G)$ by $b_1, b_2, \ldots, b_{k-2}, b_k$, from left to right. We also label the square (p_{k-1}, q_{k-1}) by b_{k-1} (or a_{k-1}), see Figure 4 for example.

Using the notation of the algorithm for ϕ_4, we label the squares containing $1's$ in G by $a_1, a_2, \ldots, a_{k-2}, a_k$, $e_1, e_2, \ldots, e_{t-q_k}$, and the squares containing $1's$ in $\beta(G)$ by $b_1, b_2, \ldots, b_{k-2}, f_1, f_2, \ldots, f_{t-q_k}, b_k$, from left to right. We also label the square (p_{k-1}, q_{k-1}) by b_{k-1} (or a_{k-1}), see Figure 5 for example.

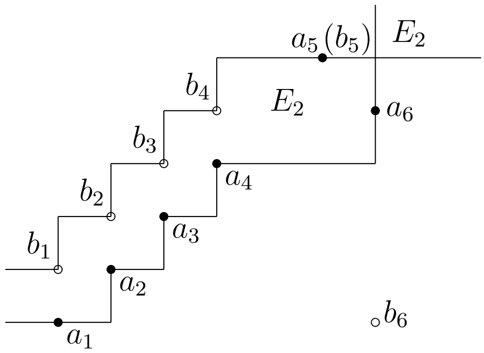

Figure 4: The labelling of squares in $\phi_3(\pi)$ for $k = 6$.

In $\phi_1(\pi)$ or $\phi_2(\pi)$, we denote by E_1 the union of the following four parts of the board: the board that is above a_1 but below b_1 and to the left of a_1, the board that is above a_{k-1} but below b_{k-1}, to the left of c_1 and to the right of a_{k-1}, the union of the rectangles with corners a_i and b_{i+1} for $i = 1, 2, \ldots, k-2$, and the board that is above a_k and to the right of c_1, see Figures 2 and 3 for example.

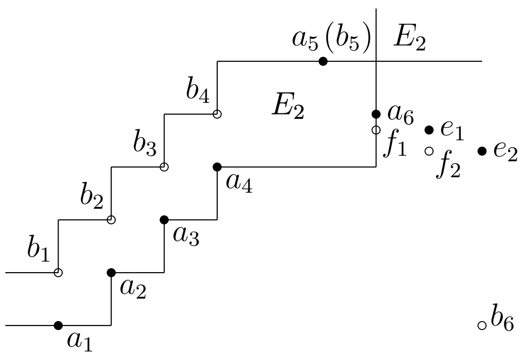

Figure 5: The labelling of squares in $\phi_4(\pi)$ for $k = 6$.

We claim that there are no $1's$ inside E_1 in π, $\phi_1(\pi)$ or $\phi_2(\pi)$. By the selection of a_k, there is no 1 to the right of c_1 and above a_k in π, $\phi_1(\pi)$ or $\phi_2(\pi)$. Suppose that there is a 1 in the rectangle with corners a_i and b_{i+1} for $i = 1, 2, \ldots, k-2$, then that 1 combining with the $1's$ positioned at $a_1, a_2, \ldots, a_i, a_{i+2}, \ldots, a_k$ would form an H_k in π, which contradicts the selection of a_{i+1}. If there is a 1 above a_1 but below b_1, then that 1, combining with the $1's$ positioned at $a_2, a_3, \ldots, a_k$ would form a H_k in π, which contradicts the selection of a_1. If there is a 1 above a_{k-1} but below b_{k-1} and to the left of c_1, then that 1, combining with the $1's$ positioned at $a_2, a_3, \ldots, a_k$ would form a H_k in π, which contradicts the selection of a_1. Thus, all the $1's$ are to the left of E_1 or to the right of E_1 in π, $\phi_1(\pi)$ or $\phi_2(\pi)$.

In $\phi_3(\pi)$ and $\phi_4(\pi)$, we denote by E_2 the union of the following four parts of the board: the board that is above a_1 but below b_1 and to the left of a_1, the board that is above and to the right of a_{k-2} but below a_{k-1}, and to the left of a_k, the union of the rectangles with corners a_i and b_{i+1} for $i = 1, 2, \ldots, k-2$, and the board that is above a_{k-1} and to the right of a_k, see Figures 4 and 5 for example.

We claim that there are no $1's$ inside E_2 in π, $\phi_3(\pi)$ and $\phi_4(\pi)$. By similar arguments in E_1, one can easily verify that there are no $1's$ inside the board that is above a_1 but below b_1 and to the left of a_1, the union of the rectangles with corners a_i and b_{i+1} for $i = 1, 2, \ldots, k-2$, and the board that is above a_{k-1} and to the right of a_k. It remains to show that there are no $1's$ inside the board that is above a_{k-2} but below a_{k-1} and to the left of a_k. According to the definition of Q_k, all of the $1's$ between a_{k-1} and a_k are above a_{k-1}. This implies that there are no $1's$ inside the board that is below and to the right of a_{k-1}, and to the left of a_k. Now suppose that there is a 1 inside the rectangle with corners a_{k-2} and a_{k-1}. Suppose that this 1 is at the square (π_g, g). If (π_g, g) is below a_k, then the $1's$ positioned at the squares $a_2, a_3, \ldots, a_{k-2}, (\pi_g, g), a_{k-1}, a_k$ would form a Q_k in π, which contradicts the selection of a_1. If (π_g, g) is above a_k, then we have two cases. If there exists a j such that $g \leqslant j < q_k$ and $j \in \mathcal{D}(\pi)$, then the $1's$ positioned at $a_1, a_2, \ldots, a_{k-2}, (\pi_g, g), a_{k-1}$ would form an H_k in π, which contradicts the selection of a_{k-1}. Otherwise, the $1's$ positioned at the squares $a_1, a_2, \ldots, a_{k-2}, (\pi_g, g), a_k$ would form a Q_k in π, which contradicts the selection of a_{k-1}. Hence, we have concluded that there are no $1's$ inside the board that is above a_{k-2} but below a_{k-1} and to the left of a_k. Hence, the claim is proved. In other words, all the $1's$ of π, $\phi_3(\pi)$ or $\phi_4(\pi)$ are to the left or to the right of E_2.

Definition 3.3. *A 1 is said to be strictly to the left (resp. right) of E_1(or E_2) if it is lying to the left (resp. right) of E_1 (or E_2) and does not belong to the boundary of E_1 (or E_2).*

In order to show that the transformation ϕ has the desired properties, which are essential in the proof of Theorem 1.5, we introduce *vertical slide algorithm* and *horizontal slide algorithm* for ϕ. Before that, we need the following useful properties that will play a crucial role in the construction of vertical slide algorithm and horizontal slide algorithm for ϕ.

Properties

(1) For any $1 \leqslant i \leqslant k-2$, the board that is above a_1 and below b_i cannot contain a J_i with all its $1's$ strictly to the left of E_1 (or E_2) in $\phi(\pi)$.

(2) For any $1 \leqslant i < j \leqslant k-2$, the rectangle with corners b_i and b_j cannot contain a J_{j-i} with all its $1's$ strictly to the left of E_1 (or E_2) in $\phi(\pi)$. Moreover, the rectangle with corners b_i and b_{k-1} cannot contain a J_{k-1-i} with all its $1's$ strictly to the left of E_1 in $\phi_1(\pi)$ (or $\phi_2(\pi)$).

Proof.

(1) If there is such a J_i below b_i in $\phi(\pi)$, then it is below a_{i+1}. Therefore these i $1's$, combining with $a_{i+1}, a_{i+2}, \ldots, a_k$, will either form an H_k or a Q_k in π, which contradicts the selection of a_1.

(2) If there is a J_{j-i} in this region, then either its leftmost 1 is to the left to b_{i+1} (and hence to the left of a_{i+1}), or else it lies to the right of b_{i+1} (and a_{i+1}). In the first

case, $a_1, a_2, \ldots, a_i$, combining with this J_{j-i} and $a_{j+1}, \ldots, a_k$, will form an H_k (or Q_k) in π, which contradicts the selection of a_{i+1}. In the second case, $a_2, a_3, \ldots, a_{i+1}$, combining with this J_{j-i} and $a_{j+1}, \ldots, a_k$, will form an H_k (or Q_k) in π, which contradicts the selection of a_1. ∎

Now we proceed to introduce the vertical slide algorithm and horizontal slide algorithm for ϕ.

Suppose that H is a J_t in $\phi(\pi)$. Label the squares containing $1's$ of H by $h_1, h_2, \ldots, h_t$, from left to right.

Vertical slide algorithm for ϕ: When H is in $\phi_1(\pi)$ (or $\phi_2(\pi)$), find the largest i such that b_i falls in H with $i \leqslant k-1$; otherwise, find the largest i such that b_i falls in H with $i \leqslant k-2$. If there is a 1 of H which is below b_i and to the right of E_1 (or E_2), find the rightmost square containing such a 1 and denote it by h_y. Find x such that h_y is to the right of b_x, and to the left of b_{x+1}. By property (2), there are at most $i-x$ $1's$ in H, which are above b_x but not above b_i, and weakly to the left of E_1 (or E_2). So we can replace these $1's$ by $b_{x+1}, b_{x+2}, \ldots, b_i$, and hence by $a_{x+1}, a_{x+2}, \ldots, a_i$.

We can repeat the vertical slide algorithm until the following two cases appear.

(1) There is no b_i that falls in H.

(2) There is such a b_i, but h_y does not exist. By Property (1), there are at most i $1's$ of H that are above a_1 but not above b_i, and weakly to the left of E_1 (or E_2). So we can replace these $1's$ by $a_1, a_2, \ldots, a_i$ to form an J_t in π.

Suppose that H is a J_t in $\phi(\pi)$. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_t$, from left to right. Assume that h_t is not above b_{k-1} when H is in $\phi_1(\pi)$ (or $\phi_2(\pi)$), and not above b_{k-2} when H is in $\phi_3(\pi)$ (or $\phi_4(\pi)$).

Horizontal slide algorithm for ϕ: When H is in $\phi_1(\pi)$ (or $\phi_2(\pi)$), find the least i such that b_i falls in H with $i \leqslant k-1$; otherwise, find the least i such that b_i falls in H with $i \leqslant k-2$. If there is a 1 of H which is above b_i and to the right of E_1 (or E_2), find the leftmost square containing such a 1 and denote it by h_y. Find x such that h_y is above b_x, and below b_{x+1}. By property (2), there are at most $x+1-i$ $1's$ in H, which are below b_{x+1} but not below b_i, and weakly to the left of E_1 (or E_2). So we can replace these $1's$ by $b_i, b_{i+1}, \ldots, b_x$, and hence by $a_{i+1}, a_{i+2}, \ldots, a_{x+1}$.

We can repeat the horizontal slide algorithm until the following two cases appear.

(1) There is no b_i that falls in H.

(2) There is such a b_i, but h_y does not exist. Find the least v such that h_t is not above b_v. By property (2), we have at most $v-i$ $1's$ of H that are below b_v but not below b_i and weakly to the left of E_1(or E_2). So we can replace these $1's$ by $a_{i+1}, a_{i+2}, \ldots, a_v$ to form an J_t in π.

Our next goal is to show that the transformation ϕ have the following properties, which are essential in the proof of Theorem 1.5.

Lemma 3.4. *If there is no F_k with at least one square in a row below a_1, then we have $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$.*

Proof. Since there are no $1's$ inside E_1 (or E_2) and no F_k with at least one square in a row below a_1, one can easily verify that $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$. The details are omitted here. ∎

Lemma 3.5. *If π contains no F_k with at least one square in a row below a_1, then $\phi(\pi)$ contains no such F_k.*

Proof. If not, suppose that H is such an F_k in $\phi(\pi)$. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_k$, from left to right. Then h_k is below a_1.

We claim that h_k must be positioned to the left of a_{k-1}. If not, then $a_1, a_2, \ldots, a_{k-1}$, h_k would form an F_k in π, which contradicts the hypothesis. From the construction of the transformation ϕ, it follows that at least one of $b_1, b_2, \ldots, b_{k-2}$ must fall in H. Otherwise, H is an F_k with at least one square in a row below a_1 in π, which contradicts the hypothesis.

By applying the vertical slide algorithm repeatedly to the J_{k-1} consisting of $h_1, h_2, \ldots, h_{k-1}$, we can get a J_{k-1} not below a_1 in π. Then, that J_{k-1} combining with h_k will form an F_k in π, which contradicts the hypothesis. ∎

In the remaining part of this subsection, we assume that π contains no F_k with at least one square in a row below a_1. By Lemma 3.4, we have $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$.

Lemma 3.6. *The board that is to the left of b_{t+1} and above a_1 cannot contain a J_t in $\phi(\pi)$ with its highest 1 below b_t for $t = 1, 2, \ldots, k-1$.*

Proof. First we aim to prove the assertion for $1 \leqslant t \leqslant k-2$. Suppose that H is such a J_t in $\phi(\pi)$. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_t$ from left to right. We claim that at least one of $b_1, b_2, \ldots, b_{t-1}$ must fall in H. Otherwise, these t $1's$, combining with $a_{t+1}, a_{t+2}, \ldots, a_k$, would form an H_k or Q_k in π. This contradicts the selection of a_1.

By applying the the horizontal slide algorithm repeatedly to H, we can get a J_t in π. It is easy to check that the obtained J_t is below and to the left of a_{t+1} and above a_1. That J_t, combining with $a_{t+1}, a_{t+2}, \ldots, a_k$, would form an H_k or Q_k in π. This contradicts the selection of a_1. Thus, we have concluded that the assertion holds for $1 \leqslant t \leqslant k-2$.

Now we proceed to show that the assertion also holds for $t = k-1$. Suppose that G is a J_{k-1} in $\phi(\pi)$, which is to the left of b_k and below b_{k-1}. We label the squares containing the $1's$ of G by $g_1, g_2, \ldots, g_{k-1}$, from left to right. We have three cases.

Case 1. G is in $\phi_1(\pi)$. By repeating the horizontal slide algorithm, we can get a J_{k-1} in π, which is to the left of b_k and above a_1. Since $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$ and $s-1 \in \mathcal{D}(\pi)$, we have $s-1 \in \mathcal{D}(\phi(\pi))$. Recall that b_k is at column s. Thus, the obtained J_{k-1} combining with a_k would form an H_k in π. This contradicts the selection of a_1.

Case 2. G is in $\phi_2(\pi)$. If $g_{k-1} \neq f_i$, we can get a J_{k-1} in π by repeating the horizontal slide algorithm, which is below and to the left of a_k and above a_1. We label the squares containing the $1's$ of this J_{k-1} by $m_1, m_2, \ldots, m_{k-1}$, from left to right. If m_{k-1} is below e_1, then $m_2, m_3, \ldots, m_{k-1}$, combining with a_k, e_1, would form a Q_k in π, which contradicts the selection of a_1. If m_{k-1} is above e_1, then it is positioned to the left of c_1 in π since all the $1's$ positioned at columns $s, s+1, \ldots, q_k - 1$ form a $J_{q_k - s}$, and $(\pi_{q_k - 1}, q_k - 1)$ is below $e_1 = (\pi_{q_k+1}, q_k + 1)$. Moreover, since $s-1 \in \mathcal{D}(\pi)$ and $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$, we have $s-1 \in \mathcal{D}(\phi(\pi))$. Recall that c_1 is at column s. Thus, $m_1, m_2, \ldots, m_{k-1}, a_k$ will form an H_k, which contradicts the selection of a_1.

If $g_{k-1} = f_i$ for some i, then we can get a J_{k-1} in π by repeating the horizontal slide algorithm and replacing $f_j's$ by $e_j's$ whenever f_j falls in G. Notice that the rightmost 1 of the obtained J_{k-1} is e_i. If $i = 1$, then this J_{k-1} combining with a_k would form a Q_k in π. For $i > 1$, this J_{k-1} combining with e_{i-1} would form a Q_k in π. In both cases, we get a Q_k that is above a_1. This contradicts the selection of a_1.

Case 3. G is in $\phi_3(\pi)$ or $\phi_4(\pi)$. When $g_{k-1} \neq f_i$, according to the definition of Q_k, there contains no $1's$ which are below and to the right of a_{k-1}, and to the left of a_k. So g_{k-1} is to the left of a_{k-1}. Recall that we have shown that there is no J_{k-2} in $\phi(\pi)$, which is to the left of b_{k-1} and below b_{k-2}. So g_{k-1} is below and to the left of b_{k-1} (and a_{k-1}), and above and to the left of b_{k-2}. By repeating the vertical slide algorithm, we can get a J_{k-1} not below a_1 in π, whose rightmost 1 is positioned at g_{k-1}. Then this J_{k-1} combining with a_{k-1} would form an H_k in π, which contradicts the selection of a_{k-1}. When $g_{k-1} = f_i$ for some $i \geqslant 1$, by the same way as in Case 2, we can get a Q_k above a_1 in π. This contradicts the selection of a_1.

Thus, we deduce that the assertion also holds for $t = k - 1$. This completes the proof. ∎

Lemma 3.7. *The rows above a_1 cannot contain an H_k or Q_k in $\phi(\pi)$.*

In order to prove Lemma 3.7, we need the following two lemmas.

Lemma 3.8. *Suppose that G is an H_k above a_1 in $\phi(\pi)$. Label the squares containing the $1's$ of G by $g_1, g_2, \ldots, g_k$, from left to right. Then the squares g_k and g_{k-1} are also filled with $1's$ in π.*

Proof. Here we only prove the assertion for $\phi_1(\pi)$ and $\phi_4(\pi)$. All the other cases can be verified by similar arguments. By Lemma 3.6, there is no J_{k-1} below b_{k-1} and to the left of b_k in $\phi_1(\pi)$ and $\phi_4(\pi)$. This implies that neither g_{k-1} nor g_k will be any of $b_i's$ for $1 \leqslant i \leqslant k-2$ in $\phi_1(\pi)$ and $\phi_4(\pi)$. Moreover, neither g_{k-1} nor g_k will be any of $f_i's$ in $\phi_4(\pi)$. Thus, we have deduced that the assertion holds for $\phi_4(\pi)$.

In order to prove the assertion for $\phi_1(\pi)$, it remains to show that neither g_k nor g_{k-1} will be any of b_{k-1} and $d_i's$ in $\phi_1(\pi)$. We have four cases.

(1) If $g_k = b_{k-1}$, then $g_1, g_2, \ldots, g_{k-1}$ form a J_{k-1}, which is to the left of b_k and below b_{k-1} in $\phi_1(\pi)$. This contradicts Lemma 3.6.

(2) If $g_{k-1} = b_{k-1}$, then g_k is above b_{k-1} and to the left of E_1. This implies the square g_k is also filled with a 1 in π. Since $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$, the $1's$ positioned at b_{k-1} and g_k belong to two different blocks of $\phi(\pi)$. This implies that those positioned at a_{k-1} and g_k also belong to two different blocks of π. Thus, $a_1, a_2, \ldots, a_{k-1}, g_k$ form an H_k in π. This contradicts the selection of a_k since g_k is above a_k.

(3) If $g_k = d_i$ for some i, then we have that g_{k-1} is to the left of b_k since $d_1, d_2, \ldots, d_{q_k-s}$ lie in consecutive columns and form a J_{q_k-s}. Thus, $g_1, g_2, \ldots, g_{k-1}$ will form a J_{k-1} in $\phi_1(\pi)$, which is to the left of b_k and below b_{k-1}. This contradicts Lemma 3.6. Hence, we have $g_k \neq d_i$ for any $i \geqslant 1$.

(4) If $g_{k-1} = d_j$ for some $j \geqslant 1$, then g_k is to the right of a_k since $d_1, d_2, \ldots, d_{q_k-s}$ lie in consecutive columns and form a J_{q_k-s}. By repeating the horizontal slide algorithm for ϕ and replacing any of $d_i's$ that falls in G by $c_i's$, we will get a J_{k-1} in π whose rightmost 1 is c_j. Since $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$ and $q_k \in \mathcal{D}(\pi)$, we have $q_k \in \mathcal{D}(\phi(\pi))$. Thus, this J_{k-1} combining with g_k will form an H_k in π. This contradicts the selection of a_1.

Hence, we have concluded that the assertion also holds for ϕ_1, which completes the proof. ∎

Lemma 3.9. *Suppose that H is a Q_k above a_1 in $\phi(\pi)$, in which the last two $1's$ lie in two consecutive columns. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_k$, from left to right. Then the squares h_k and h_{k-1} are also filled with $1's$ in π.*

Proof. Here we only prove the assertion for ϕ_1 and ϕ_4. All the other cases can be verified by similar arguments. By Lemma 3.6, there is no J_{k-1} below b_{k-1} and to the left of b_k in $\phi_1(\pi)$. This implies that neither h_{k-1} nor h_k will be any of $b_i's$ for $1 \leqslant i \leqslant k-2$ in $\phi_1(\pi)$ and $\phi_4(\pi)$. Moreover, neither g_{k-1} nor g_k will be any of $f_i's$ in $\phi_4(\pi)$. Thus, we deduce that the assertion holds for ϕ_4.

In order to prove the assertion for ϕ_1, it remains to show that neither h_k nor h_{k-1} will be any of b_{k-1} and $d_i's$ in $\phi_1(\pi)$. Since $b_k, d_1, d_2, \ldots, d_{q_k-s}$ lie in consecutive columns and form a J_{q_k-s+1} in $\phi_1(\pi)$, neither of $d_i's$ can be h_k. Moreover, neither of $d_i's$ can be h_{k-1} for $1 \leqslant i \leqslant q_k - s - 1$. Thus we have $h_{k-1} = d_{q_k-s}$, $h_{k-1} = b_{k-1}$ or $h_k = b_{k-1}$.

If $h_{k-1} = d_{q_k-s}$, then by applying the horizontal slide algorithm for ϕ repeatedly to $h_1, h_2, \ldots, h_{k-2}, h_k$ and replacing any of $d_i's$ that falls in $h_1, h_2, \ldots, h_{k-2}, h_k$ by $c_i's$, we will get a J_{k-1} above a_1 in π. Notice that the rightmost 1 of the obtained J_{k-1} is h_k. This J_{k-1}, combining with a_k, will form a Q_k in π, which contradicts the selection of a_1. If $h_k = b_{k-1}$, then $a_1, a_2, \ldots, a_{k-2}$, combining with h_{k-1} and a_{k-1}, will form a Q_k in π, which contradicts the selection of a_{k-1}. If $h_{k-1} = b_{k-1}$, then h_k is below a_{k-1} and to the left of b_k. Then $h_1, h_2, \ldots, h_{k-2}, h_k$ form a J_{k-1} in $\phi_1(\pi)$, which is to the left of b_k and below b_{k-1}. This contradicts Lemma 3.6. Hence, we have proved that the assertion also holds for ϕ_1. ∎

The proof of Lemma 3.7. If not, suppose that G is an H_k above a_1 in $\phi(\pi)$. Label the squares containing $1's$ of G by $g_1, g_2, \ldots, g_k$, from left to right. Moreover, let H be a Q_k above a_1 in $\phi(\pi)$ such that the rightmost two $1's$ lie in two consecutive columns. We label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_k$. According to the definition of Q_k, there is a Q_k above a_1 in $\phi(\pi)$ if and only if there exists such an H.

We wish to replace some $1's$ of G (resp. H) to form an H_k (resp. Q_k) in π. Here we only consider the case when G (resp. H) is in $\phi_1(\pi)$. The other cases can be verified by the similar arguments. Since the transformation ϕ_1 does not change the positions of any other $1's$, one of $b_i's$ and $d_i's$ must fall in G (resp. H).

First, replace each d_i by c_i whenever d_i falls in $g_1, g_2, \ldots, g_{k-2}$ (resp. $h_1, h_2, \ldots, h_{k-2}$). Then, find the largest i such that b_i falls in $g_1, g_2, \ldots, g_{k-2}$ (resp. $h_1, h_2, \ldots, h_{k-2}$). We can apply the vertical slide algorithm repeatedly to $g_1, g_2, \ldots, g_{k-2}$ (resp. $h_1, h_2, \ldots, h_{k-2}$)

until the following two cases appear.
(1) There is no b_i that falls in $g_1, g_2, \ldots, g_{k-2}$ (resp. $h_1, h_2, \ldots, h_{k-2}$).
(2) There is such a b_i, but there is no 1 positioned at the squares $g_1, g_2, \ldots, g_{k-2}$ (resp. $h_1, h_2, \ldots, h_{k-2}$) that is to the left of b_i and to the right of E_1. Since there are at most i $1's$ positioned at $g_1, g_2, \ldots, g_{k-2}$ (resp. $h_1, h_2, \ldots, h_{k-2}$) that are not above b_i and to the left of E_1, we can replace these $1's$ by $a_1, a_2, \ldots, a_i$.

In both cases, we get a J_{k-2} not below a_1 in π. From Lemmas 3.8 and 3.9, the squares g_k and g_{k-1} (resp. h_k and h_{k-1}) are also filled with $1's$ in π. Recall that $\mathcal{D}(\pi) = \mathcal{D}(\phi(\pi))$ and the $1's$ positioned at g_{k-1} and g_k belong to two different blocks of $\phi(\pi)$. This yields that the $1's$ positioned at g_{k-1} and g_k also belong to two different blocks of π. Thus, the obtained J_{k-2}, combining with g_{k-1} and g_k (resp. h_{k-1} and h_k) forms an H_k (resp. Q_k) in π. In the first case, the obtained H_k (resp. Q_k) is above a_1, which contradicts the selection of a_1. In the second case, suppose that g_z(resp. h_z) is the first square containing a 1 of the obtained H_k (resp. Q_k) that is to the right of a_i. Clearly, g_z (resp. h_z) is above b_i and a_{i+1}. If g_z (resp. h_z) is to the left of a_{i+1}, then the obtained H_k (or Q_k) contradicts the selection of a_{i+1}. Otherwise, $a_2, a_3, \ldots, a_{i+1}$, combining with the $1's$ of the obtained H_k (resp. Q_k) that are to the right of a_i, would form an H_k (or Q_k) in π. This contradicts the selection of a_1, which completes the proof. ∎

3.2 The map Ψ from the set $\mathcal{S}_n(H_k, Q_k)$ to the set $\mathcal{S}_n(F_k)$

Before we describe the map Ψ we define three transformations, which will play an essential role in the construction of the map Ψ.

Let $\sigma = \{(\sigma_1, 1), (\sigma_2, 2), \ldots, (\sigma_n, n)\}$. Suppose that G is the submatrix of σ at columns $c_1 < c_2 < \ldots < c_k < c_k + 1 < c_k + 2 < \ldots < t$ and rows $r_1 < r_2 < \ldots < r_{k-1} > r_k < \sigma_{c_k+1} < \sigma_{c_k+2} < \ldots < \sigma_t$, in which the squares (r_i, c_i) are filled with $1's$ for all $i = 1, 2, \ldots, k$. Let $\delta(G)$ be the submatrix at the same rows and columns as G, such that the squares (r_k, c_1), (r_1, c_2), $\ldots$, (r_{k-2}, c_{k-1}), (σ_{c_k+1}, c_k), $(\sigma_{c_k+2}, c_k + 1)$, $\ldots$, $(\sigma_t, t-1)$, (r_{k-1}, t) are filled with $1's$ and all the other squares are filled with $0's$.

Suppose that H is the submatrix of σ at columns $c_1 < c_2 < \ldots < c_{k-1} < t < t+1 < \ldots < c_k$ and rows $r_1 < r_2 < \ldots < r_{k-1} > \sigma_t > \sigma_{t+1} > \ldots > \sigma_{c_k} = r_k$, in which the squares (r_i, c_i) are filled with $1's$ for all $i = 1, 2, \ldots, k$. Define $\gamma(H)$ to be the submatrix at the same columns and rows as H, such that the squares (r_k, c_1), (r_1, c_2), $\ldots$, (r_{k-2}, c_{k-1}), (r_{k-1}, t), $(\sigma_t, t+1)$, $(\sigma_{t+1}, t+2)$, $\ldots$, (σ_{c_k-1}, c_k) are filled with $1's$ and all the other squares are filled with $0's$.

The transformation ψ: Suppose that $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ is a permutation in $\mathcal{S}_n$. First, find the lowest square (p_k, q_k) containing a 1, such that there is an F_k in σ in which the 1 positioned at (p_k, q_k) is its rightmost 1. Then, find the lowest square (p_{k-1}, q_{k-1}) containing a 1, such that there is an F_k in σ in which the $1'$ positioned at (p_k, q_k) and (p_{k-1}, q_{k-1}) are the rightmost two $1's$. Find $(p_{k-2}, q_{k-2}), (p_{k-3}, q_{k-3}), \ldots, (p_1, q_1)$ one by one as (p_{k-1}, q_{k-1}). Assume that there is no H_k or Q_k above row p_k in σ.

If $\sigma_{q_k-1} > \sigma_{q_k+1}$, then we wish to generate a permutation π from σ by the considering the following two cases.

Case 1. $p_{k-1} = \sigma_{q_{k-1}} < \sigma_{q_{k-1}+1} < \ldots < \sigma_{q_k-1} > \sigma_{q_k} = p_k$ and $\sigma_{q_k+1} > p_{k-2}$. In this case, let G be the submatrix of σ at columns $q_1 < q_2 < \ldots < q_{k-2} < q_k$ and rows $p_k < p_1 < p_2 < \ldots < p_{k-2}$. Replace G by $\theta^{-1}(G)$ and leave all the other rows and columns fixed.

Case 2. Otherwise, find the least t such that $t > q_k$ and $t \in \mathcal{D}(\sigma)$. If such t does not exist, set $t = n$. In this case, let G be the submatrix of σ at columns $q_1 < q_2 < \ldots < q_k < q_k + 1 < \ldots < t$ and rows $p_1 < p_2 < \ldots p_{k-1} > p_k < \sigma_{q_k+1} < \sigma_{q_k+2} < \ldots < \sigma_t$. Replace G by $\delta(G)$ the other rows and columns fixed.

If $q_k = n$ or $\sigma_{q_k-1} < \sigma_{q_k+1}$, then we wish to generate a permutation π from σ by considering the following two cases.

Case 3. If there exists an s such that $q_{k-1} < s < q_k$ and $\sigma_{s-1} > \sigma_s < \sigma_{s+1}$. Find the largest t such that $q_{k-1} < t \leqslant q_k$ and $t - 1 \in \mathcal{A}(\sigma)$. Let G be the submatrix of σ at columns $q_1 < q_2 < \ldots < q_{k-1} < t < t+1 < \ldots < q_k$ and rows $p_1 < p_2 < \ldots < p_{k-1} > \sigma_t > \sigma_{t+1} > \ldots > p_k$. Replace G with $\gamma(G)$ and leave all the other rows and columns fixed.

Case 4. Otherwise, we have $p_{k-1} = \sigma_{q_{k-1}} < \sigma_{q_{k-1}+1} < \ldots < \sigma_{t-1} > \sigma_t > \sigma_{t+1} > \ldots > \sigma_{q_k} = p_k$ for some t with $q_{k-1} < t \leqslant q_k$. Let G be the submatrix of σ at columns $q_1 < q_2 < \ldots < q_{k-2} < q_{k-1} < t < t+1 < \ldots < q_k$ and rows $p_1 < p_2 < \ldots < p_{k-2} < p_{k-1} > \sigma_t > \sigma_{t+1} > \ldots > p_k$. Replace G with $\gamma(G)$ and leave all the other rows and columns fixed.

Remark 3.10. *In Case* 2*, the selection of* (p_k, q_k) *ensures that* $p_k < \sigma_{q_k+1}$*. If not, the the* $1's$ *positioned at* $(p_1, q_1), (p_2, q_2), \ldots, (p_{k-1}, q_{k-1}), (\sigma_{q_k+1}, q_k + 1)$ *would form an* F_k*, which contradicts the selection of* (p_k, q_k)*. In Case* 3*, the existence of such* s *and the hypothesis that there is no* H_k *above row* p_k *ensure that* $p_{k-1} > \sigma_t$*. If not, then the* $1's$ *positioned at* $(p_2, q_2), (p_3, q_3), \ldots, (p_{k-1}, q_{k-1}), (\sigma_t, t)$ *would form an* H_k *above row* p_k *in* σ*. In Case* 4*, the hypothesis that there is no* Q_k *above row* p_k *ensures that* $p_{k-1} > \sigma_t$*. If not, then the* $1's$ *positioned at* $(p_2, q_2), (p_3, q_3), \ldots, (p_{k-1}, q_{k-1}), (\sigma_{t-1}, t-1), (\sigma_t, t)$ *would form a* Q_k *above row* p_k *in* σ*.*

Remark 3.11. *We denote the resulting permutation in Case* 1*, Case* 2*, Case* 3 *and Case* 4 *by* $\psi_1(\sigma)$*,* $\psi_2(\sigma)$*,* $\psi_3(\sigma)$ *and* $\psi_4(\sigma)$*, respectively.*

It is obvious that the transformation ψ changes every occurrence of F_k to an occurrence of H_k (or Q_k). Denote by Ψ the iterated transformation, that recursively transforms every occurrence of F_k into H_k (or Q_k).

Using the notation of the algorithm for ψ_1, we label the squares containing $1's$ in G by $b_1, b_2, \ldots, b_k$, and the squares containing $1's$ in $\theta^{-1}(G)$ by $a_1, a_2, \ldots, a_{k-1}, a_k$, from left to right, see Figure 4 for example.

Using the notation of the algorithm for ψ_2, we label the squares containing $1's$ in G by $b_1, b_2, \ldots, b_{k-1}, b_k, d_1, d_2, \ldots, d_{t-q_k}$, and the squares containing $1's$ in $\delta(G)$ by $a_1, a_2, \ldots, a_{k-1}, c_1, c_2, \ldots, c_{t-q_k}, a_k$, from left to right, see Figure 2 for example.

Using the notation of the algorithm for ψ_3, we label the squares containing $1's$ in G by $b_1, b_2, \ldots, b_{k-1}, f_1, f_2, \ldots, f_{q_k-t}, b_k$, and the squares containing $1's$ in $\gamma(G)$ by $a_1, a_2, \ldots, a_{k-1}, a_k, e_1, e_2, \ldots, e_{q_k-t}$, from left to right. We also label the minimum entry of the block to which f_1 belongs by c_1, see Figure 3 for example.

Using the notation of the algorithm for ψ_4, we label the squares containing $1's$ in G by $b_1, b_2, \ldots, b_{k-1}, f_1, f_2, \ldots, f_{q_k-t}, b_k$, and the squares containing $1's$ in $\gamma(G)$ by $a_1, a_2, \ldots, a_{k-1}, a_k, e_1, e_2, \ldots, e_{q_k-t}$, from left to right, see Figure 5 for example.

In $\psi_2(\sigma)$ and $\psi_3(\sigma)$, let E_1 be the same board defined in $\phi_1(\pi)$ and $\phi_2(\pi)$. Similarly, in $\psi_1(\sigma)$ and $\psi_4(\sigma)$, let E_2 be the same board defined in $\phi_3(\pi)$ and $\phi_4(\pi)$. From the selection of $b_i's$ and the hypothesis that there is no H_k above a_1, it follows that there are no $1's$ inside E_1 (or E_2). In other words, all the $1's$ are to the left or to the right of E_1 (or E_2) in $\psi(\pi)$.

Now we proceed to prove that the transformation ψ have the following properties, which are essential in the proof of Theorem 1.5.

Lemma 3.12. *If there is no H_k or Q_k above a_1 in σ, then we have $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$.*

Proof. Since there are no $1's$ inside E_1 (or E_2) and no H_k or Q_k above a_1, one can easily verify that $\mathcal{D}(\pi) = \mathcal{D}(\psi(\sigma))$. The details are omitted here. ∎

Properties

(1′) For any $1 \leqslant i < j \leqslant k-1$, the rectangle with corners a_i and a_j cannot contain a J_{j-i} with all its $1's$ strictly to the right of E_1 (or E_2) in $\psi(\sigma)$.

(2′) For any $1 \leqslant i \leqslant k-2$, the rectangle with corners a_i and a_k cannot contain a J_{k-i-1} with all its $1's$ strictly to the right of E_2 in $\psi_1(\sigma)$ (or $\psi_4(\sigma)$).

Proof.

(1′) If there is a J_{j-i} in this region, then $b_1, b_2, \ldots, b_{i-1}$, combining with this J_{j-i} and $b_j, b_{j+1}, b_{j+2}, \ldots, b_k$, will form an F_k in σ, which contradicts the selection of b_{j-1}.

(2′) If there is a J_{k-1-i} in this region, then the rightmost 1 of this J_{k-1-i} is to the left of b_{k-1} since all the $1's$ lying between b_{k-1} and a_k are to the left of E_2. Clearly, the rightmost 1 of this J_{k-1-i} is below b_{k-2}. So $b_1, b_2, \ldots, b_{i-1}$, combining with this J_{k-1-i} and b_{k-1}, b_k, will form an F_k in σ, which contradicts the selection of b_{k-2}.

Lemma 3.13. *$\psi(\sigma)$ contains no F_k with at least one square in a row below a_1.*

Proof. If not, suppose H is such an F_k in $\psi(\sigma)$. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_k$ from left to right. Then h_k is below a_1. As in the proof of Lemma 3.5, we shall replace some $1's$ of H (except h_k) to form an F_k in π, which contradicts the selection of b_k.

By the selection of b_k, we have that h_k must be at the left side of b_{k-1}. From the construction of $\psi(\sigma)$, at least one of $a_1, a_2, \ldots, a_{k-2}$ must fall in H. Otherwise, H is also an F_k in σ, which contradicts the selection of b_k.

Find the least i such that a_i falls in H.

Vertical slide algorithm for ψ: If there is a 1 of H which is above a_i and to the left of E, find the leftmost square containing such a 1 and denote it by h_y. find x such that h_y is to the right of a_x and to the left of a_{x+1}. Then by property $(1')$, there are at most $x+1-i$ $1's$ in H that are below a_{x+1} but not below a_i, and to the right of E_1 (or E_2). So we can replace these $1's$ by $a_i, a_{i+1}, \ldots, a_x$, and hence by those positioned at $b_i, b_{i+1}, \ldots, b_x$.

We can repeat the vertical slide algorithm until one of the following two cases appears.

(1) There is no a_i that falls in H. This ends the proof.

(2) There is such an a_i, but h_y does not exist. Then suppose a_v is the first square to the right of h_k. By property $(1')$, there are at most $v-i$ $1's$ in H that are below and to the left of a_v, but not below a_i, and to the right of E_1 (or E_2). So we can replace these $1's$ by $a_i, a_{i+1}, \ldots, a_{v-1}$, and hence by $b_i, b_{i+1}, \ldots, b_{v-1}$. Then we have an F_k in σ with a square h_k below a_1. ∎

Lemma 3.14. *If σ contains no H_k or Q_k that is above a_1, neither does $\psi(\sigma)$.*

In order to prove Lemma 3.14, we need the following two lemmas.

Lemma 3.15. *Suppose that G is an H_k above a_1 in $\psi(\sigma)$. Label the squares containing the $1's$ of G by $g_1, g_2, \ldots, g_k$, from left to right. If σ contains no H_k or Q_k that is above a_1, then the squares g_k and g_{k-1} are also filled with $1's$ in σ.*

Lemma 3.16. *Suppose that H is a Q_k above a_1 in $\psi(\sigma)$, in which the last two $1's$ lie in two consecutive columns. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_k$, from left to right. If σ contains no H_k or Q_k that is above a_1, then the squares h_k and h_{k-1} are also filled with $1's$ in σ.*

Before we prove Lemmas 3.15 and 3.16, we introduce the following *horizontal slide algorithm* for ψ.

Suppose H is a J_k in $\psi(\sigma)$. Label the squares containing the $1's$ of H by $h_1, h_2, \ldots, h_k$ from left to right.

Horizontal slide algorithm for ψ_2 (or ψ_3): Find the largest i such that a_i falls in H with $i \leqslant k-1$. If there is a 1 of H which is below a_i to the left of E_1, find the rightmost squares containing such a 1 and denote it by h_y. Find x such that h_y is below a_x, and above a_{x-1}. Then by property $(1')$, there are $i-x+1$ $1's$ in H that are above a_{x-1} but not above a_i, and to the right of E_1. So we can replace these $1's$ by $a_x, a_{x+1}, \ldots, a_i$, and hence by $b_{x-1}, b_x, \ldots, b_{i-1}$.

We can repeat this horizontal slide algorithm until one of the following two cases appears.

(1) There is no a_i that falls in H.

(2) There is such an a_i, but h_y does not exist. Find x such that h_1 is below a_{x+1} and above a_x. Then by property $(1')$, there are $i-x$ $1's$ in H that are above a_x but not above a_i. So we can replace these $1's$ by $a_{x+1}, a_{x+2}, \ldots, a_i$, and hence by $b_x, b_{x+1}, \ldots, b_{i-1}$.

Horizontal slide algorithm for ψ_1 (or ψ_4). Find the largest i such that a_i falls in H with $i \leqslant k-2$ or $i = k$. If there is a 1 of H which is below a_i to the left of E_2, find the rightmost square containing such a 1 and denote it by h_y. Find x such that h_y is below a_x, and above a_{x-1}. If $i \leqslant k-2$, then by property $(1')$, there are $i-x+1$ $1's$ in H that are above a_{x-1} but not above a_i, and to the right of E_2. So we can replace these $1's$ by those positioned at $a_x, a_{x+1}, \ldots, a_i$, and hence by $b_{x-1}, b_x, \ldots, b_{i-1}$. If $i = k$, then by property $(2')$, there are $k-x$ $1's$ in H that are above a_{x-1} but not above a_k, and to the right of E_2. So we can replace these $1's$ by $a_x, a_{x+1}, \ldots, a_{k-2}, a_k$, and hence by $b_{x-1}, b_x, \ldots, b_{k-2}$.

We can repeat this horizontal slide algorithm until one of the following two cases appears.

(1) There is no a_i that falls in H.

(2) There is such an a_i, but h_y does not exist. Find x such that h_1 is below a_{x+1} and above a_x. If $i < k-2$, then by property $(1')$, there are $i-x$ $1's$ in H that are above a_x but not above a_i. So we can replace these $1's$ by $a_{x+1}, a_{x+2}, \ldots, a_i$, and hence by $b_x, b_{x+1}, \ldots, b_{i-1}$. If $i = k$, then by property $(2')$, there are $k-1-x$ $1's$ in H that are above a_x but not above a_k. So we can replace these $1's$ by $a_{x+1}, a_{x+2}, \ldots, a_{k-2}, a_k$, and hence by $b_x, b_{x+1}, \ldots, b_{k-2}$.

The proof of Lemma 3.15. Here we only prove the assertion for $\psi_2(\sigma)$ and $\psi_4(\sigma)$. The other cases can be verified by similar arguments. In order to prove the assertion, it suffices to show that neither g_k nor g_{k-1} will be any of the $a_i's$ and $c_i's$ in $\psi_2(\sigma)$, and be any of the $a_i's$ for $i = 1, 2, \ldots, k-2, k$ and $e_i's$ in $\psi_4(\sigma)$.

We claim there is no J_{k-1} which is below b_{k-1} but above a_1, and not to the right of b_k in $\psi_2(\sigma)$ (or $\psi_4(\sigma)$). If not, suppose that R is such a J_{k-1}. When J_{k-1} is in $\psi_2(G)$, we can get a J_{k-1} from R by repeating the horizontal slide algorithm for ψ_2. When J_{k-1} is in $\psi_4(G)$, we can get a J_{k-1} from R by repeating the horizontal slide algorithm for ψ_4 and replacing any e_i by f_i whenever e_i fall in R. In both cases, the obtained J_{k-1} is below b_{k-1} but above a_1, and to the left of b_k. Then Then this J_{k-1} combining with b_k will form an F_k in σ, which contradicts the selection of b_{k-1}. Hence, the claim is proved.

From the claim, it follows that neither g_k nor g_{k-1} will be any of the $a_i's$ in $\psi_2(\sigma)$ for $i \leqslant k-1$. In order to prove the assertion for $\psi_2(\sigma)$, it remains to show that neither g_k or g_{k-1} will be any of a_k and $c_i's$ in $\psi_2(\sigma)$. Clearly, g_{k-1} cannot be a_k since there is no $1's$ above and to the right of a_k.

(1) If g_k is either a_k or one of $c_i's$ in $\psi_2(\sigma)$, then $g_1, g_2, \ldots, g_{k-1}$ form a J_{k-1} which is to the left of b_k and below b_{k-1} in $\phi_2(\sigma)$ since $c_1, c_2, \ldots, c_{t-q_k}, a_k$ lie in consecutive columns and form a J_{t-q_k+1}. This contradicts the claim proved above.

(2) If g_{k-1} is one of $c_i's$ in $\psi_2(\sigma)$, then g_k is to the right of a_k since $c_1, c_2, \ldots, c_{t-q_k}$, a_k lie in consecutive columns and form a J_{t-q_k+1}. By repeating the horizontal slide algorithm for ψ_2 and replacing any c_i falling in G by d_i, we can get a J_k above a_1 in σ from G. Notice that if $g_{k-1} = c_j$, then the rightmost two $1's$ of the obtained J_k are g_k and d_j. Recall that a_k is positioned at column t. Since σ contains no H_k or Q_k

that is above a_1, we have $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$ by Lemma 3.12. The fact that $t \in \mathcal{D}(\sigma)$ ensures that $t \in \mathcal{D}(\psi(\sigma))$. Thus, the obtained J_k is an H_k. This contradicts the fact that there contains no H_k above a_1 in σ.

Hence, we have concluded that the assertion holds for $\psi_2(\sigma)$.

From the claim that is no J_{k-1} which is below b_{k-1} but above a_1, and not to the right of b_k in $\psi_4(\sigma)$, it follows that neither g_k nor g_{k-1} will be any of the $a_i's$ for $i = 1, 2, \ldots, k-2, k$ and $e_i's$ in $\psi_1(\sigma)$. Hence, we deduce that the assertion also holds for $\psi_4(\sigma)$, which completes the proof. ∎

The proof of Lemma 3.16. Here we only prove the assertion for $\psi_2(\sigma)$ and $\psi_4(\sigma)$. The other cases can be verified by similar arguments. In order to prove the assertion, it suffices to show that neither h_k nor h_{k-1} will be any of the $a_i's$ and $c_i's$ in $\psi_2(\sigma)$, and be any of the $a_i's$ for $i = 1, 2, \ldots, k-2, k$ and $e_i's$ in $\psi_4(\sigma)$.

Recall that we have proved the claim in the proof of Lemma 3.15 that there is no J_{k-1} which is below b_{k-1} but above a_1, and not to the right of b_k in $\psi_4(\sigma)$. It follows that neither h_k nor h_{k-1} will be any of the $a_i's$ for $i = 1, 2, \ldots, k-2, k$ and $e_i's$ in $\psi_4(\sigma)$. Thus, the assertion holds for $\psi_4(\sigma)$.

Similarly, from the claim proved in the proof of Lemma 3.15, it follows that neither h_k nor h_{k-1} will be any of the $a_i's$ for $i = 1, 2, \ldots, k-1$ in $\psi_2(\sigma)$. In order to prove the assertion for $\psi_2(\sigma)$, it remains to verify that neither h_k nor h_{k-1} will be any of a_k and $c_i's$ in $\psi_2(\sigma)$. Recall that $c_1, c_2, \ldots, c_{q_k-t}, a_k$ lie in consecutive columns and form a J_{q_k-t+1} in $\psi_1(\sigma)$. It implies that if h_k or h_{k-1} is one of $c_1, c_2, \ldots, c_{q_k-t}, a_k$, then we have either $h_{k-1} = a_k$ or $h_k = c_1$.

In the former case, we can get a J_{k-1} above a_1 in σ from the J_{k-1} consisting of $h_1, h_2, \ldots, h_{k-2}, h_k$, by repeating the horizontal slide algorithm for ψ_2 and replacing any c_i by d_i. Since σ contains no H_k or Q_k above a_1, we have $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$ by Lemma 3.12. Recall that a_k is above h_k. From the equality $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$, it follows that d_{t-q_k} is above h_k. Notice that the rightmost 1 of the obtained J_{k-1} is h_k. Thus, this J_{k-1}, combining with d_{t-q_k}, will form a Q_k above a_1 in σ, which contradicts the hypothesis that σ contains no Q_k above a_1.

In the latter case, h_{k-2} is not above a_{k-1} since c_1 is below a_k (and b_{k-1}) and there is no $1's$ inside E_1. If h_{k-2} is to the left of a_{k-1} (and b_{k-1}), then by repeating the horizontal slide algorithm, we can obtain a J_{k-2} above a_1 in σ from the J_{k-2} consisting of $h_1, h_2, \ldots, h_{k-2}$. Notice that the rightmost 1 of the resulting J_{k-2} is below b_{k-2} and to the left of b_{k-1}. Then, this J_{k-2}, combining with b_{k-1} and b_k, will form an F_k in σ. This contradicts the selection of b_{k-2}.

Now suppose that h_{k-2} is either equal to a_{k-1} or is at the right of a_{k-1} (and b_{k-1}), then by the claim obtained in the proof of Lemma 3.15, h_{k-1} is above b_{k-1} and to the left of E_1. If $h_{k-2} \neq a_{k-1}$, then $b_1, b_2, \ldots, b_{k-1}, h_{k-1}$ form an H_k above a_1 in σ, which contradicts the hypothesis. If $h_{k-2} = a_{k-1}$, then we have c_1 is above b_{k-2} (and a_{k-1}). Thus, according to the definition of ψ_2, there must exists s such that $s \in \mathcal{D}(\sigma)$ and $q_{k-1} \leqslant s < q_k - 1$. Recall that c_1 and h_{k-1} lie in columns q_k and q_{k-1}, respectively. From the equality $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$, it follows that $b_1, b_2, \ldots, b_{k-1}, h_{k-1}$ form an H_k above a_1

in σ, which contradicts the hypothesis. Hence, we have concluded that the assertion also holds for $\psi_2(\sigma)$. ∎

The proof of Lemma 3.14. If not, suppose that G is an H_k above a_1 in $\psi(\sigma)$. Label the $1's$ in G by $g_1, g_2, \ldots, g_k$, from left to right. Moreover, let H be a Q_k above a_1 in $\psi(\sigma)$ such that the rightmost two $1's$ lie in two consecutive columns. We label its $1's$ in H by $h_1, h_2, \ldots, h_k$. According to the definition of Q_k, there is a Q_k above a_1 in $\psi(\sigma)$ if and only if there exists such an H.

We wish to replace some $1's$ of G (resp. H) to form an H_k (resp. Q_k) above a_1 in σ, which contradicts the hypothesis that there is no H_k (resp. Q_k) above a_1 in σ. Here we only consider the case when G (resp. H) is in $\psi_2(\sigma)$. The other cases can be verified by the similar arguments. Since the map ψ_2 does not change the positions of any other $1's$, one of $a_i's$ and $c_i's$ must fall in G (resp. H).

We can get a J_{k-2} above a_1 in σ from the J_{k-2} consisting of $g_1, g_2, \ldots, g_{k-2}$ (resp, $h_1, h_2, \ldots, h_{k-2}$), by repeating the horizontal slide algorithm and replacing each c_j by d_j whenever c_j falls in G (resp. H). From Lemmas 3.15 and 3.16, it follows that the squares g_k and g_{k-1} (resp. h_k and h_{k-1}) are also filled with $1's$ in σ. Hence, the obtained J_{k-2} combining with g_k and g_{k-1} (resp. h_k and h_{k-1}) will form a J_k (resp, G_k) in σ. Since h_{k-1} and h_k lie in two consecutive columns, the obtained G_k is a Q_k. Recall that $\mathcal{D}(\pi) = \mathcal{D}(\psi(\sigma))$ and the $1's$ positioned at g_{k-1} and g_k belong to two different blocks of $\psi(\sigma)$. This yields that the $1's$ positioned at g_{k-1} and g_k also belong to two different blocks of π. Thus, the obtained J_k is an H_k. This completes the proof. ∎

Lemma 3.17. *If σ contains no H_k or Q_k that is above a_1, then*

(1) there exists no 1 that is above and to the left of a_k such that this 1, combining with $a_1, a_2, \ldots, a_{k-1}$, forms an H_k in $\psi(\sigma)$;

(2) there exists no 1 that is to the left of a_k in $\psi_1(\sigma)$ (or $\psi_4(\sigma)$), such that this 1, combining with $a_1, a_2, \ldots, a_{k-1}$, forms a Q_k in $\psi_1(\sigma)$ (or $\psi_4(\sigma)$);

(3) for $1 \leqslant t \leqslant k-2$, the board that is above and to the right of a_t cannot contain an H_{k-t} or Q_{k-t} in $\psi(\sigma)$ such that the lowest 1 of this H_{k-t} or Q_{k-t} is to the left of a_{t+1}, and this H_{k-t} or Q_{k-t} , combining with $a_1, a_2, \ldots, a_t$, forms an H_k or Q_k in $\psi(\sigma)$.

Proof. (1) Since σ contains no H_k or Q_k that is above a_1, we have $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$ by Lemma 3.12. If there is such a 1, then this 1, combining with $b_1, b_2, \ldots, b_{k-1}$, forms an H_k in σ since $\mathcal{D}(\sigma) = \mathcal{D}(\psi(\sigma))$. This contradicts the hypothesis that there is no H_k above a_1 in σ.

(2) The result follows immediately from the fact that there is no $1's$ below and to the right of b_{k-1} (and a_{k-1}), and the left of a_k.

(3) If not, suppose that G is such an H_{k-t} (or Q_{k-t}) in $\psi(T)$. Label its $1's$ by $g_{t+1}, g_{t+2}, \ldots, g_k$, from left to right. By hypothesis, g_{t+1} is to the left of a_{t+1}. By the same reasoning as in the proof of Lemmas 3.15 and 3.16, one can verify that both the

squares g_k and g_{k-1} are also filled with $1's$ in σ. This ensures that by repeating the horizontal slide algorithm and replacing each c_j (resp. e_j) by d_j (resp. f_j) in $\psi_2(\sigma)$ (resp. $\psi_3(\pi)$ and $\psi_4(\pi)$), we can get an H_{k-t} (or Q_{k-t}) in σ, in which g_{t+1} is leftmost 1. This H_{k-t} (or Q_{k-t}), combining with $b_1, b_2, \ldots, b_t$, forms an H_k (or Q_k) in σ, which is above a_1. This contradicts the hypothesis that there is no H_k or Q_k above a_1. This completes the proof. ∎

3.3 Correctness of the bijection

First, we aim to show that the map Φ is well defined, that is, after finitely many iterations of ϕ, there will be no occurrences of H_k or Q_k. Suppose that we start with some $\tau \in \mathcal{S}_n(F_k)$. At the tth application of ϕ we select a copy of H_k (or Q_k) in $\phi^{t-1}(\tau)$. This has its lowest 1 in some row r. By Lemma 3.7, the H_k (or Q_k) we will select in $\phi^t(\tau)$ cannot have its lowest 1 anywhere above row r. If it is in row r, then we know it is further to the right than at the previous iteration, because there is only one 1 in that row, and we have just moved it to the right,from a_1 to b_k. It follows that each iteration the selection of a_1 can only go down or slide right, and therefore the map Φ is well defined.

Next we aim to show that $\mathcal{D}(\tau) = \mathcal{D}(\Phi(\tau))$. We prove by induction on t. Suppose that for any $j < t$, we have $\mathcal{D}(\phi^{j-1}(\tau)) = \mathcal{D}(\phi^j(\tau))$. We wish to show that $\mathcal{D}(\phi^{t-1}(\tau)) = \mathcal{D}(\phi^t(\tau))$. At the tth application of ϕ we select a copy of H_k (or Q_k) in $\phi^{t-1}(\tau)$. This has its lowest 1 in some row a_1. Recall that we have shown that each iteration the selection of lowest square of the selected H_k (or Q_k) can only go down or slide right. By Lemma 3.5, there is no F_k with at least one square below a_1 in $\phi^{t-1}(\tau)$. From Lemma 3.4, it follow that $\mathcal{D}(\phi^{t-1}(\tau)) = \mathcal{D}(\phi^t(\tau))$.

Now we proceed to show that the map Ψ is the inverse of the map Φ. To this end, it suffices to show that $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$. For our convenience, let $\pi = \phi^{t-1}(\tau)$ and $\sigma = \phi^t(\tau)$. Suppose that at the tth application of ϕ we select a copy of H_k (or Q_k) in π, in which the $1's$ are positioned in the squares $(p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)$, from left to right. We have four cases.

Case 1. The selected $1's$ form a copy of H_k, and $\pi_{q_k-1} > \pi_{q_k+1}$ or $q_k = n$. In this case, find the largest s such that $q_{k-1} < s < q_k$ and $s-1 \in \mathcal{D}(\pi)$. By the construction of the transformation ϕ, the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-1}), (p_1, s), (\pi_s, s+1), \ldots, (\pi_{q_k-1}, q_k)$ are filled with $1's$ in σ, and all the other rows and columns are the same as π. Note that the $1's$ positioned at the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-1}), (p_1, s)$ form an F_k in σ. Lemmas 3.5 and 3.6 ensure that when we apply the map ψ to σ, the squares we selected are just $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-1}), (p_1, s)$. By Lemma 3.7, there is no H_k or Q_k above row p_1. This implies that $\psi(\sigma)$ is well defined. Suppose that $\sigma = \{(\sigma_1, 1), (\sigma_2, 2), \ldots, (\sigma_n, n)\}$. Clearly, we have $\sigma_{q_i} = p_{i+1}$ for $i = 1, 2, \ldots, k-1$, $\sigma_s = p_1$ and $\sigma_j = \pi_{j-1}$ for $j = s+1, s+2, \ldots, q_k$.

We claim that $\sigma_{s-1} > \sigma_{s+1}$. If $s-1 \neq q_{k-1}$, then we have $\sigma_{s-1} = \pi_{s-1}$. Since $s-1 \in \mathcal{D}(\pi)$, we have $\pi_{s-1} > \pi_s$. In this case, we have $\sigma_{s-1} = \pi_{s-1} > \pi_s = \sigma_{s+1}$. If $s-1 = q_{k-1}$, then we have $\sigma_{s-1} = p_k$. Recall that we have $\pi_s < \pi_{s+1} < \ldots < p_k$. This implies that $\sigma_{s-1} = p_k > \pi_s = \sigma_{s+1}$. Hence, we have concluded that $\sigma_{s-1} > \sigma_{s+1}$.

We claim that if $\sigma_{q_{k-1}} < \sigma_{q_{k-1}+1} < \ldots < \sigma_{s-1} > \sigma_s$, then we have $\sigma_{q_{k-2}} > \sigma_{s+1}$. If not, since $\sigma_{q_{k-2}} = p_{k-1} = \pi_{q_{k-1}}$ and $s-1 \in \mathcal{D}(\pi)$, we have $s-1 \neq q_{k-1}$. Then the $1's$ positioned at the squares $(p_2, q_2), (p_3, q_3), \ldots, (p_{k-1}, q_{k-1}), (\pi_{s-1}, s-1), (\pi_s, s)$ will form a Q_k above row p_1 in π, which contradicts the selection of (p_1, q_1). Hence the claim is proved.

Then, according to the definition of map ψ, we have $\psi(\sigma) = \psi_2(\sigma)$. Since we have $\mathcal{D}(\pi) = \mathcal{D}(\sigma)$ and $\pi_s < \pi_{s+1} < \ldots < p_k$, we have $\sigma_s < \sigma_{s+1} < \ldots < \sigma_{q_k}$. Recall that there are no $1's$ inside E_1, we have either $q_k = n$ or $q_k \in \mathcal{D}(\pi)$. This yields that we have either $q_k = n$ or $q_k \in \mathcal{D}(\phi(\pi))$ Hence, when we apply the the map ψ_2 to σ, q_k is the largest integer m such that $m > s$ and $m \in \mathcal{D}(\sigma)$ or $m = n$. Thus, it is easily seen that $\psi_2(\sigma) = \pi$, that is, $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$.

Case 2. The selected $1's$ form a copy of H_k and $\pi_{q_k-1} < \pi_{q_k+1}$. In this case, find the least t such that $t > q_k$ and $t \in \mathcal{A}(\pi)$ or $t = n$. By the construction of the map ϕ, the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-1}), (\pi_{q_k+1}, q_k), (\pi_{q_k+2}, q_k+1), \ldots, (\pi_t, t-1)(p_1, t)$ are filled with $1's$ in σ, and all the other rows and columns are the same as π. Note that the $1's$ positioned at the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-1}), (p_1, t)$ form an F_k in σ. Lemmas 3.5 and 3.6 ensure that when we apply the map ψ to σ, the squares we selected are just $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-1}), (p_1, t)$. By Lemma 3.7, there is no H_k or Q_k above row p_1. This implies that $\psi(\sigma)$ is well defined. Clearly, we have $\sigma_{q_i} = p_{i+1}$ for $i = 1, 2, \ldots, k-1$, $\sigma_t = p_1$ and $\sigma_j = \pi_{j+1}$ for $j = q_k, q_k+1, \ldots, t-1$.

Since $t \in \mathcal{A}(\pi)$ or $t = n$, we have $\pi_t < \pi_{t+1}$ or $t = n$. This implies that $\sigma_{t-1} = \pi_t < \pi_{t+1} = \sigma_{t+1}$ or $t = n$. By Remark 3.1, we see that there exits an s such that $s-1 \in \mathcal{D}(\pi)$ and $q_{k-1} < s < q_k$. This implies that $\pi_{s-1} > \pi_s < \pi_{s+1}$. Since $\mathcal{D}(\pi) = \mathcal{D}(\sigma)$, we have $\sigma_{s-1} > \sigma_s < \sigma_{s+1}$ and $p_{k-1} < s < q_k < t$. Then, according to the definition of map ψ, we have $\psi(\sigma) = \psi_3(\sigma)$. When we apply the the map ψ_3 to σ, since we have $\mathcal{D}(\pi) = \mathcal{D}(\sigma)$ and $\pi_{q_k-1} < \pi_{q_k} > \pi_{q_k+1} > \ldots > \pi_t$, q_k is the largest integer m such that $m-1 \in \mathcal{A}(\sigma)$ and $q_{k-1} < m \leqslant t$. Thus, it is easily seen that $\psi_3(\sigma) = \pi$, that is, $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$.

Case 3. The selected $1's$ form a copy of Q_k and $q_k \in \mathcal{A}(\pi)$. By the construction of the map ϕ, the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-2}), (p_1, q_k)$ are filled with $1's$ in σ, and all the other rows and columns are the same as π. Note that the $1's$ positioned at the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-2}), (p_1, q_k)$ form an F_k in σ. Lemmas 3.5 and 3.6 ensure that when we apply the map ψ to σ, the squares we selected are just $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-2}), (p_1, q_k)$. By Lemma 3.7, there is no H_k or Q_k above row p_1. This implies that $\psi(\sigma)$ is well defined. Clearly, we have $\sigma_{q_i} = p_{i+1}$ for $i = 1, 2, \ldots, k-3$, $\sigma_{q_{k-2}} = p_k$ and $\sigma_{q_k} = p_1$.

According to the definition of Q_k, we have $\pi_{q_{k-1}} < \pi_{q_{k-1}+1} < \ldots < \pi_{q_k-1} > \pi_{q_k}$. Moreover, we have $\sigma_j = \pi_j$ for $j = q_{k-1}, q_{k-1}+1, \ldots, q_k-1$. Thus, we have $\sigma_{q_{k-1}} < \sigma_{q_{k-1}+1} < \ldots < \sigma_{q_k-1} > p_1 = \sigma_{q_k}$ and $\sigma_{q_{k-2}} = p_k = \pi_{q_k} < \pi_{q_k+1} = \sigma_{q_k+1}$. Then, according to the definition of map ψ, we have $\psi(\sigma) = \psi_1(\sigma)$. Thus, it is easily seen that $\psi_1(\sigma) = \pi$, that is, $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$.

Case 4. The selected $1's$ form a copy of Q_k, and $\pi_{q_k} > \pi_{q_k+1}$ or $q_k = n$. In this case, let t be the least such that $t > q_k$ and $t \in \mathcal{A}(\pi)$ or $t = n$. By the construction of the map ϕ, the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-2}), (\pi_{q_k+1}, q_k), (\pi_{q_k+2}, q_k+1), \ldots, (\pi_t, t-1)(p_1, t)$ are filled with $1's$ in σ, and all the other rows and columns are the same as π. Note that the $1's$ positioned at the squares $(p_2, q_1)(p_3, q_2), \ldots, (p_k, q_{k-2}), (p_1, t)$ form an F_k in σ. Lemmas 3.5 and 3.6 ensure that when we apply the map ψ to σ, the squares we selected are just $(q_2, p_1)(q_3, p_2), \ldots, (p_k, q_{k-1}), (p_1, t)$. By Lemma 3.7, there is no H_k or Q_k above row p_1. This implies that $\psi(\sigma)$ is well defined. Clearly, we have $\sigma_{q_i} = p_{i+1}$ for $i = 1, 2, \ldots, k-3$, $\sigma_{q_{k-2}} = p_k$, $\sigma_t = p_1$ and $\sigma_j = \pi_{j+1}$ for $j = q_k, q_k+1, \ldots, t-1$.

Since $t \in \mathcal{A}(\pi)$ or $t = n$, we have $\pi_t < \pi_{t+1}$ or $t = n$. This implies that $\sigma_{t-1} = \pi_t < \pi_{t+1} = \sigma_{t+1}$ or $t = n$. According to the definition of Q_k, we have $\pi_{q_{k-1}} < \pi_{q_{k-1}+1} < \ldots < \pi_{q_k-1} > \pi_{q_k}$. Thus, we have $\sigma_{q_{k-1}} < \sigma_{q_{k-1}+1} < \ldots < \sigma_{q_k-1} > \sigma_{q_k} > \sigma_{q_k+1} > \ldots > \sigma_t = p_1$. Then, according to the definition of map ψ, we have $\psi(\sigma) = \psi_4(\sigma)$. Thus, it is easily seen that $\psi_4(\sigma) = \pi$, that is, $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$.

So far, we have deduced that $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$.

Now we proceed to to show that the map Ψ is well defined, that is, after finitely many iterations of ψ, there will be no occurrences of F_k. Suppose that we start with some $\tau \in \mathcal{S}_n(H_k, Q_k)$. At the tth application of ψ we select a copy of F_k in $\psi^{t-1}(\tau)$. This has its lowest 1 in some row r. By Lemma 3.13, the F_k we will select in $\phi^t(\tau)$ cannot have its lowest 1 anywhere below row r. If it in row r, then we know it is further to the left than at the previous iteration, because there is only one 1 in that row, and we have just moved it to the left, from b_k to a_1. It follows that at each iteration the selection of b_k can only go up or slide left. Moreover, Lemma 3.14 implies that there is no H_k or Q_k above b_k. Therefore, after finitely many iterations of ψ, there will be no occurrences of F_k.

Next we aim to show that $\mathcal{D}(\tau) = \mathcal{D}(\Psi(\tau))$. We prove by induction on t. Suppose that for any $j < t$, we have $\mathcal{D}(\psi^{j-1}(\tau)) = \mathcal{D}(\psi^j(\tau))$. We wish to show that $\mathcal{D}(\psi^{t-1}(\tau)) = \mathcal{D}(\psi^t(\tau))$. At the tth application of ψ we select a copy of F_k in $\psi^{t-1}(\tau)$. This has its lowest 1 in some row b_k. Recall that we have shown that each iteration the selection of lowest square of the selected F_k can only go up or slide left. By Lemma 3.14, there is no H_k or Q_k above b_k in $\psi^{t-1}(\tau)$. Hence, from Lemma 3.12, it follow that $\mathcal{D}(\psi^{t-1}(\tau)) = \mathcal{D}(\psi^t(\tau))$.

By the same reasoning as in the proof of the equality $\psi(\phi^t(\tau)) = \phi^{t-1}(\tau)$, we can prove the equality $\phi(\psi^t(\tau)) = \psi^{t-1}(\tau)$ relying on Lemmas 3.14 and 3.17, and the equality $\mathcal{D}(\psi^{t-1}(\tau)) = \mathcal{D}(\psi^t(\tau))$. The details are omitted here.

So far, we have concluded that the maps Φ and Ψ are well defined and preserve the descent set. Moreover, the map Φ and Ψ are inverses of each other. Thus, the map Φ is the desired bijection between $\mathcal{S}_n(F_k)$ and $\mathcal{S}_n(H_k, Q_k)$ as claimed in Theorem 1.5.

To conclude this section, we remark that the method presented here seems not so attractive for the purpose of establishing Conjecture 1.1 for all $m > 1$. Our proof of Conjecture 1.1 for all $k \geqslant 1$ and $m = 1$ relies on the descent set preserving bijection f between the set $\mathcal{S}_n(G_k)$ and the set $\mathcal{S}_n(H_k, Q_k)$, and the descent set preserving bijection Φ between the set $\mathcal{S}_n(F_k)$ and the set $\mathcal{S}_n(H_k, Q_k)$. However, it is difficult to figure out whether there exist analogous descent set preserving bijections for the case when $m > 1$.

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