On Commuting Graphs for Elements of Order 3 in Symmetric Groups

Athirah Nawawi and Peter Rowley

School of Mathematics
University of Manchester
Manchester, M13 6PL, U.K.

peter.j.rowley@manchester.ac.uk

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Abstract

The commuting graph $\mathcal{C}(G, X)$, where G is a group and X is a subset of G, is the graph with vertex set X and distinct vertices being joined by an edge whenever they commute. Here the diameter of $\mathcal{C}(G, X)$ is studied when G is a symmetric group and X a conjugacy class of elements of order 3.

Keywords: Commuting graph, Symmetric group, Order 3 elements, Diameter

1 Introduction

Suppose that G is a finite group and X is a subset of G. The *commuting graph* $\mathcal{C}(G, X)$ is the graph with X as the vertex set and two distinct elements of X being joined by an edge if they are commuting elements of G. This type of graph has been studied for a wide variety of groups G and selection of subsets of G. One of the earliest investigations occurred in Brauer and Fowler [8] in which $X = G\backslash\{1\}$. This particular case has recently been the subject of further study by Segev [14], [15] and Segev and Seitz [16]. A great deal of attention has been focussed on the case when X is a conjugacy class of involutions – the so-called commuting involution graphs. Pioneering work on such graphs appeared in Fischer [13] which led to the construction of the three Fischer groups. Recently various properties of other commuting involution graphs have been studied; see, for example, [2], [3], [4], [5], [11] and [12]. When X is a conjugacy class of non-involutions, $\mathcal{C}(G, X)$ has to date received less attention. Never-the-less graphs of this type can be of interest – witness the computer-free uniqueness proof of the Lyon's simple group by Aschbacher and Segev [1] which employed a commuting graph whose vertex set consisted of the 3-central subgroups of order 3. Also see Baumeister and Stein [7], the results obtained there being used to describe the structure of Bruck loops and Bol loops of exponent 2. Further, commuting graphs when G is a symmetric group have been investigated in Bates, Bundy, Perkins and Rowley [6] and Bundy[9]. The former paper concentrates on the structure of discs (around some fixed vertex) and the diameter of the graph while the latter gives a complete answer as to when $\mathcal{C}(G, X)$ is a connected graph.

In the present paper we shall determine the diameters of $\mathcal{C}(G, X)$ when G is a symmetric group and X is a G-conjugacy class of elements of order 3. So for the rest of this paper we assume $G = Sym(\Omega) = Sym(n)$ with G acting upon the set $\Omega = \{1, \ldots, n\}$ in the usual manner. Also let

$$t = (1, 2, 3)(4, 5, 6)(7, 8, 9) \ldots (3r - 2, 3r - 1, 3r).$$

Thus t has order 3 and cycle type $1^{n-3r}3^r$. Set $X = t^G$, the G-conjugacy class of t, and let Diam $(\mathcal{C}(G, X))$ denote the diameter of the commuting graph $\mathcal{C}(G, X)$. Our main results are as follows.

Theorem 1.1. *If* $n \geqslant 8r$*, then* Diam $(\mathcal{C}(G, X)) = 2$.

Theorem 1.2. *If* $6r < n < 8r$*, then* Diam $(\mathcal{C}(G, X)) = 3$.

Our last theorem only gives a bound on Diam $(\mathcal{C}(G, X))$.

Theorem 1.3. *If* $r > 1$ *and* $n = 6r$*, then* Diam $(\mathcal{C}(G, X)) \leqslant 4$.

Consulting Table 1 (or Table 1 of [6]) we see that for $r = 1, n = 7$ or $r = 2, n = 15$ we have that Diam $(\mathcal{C}(G, X)) = 3$ and so Theorem 1.1 is sharp. For $r = 2$ the same table gives Diam $(\mathcal{C}(G, X)) = 4$ when $n = 12$ and 2 when $n = 16$, so Theorems 1.2 and 1.3 are also sharp. We note that for $r = 1$ and $n = 6$, $\mathcal{C}(G, X)$ is disconnected which explains the assumption $r > 1$ in Theorem 1.3. All the graphs we consider here are connected – see [9]. For $g \in G$, $supp(g)$ denotes the set of points of Ω not fixed by g. We use $d(,)$ for the usual distance metric on the graph $\mathcal{C}(G, X)$. For $x \in X$, the i^{th} disc, $\Delta_i(x)$, is defined as follows

$$\Delta_i(x) = \{y \mid y \in X \text{and } d(x, y) = i\}.$$

The proofs of Theorems 1.1, 1.2 and 1.3 adopt a similar, somewhat direct, approach. Since G acting by conjugation upon X induces graph automorphisms on $\mathcal{C}(G, X)$ and of course is transitive on X, it suffices to determine (or bound) $d(t, x)$ for an arbitrary vertex x of X. This we do by writing down explicit paths in $\mathcal{C}(G, X)$.

2 Diameter of $\mathcal{C}(G, X)$

We begin by establishing Theorem 1.1.

Proof of Theorem 1.1

Let $x \in X$. Set $\Lambda = supp(t) \cup supp(x)$ and $s = |supp(t) \cap supp(x)|$. Then $|\Lambda| = 6r - s$. If

$s \geqslant r$, then $|\Lambda| \leqslant 5r$. Hence there exists $y \in X$ with $supp(t) \cap supp(y) = \emptyset = supp(x) \cap supp(y)$ and so $d(t, x) \leqslant 2$. Now consider the case $s < r$, and set $e = r - s$. Without loss of generality we may suppose that $supp(t) \cap supp(x) \subseteq \{1, 2, 3, \ldots, 3s-2, 3s-1, 3s\}$. Put $y_1 = (3s+1, 3s+2, 3s+3) \ldots (3r-2, 3r-1, 3r)$ (so y_1 is the product of the "last" $r - s = e$ 3-cycles of t). Since $|\Omega \setminus \Lambda| = 8r - (6r - s) = 2r + s > 3s$ and $s < r$, we may select y_2 with $supp(y_2) \subseteq \Omega \setminus \Lambda$ and y_2 is a product of s pairwise disjoint 3-cycles. So $y = y_1 y_2 \in X$, $ty = yt$ and $xy = yx$. Thus $d(t, x) \leqslant 2$. Clearly Diam $(\mathcal{C}(G, X)) \geqslant 2$, and so the theorem follows.

Before proving Theorems 1.2 and 1.3 we introduce some notation and certain permutations of $Sym(\Omega)$. These permutations, though elements of order 3, are not in general in X. We will assume that $|\Omega| \geqslant 6r$. For $x \in X$, we let $\{\vartheta_i(x)\}_{i=1,\ldots,r}$ denote the orbits of $\langle x \rangle$ on Ω of size 3. So $supp(x) = \bigcup_{i=1}^{r} \vartheta_i(x)$. Write $t = t_1 t_2 \ldots t_r$ where $t_i = (3i-2, 3i-1, 3i)$. So $\vartheta(t_i) = \vartheta_i(t) = \{3i-2, 3i-1, 3i\}$.

Let $x \in X$. Denote the product of the t_i's for which $\vartheta_i(t) \cap supp(x) = \emptyset$ by τ_0 and let τ_3 be the product of the t_i's for which $\vartheta_i(t) \subseteq supp(x)$. Also let τ_1 be the product of r_1 t_i's where $|\vartheta_i(t) \cap supp(x)| = 1$, $3 \mid r_1$ and r_1 is as large as possible. Analogously, τ_2 is the product of r_2 t_i's where $|\vartheta_i(t) \cap supp(x)| = 2$, $3 \mid r_2$ and r_2 is as large as possible. Setting $\tau_* = t\tau_0^{-1}\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}$ we have $t = \tau_* \tau_0 \tau_1 \tau_2 \tau_3$. Let r_* be the number of t_i's in τ_*, r_0 the number of t_i's in τ_0 and r_3 the number of t_i's in τ_3. Observe that the maximality of r_1 and r_2 means $r_* \leqslant 4$ and that at most two of the t_i's in τ_* will have $|\vartheta_i(t) \cap supp(x)| = 1$ and at most two will have $|\vartheta_i(t) \cap supp(x)| = 2$. Evidently $r = r_* + r_0 + r_1 + r_2 + r_3$ and, for $i = 0, 1, 2, 3, |supp(x) \cap supp(\tau_i)| = ir_i$. Putting $s_* = |supp(x) \cap supp(\tau_*)|$, we also have

$$|supp(t) \cap supp(x)| = s_* + r_1 + 2r_2 + 3r_3.$$

Set $\Lambda = \Omega \setminus (supp(t) \cup supp(x))$. Since

$$\begin{aligned} |supp(t) \cup supp(x)| &= 3r + 3r - (s_* + r_1 + 2r_2 + 3r_3) \\ &= 6r - (s_* + r_1 + 2r_2 + 3r_3) \end{aligned}$$

it follows that

$$|\Lambda| = s_* + r_1 + 2r_2 + 3r_3 \text{ if } n = 6r \text{ and}$$

$$|\Lambda| \geqslant 1 + s_* + r_1 + 2r_2 + 3r_3 \text{ if } n > 6r.$$

Since 3 divides r_1, we may write

$$\tau_1 = \prod \mu_{i_1 i_2 i_3}$$

where the product of the $\mu_{i_1 i_2 i_3} = t_{i_1} t_{i_2} t_{i_3}$ is pairwise disjoint. For each $\mu_{i_1 i_2 i_3} = t_{i_1} t_{i_2} t_{i_3} = (3i_1 - 2, 3i_1 - 1, 3i_1)(3i_2 - 2, 3i_2 - 1, 3i_2)(3i_3 - 2, 3i_3 - 1, 3i_3)$ we may without loss, suppose that $supp(\mu_{i_1 i_2 i_3}) \cap supp(x) = \{3i_1 - 2, 3i_2 - 2, 3i_3 - 2\}$. Put

$$\lambda_{i_1 i_2 i_3} = (3i_1 - 2, 3i_2 - 2, 3i_3 - 2)(3i_1 - 1, 3i_2 - 1, 3i_3 - 1)(3i_1, 3i_2, 3i_3).$$

Then $\lambda_{i_1 i_2 i_3}$ commutes with $\mu_{i_1 i_2 i_3}$. Let

$$\rho_1 = \prod \lambda_{i_1 i_2 i_3}$$

and observe that ρ_1 commutes with t and will be a pairwise disjoint product of r_1 3-cycles. Further, $\frac{r_1}{3}$ of the 3-cycles in ρ_1 will have their support contained in $supp(x)$ while the remaining $\frac{2r_1}{3}$ 3-cycles in ρ_1 will have their support intersecting $supp(x)$ in the empty set. Also, as 3 divides r_2, we may express

$$\tau_2 = \prod \eta_{j_1 j_2 j_3}$$

where $\eta_{j_1 j_2 j_3} = t_{j_1} t_{j_2} t_{j_3}$ with the product being pairwise disjoint. For each $\eta_{j_1 j_2 j_3}$ we may suppose that $supp(\eta_{j_1 j_2 j_3}) \cap supp(x) = \{3j_1 - 2, 3j_1 - 1, 3j_2 - 2, 3j_2 - 1, 3j_3 - 2, 3j_3 - 1\}$. Define

$$\delta_{j_1 j_2 j_3} = (3j_1, 3j_2, 3j_3)(3j_1 - 2, 3j_2 - 2, 3j_3 - 2)(3j_1 - 1, 3j_2 - 1, 3j_3 - 1),$$

and let

$$\rho_2 = \prod \delta_{j_1 j_2 j_3}.$$

Evidently ρ_2 commutes with t and ρ_2 is a pairwise disjoint product of r_2 3-cycles. Moreover, $\frac{2r_2}{3}$ of the 3-cycles in ρ_2 will have their support contained in $supp(x)$ and the remaining $\frac{r_2}{3}$ have supports intersecting $supp(x)$ in the empty set.
Let σ_1 (respectively σ_2) be the product of the $\frac{2r_1}{3}$ (respectively $\frac{r_2}{3}$) 3-cycles in ρ_1 (respectively ρ_2) whose support intersects $supp(x)$ in the empty set. Also let σ_4 be a pairwise disjoint product of $(\frac{r_1}{3} + \frac{2r_2}{3} + r_3)$ 3-cycles with $supp(\sigma_4) \subseteq \Lambda$. Put $\Delta = \Lambda \backslash supp(\sigma_4)$.

We now summarize the pertinent properties of the permutations just introduced.

Lemma 2.1. *(i) $supp(\tau_0\rho_1\rho_2\tau_3) \subseteq supp(t)$, $\tau_0\rho_1\rho_2\tau_3$ commutes with t and is the product of $r - r_*$ pairwise disjoint 3-cycles.*

(ii) $\sigma_1\sigma_2\tau_0\sigma_4$ commutes with $\tau_0\rho_1\rho_2\tau_3$ and is the product of $r - r_*$ pairwise disjoint 3-cycles. Moreover $supp(\sigma_1\sigma_2\tau_0\sigma_4) \cap supp(x) = \emptyset$.

(iii) $|\Delta| = s_*$ if $n = 6r$ and $|\Delta| \geqslant 1 + s_*$ if $n \geqslant 6r$.

Proof. (i) Since $supp(\rho_1\rho_2) = supp(\tau_1\tau_2)$, $\tau_0\rho_1\rho_2\tau_3$ is the product of pairwise disjoint 3-cycles, and the number of such 3-cycles is $r - r_*$. Because ρ_1 and ρ_2 both commute with t, $\tau_0\rho_1\rho_2\tau_3$ commutes with t.
(ii) Since $supp(\sigma_4) \subseteq \Delta$ and $supp(\tau_0\rho_1\rho_2\tau_3) \subseteq supp(t)$, σ_4 commutes with $\tau_0\rho_1\rho_2\tau_3$. While $\sigma_1\sigma_2\tau_0$ is a product of 3-cycles which appear in $\tau_0\rho_1\rho_2\tau_3$ and therefore $\sigma_1\sigma_2\tau_0\sigma_4$ commutes with $\tau_0\rho_1\rho_2\tau_3$. By construction $\sigma_i \cap supp(x) = \emptyset (i = 1, 2)$, $supp(\tau_0) \cap supp(x) = \emptyset$ by definition and because we chose σ_4 so as $supp(\sigma_4) \subseteq \Lambda$ we get $supp(\sigma_1\sigma_2\tau_0\sigma_4) \cap supp(x) = \emptyset$.
(iii) Part (iii) follows from $|supp(\sigma_4)| = r_1 + 2r_2 + 3r_3$ and $\Delta = \Lambda \backslash supp(\sigma_4)$. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $y \in X$ be such that $|supp(y) \cap \vartheta_i(t)| = 1 = |supp(t) \cap \vartheta_i(y)|$ for $i = 1, \ldots, r$. Then $C_G(t) \cap C_G(y) = Sym(\Psi)$ where $\Psi = \Omega \setminus (supp(t) \cup supp(y))$. Now $|supp(t) \cup supp(y)| = 3r + 3r - r = 5r$ and so $|\Psi| = n - 5r < 8r - 5r = 3r$. Thus $X \cap C_G(t) \cap C_G(y) = \emptyset$ and consequently $d(t, y) \geqslant 3$. Hence Diam $(\mathcal{C}(G, X)) \geqslant 3$.

Let $x \in X$. We aim to show that $d(t, x) \leqslant 3$. On account of $C_G(t)$ having shape $3^r Sym(r) \times Sym(n - 3r)$ there is no loss in supposing $\tau_* = t_1 \ldots t_{r_*}$ where $0 \leqslant r_* \leqslant 4$ ($r_* = 0$ meaning $\tau_* = 1$). Depending on τ_* we define two elements ρ_* and σ_* which will be the product of r_* pairwise disjoint 3-cycles.

(**1**) $r_* = 4$

Then we have $\tau_* = t_1t_2t_3t_4 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$, $s_* = 6$ and we may, without loss, assume $supp(\tau_*) \cap supp(x) = \{1, 4, 7, 8, 10, 11\}$. Observe that $|supp(x) \backslash supp(t)| \geqslant 6$ and so we may select $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in supp(x) \backslash supp(t)$. Also by Lemma 2.1(iii), as $s_* = 6$, $|\Delta| \geqslant 7$. Thus we may also select $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \Delta$. Define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \alpha_6)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6)$$

and

$$\sigma_* = (2, 3, 5)(6, 9, 12)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6).$$

(**2**) $r_* = 3$

So $\tau_* = t_1t_2t_3 = (1, 2, 3)(4, 5, 6)(7, 8, 9)$. First we examine the case when $s_* = 4$, and may suppose that $supp(\tau_*) \cap supp(x) = \{1, 4, 7, 8\}$. Here we have $|supp(x) \backslash supp(t)| \geqslant 5$ and $|\Delta| \geqslant 5$. Choose $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in supp(x) \backslash supp(t)$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \Delta$, and define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \beta_1)(\beta_2, \beta_3, \beta_4)$$

and

$$\sigma_* = (2, 3, 5)(6, 9, \beta_5)(\beta_2, \beta_3, \beta_4).$$

We move onto the case when $s_* = 5$ and, without loss of generality, assume $supp(\tau_*) \cap supp(x) = \{1, 2, 4, 5, 7\}$. Since $|supp(x) \backslash supp(t)| \geqslant 4$ and $|\Delta| \geqslant 6$, we may select $\alpha_1, \alpha_2, \alpha_3 \in supp(x) \backslash supp(t)$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \Delta$. Then we take

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6)$$

and

$$\sigma_* = (3, 6, 8)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6).$$

(**3**) $r_* = 2$

So $\tau_* = t_1 t_2 = (1,2,3)(4,5,6)$ with $s_* = 2$, 3 or 4. First we look at the case when $s_* = 2$ or 3. Then we have $|supp(x)\backslash supp(t)| \geqslant 3$, $|supp(t)\backslash supp(x)| \geqslant 3$ and $|\Delta| \geqslant 3$. Choosing $\alpha_1, \alpha_2, \alpha_3 \in supp(x)\backslash supp(t)$, $\beta_1, \beta_2, \beta_3 \in \Delta$ and $\gamma_1, \gamma_2, \gamma_3 \in supp(t)\backslash supp(x))$, we let

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)$$

and

$$\sigma_* = (\gamma_1, \gamma_2, \gamma_3)(\beta_1, \beta_2, \beta_3).$$

Now assume that $s_* = 4$, and, without loss, that $supp(\tau_*) \cap supp(x) = \{1,2,4,5\}$. Because $|supp(x)\backslash supp(t)| \geqslant 2$ and $|\Delta| \geqslant 5$ we may choose $\alpha_1, \alpha_2 \in supp(x)\backslash supp(t)$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \Delta$ and then define

$$\rho_* = (\alpha_1, \alpha_2, \beta_1)(\beta_2, \beta_3, \beta_4)$$

and

$$\sigma_* = (3, 6, \beta_5)(\beta_2, \beta_3, \beta_4).$$

(**4**) $r_* = 1$

Then $\tau_* = t_1 = (1,2,3)$ and $s_* = 1$ or 2. Suppose $s_* = 1$ with $supp(\tau_*) \cap supp(x) = \{1\}$. So $|supp(x)\backslash supp(t)| \geqslant 2 \leqslant |\Delta|$. Selecting $\alpha_1, \alpha_2 \in supp(x)\backslash supp(t)$ and $\beta_1, \beta_2 \in \Delta$, we set

$$\rho_* = (\alpha_1, \alpha_2, \beta_1)$$

and

$$\sigma_* = (2, 3, \beta_2).$$

While if $s_* = 2$, then $|\Delta| \geqslant 3$ and selecting $\beta_1, \beta_2, \beta_3 \in \Delta$ we set

$$\rho_* = \sigma_* = (\beta_1, \beta_2, \beta_3).$$

(**5**) $r_* = 0$

Here we take $\rho_* = 1 = \sigma_*$.

Put $y = \rho_* \tau_0 \rho_1 \rho_2 \tau_3$. Since y is the product of $r_* + r_0 + r_1 + r_2 + r_3 = r$ disjoint 3-cycles, $y \in X$. Further we have that $ty = yt$ by Lemma 2.1(i). Next we consider $z = \sigma_* \sigma_1 \sigma_2 \tau_0 \sigma_4$. Each of $\sigma_* \sigma_1, \sigma_2, \tau_0$ and σ_4 are pairwise disjoint. Recalling that σ_1, σ_2 and σ_4 are, respectively, the product of $\frac{2r_1}{3}, \frac{r_2}{3}, (\frac{r_1}{3} + \frac{2r_2}{3} + r_3)$ disjoint 3-cycles, we see that $z \in X$. It may be further checked using Lemma 2.1(ii) that $yz = zy$ and $xz = zx$, and consequently $d(t,x) \leqslant 3$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $x \in X$. Our objective here is to show that $d(t,x) \leqslant 4$ from which it will follow that Diam $(\mathcal{C}(G,X)) \leqslant 4$. We proceed in a similar fashion to that in the proof of Theorem 1.1 though here, except for some cases, we will define three permutations ρ_*, σ_*, ξ_*, each a product of r_* pairwise disjoint cycles.

(6) $r_* = 4$

So $\tau_* = t_1t_2t_3t_4 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$ with $s_* = 6$. Assume, without loss, that $supp(\tau_*) \cap supp(x) = \{1,4,7,8,10,11\}$. Since $|supp(x)\backslash\ supp(t)| \geqslant 6$ and so we may choose $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in supp(x)\backslash supp(t)$. Further, as $|\Delta| = s_* = 6$ by Lemma 2.1(iii), we may also choose $\beta_1, \beta_2, \beta_3, \beta_4,\ \beta_5, \beta_6 \in \Delta$. Now define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \alpha_6)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6)$$

and

$$\sigma_* = (2,3,5)(6,9,12)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6).$$

(7) $r_* = 3$

So $\tau_* = t_1t_2t_3 = (1,2,3)(4,5,6)(7,8,9)$. If $s_* = 4$ we may suppose without loss that $supp(\tau_*) \cap supp(x) = \{1,4,7,8\}$. Here we have $|supp(x)\backslash supp(t)| \geqslant 5$ and $|\Delta| = s_* = 4$ by Lemma 2.1(iii). Choose $\alpha_1, \alpha_2, \alpha_3 \in supp(x)\backslash supp(t)$ and $\beta_1, \beta_2, \beta_3 \in \Delta$, and define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)(1,2,3),$$

$$\sigma_* = (5,6,9)(\beta_1, \beta_2, \beta_3)(1,2,3)$$

and

$$\xi_* = (5,6,9)(\beta_1, \beta_2, \beta_3)(\alpha, \beta, \gamma),$$

where (α, β, γ) is a 3-cycle of x for which $1 \notin \{\alpha, \beta, \gamma\}$. Note that $\{\alpha, \beta, \gamma\} \cap supp(\sigma_*) = \emptyset$. For the case when $s_* = 5$, without loss of generality, we assume $supp(\tau_*) \cap supp(x) = \{1,4,5,7,8\}$. Since $|supp(x)\backslash supp(t)| \geqslant 4$ and $|\Delta| = s_* = 5$, we may select $\alpha_1, \alpha_2, \alpha_3 \in supp(x)\backslash supp(t)$ and $\beta_1, \beta_2, \beta_3 \in \Delta$. Then we take

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)(4,5,6),$$

$$\sigma_* = (2,3,9)(\beta_1, \beta_2, \beta_3)(4,5,6)$$

and

$$\xi_* = (2,3,9)(\beta_1, \beta_2, \beta_3)(\alpha, \beta, \gamma),$$

where (α, β, γ) is a 3-cycle of x chosen so as $\{4,5\} \cap \{\alpha, \beta, \gamma\} = \emptyset$. Since $r \geqslant r_* = 3$ such a choice is possible.

Before dealing with $r_* = 2$ we analyze a number of small cases.

(8) Suppose that $t = (1,2,3)(4,5,6)$ (so $r = 2$ and $n = 12$).

(i) If $x = (1,7,8)(4,9,10)$ or $x = (1,4,7)(2,5,8)$, then $d(t,x) \leqslant 4$.

(ii)If $x = (1,4,7)(8,9,10)$, then $d(t,x) \leqslant 3$.

Assume that $x = (1, 7, 8)(4, 9, 10)$, and let

$$x_1 = (7, 8, 11)(9, 10, 12), \quad x_2 = (2, 3, 5)(9, 10, 12), \quad x_3 = (2, 3, 5)(1, 7, 8).$$

Then $x_1, x_2, x_3 \in X$ and (t, x_1, x_2, x_3, x) is a path in $\mathcal{C}(G, X)$ whence $d(t, x) \leqslant 4$. In the case $x = (1, 4, 7)(2, 5, 8)$ we take $x_1 = (7, 8, 9)(10, 11, 12), x_2 = (1, 3, 6)(10, 11, 12)$ and $x_3 = (2, 5, 8)\ (10, 11, 12)$. It is easily checked that (t, x_1, x_2, x_3, x) is also a path in $\mathcal{C}(G, X)$, so proving part (i). For $x = (1, 4, 7)(8, 9, 10)$ taking $x_1 = (1, 2, 3)(8, 9, 10)$ and $x_2 = (5, 6, 11)(8, 9, 10)$ gives a path (t, x_1, x_2, x) in $\mathcal{C}(G, X)$. So (ii) holds and (8) is proved.

(**9**) Suppose $t = (1, 2, 3)(4, 5, 6)(7, 8, 9)$ with $\tau_* = (1, 2, 3)(4, 5, 6)$ (so $r = 3$ and $n = 18$). Let $x \in X$ be such that $supp(\tau_*) \cap supp(x) = \{1, 4\}$ and assume 1 and 4 are in different 3-cycles of x. Then $d(t, x) \leqslant 4$.

By assumption $x = (1, *, *)(4, \delta, \epsilon)(\alpha, \beta, \gamma)$ with $\{1, 4\} \cap \{\alpha, \beta, \gamma\} = \emptyset$. Because $\tau_* = (1, 2, 3)(4, 5, 6)$ we must have $supp(t) \cap supp(x) = \{1, 4\}$ or $\{1, 4, 7, 8, 9\}$. Suppose the former holds and set $x_1 = (1, 2, 3)(\alpha, \beta, \gamma)(7, 8, 9)$ and $x_2 = (4, \delta, \epsilon)(\alpha, \beta, \gamma)(7, 8, 9)$. Then (t, x_1, x_2, x) is a path in $\mathcal{C}(G, X)$. Hence $d(t, x) \leqslant 3$. Turning to the latter case we have $|supp(t) \cup supp(x)| = 13$. So we may choose, say, $16, 17, 18 \in \Lambda$ and then take $x_1 = (1, 2, 3)(4, 5, 6)\ (16, 17, 18), x_2 = (1, 2, 3)(\alpha, \beta, \gamma)(16, 17, 18)$ and $x_3 = (4, \delta, \epsilon)(\alpha, \beta, \gamma)(16, 17,\ 18)$, giving a path (t, x_1, x_2, x_3, x) in $\mathcal{C}(G, X)$. Thus $d(t, x) \leqslant 4$, so proving (9).

(**10**) $r_* = 2$

So we have $\tau_* = t_1 t_2 = (1, 2, 3)(4, 5, 6)$ with $s_* = 2, 3$ or 4. First we consider the case $s_* = 2$, and assume $supp(\tau_*) \cap supp(x) = \{1, 4\}$. For the moment also assume that $r = 2$ (so $t = \tau_*$). Then, without loss, x is either $(1, 7, 8)(4, 9, 10)$ (1 and 4 in different 3-cycles of x) or $(1, 4, 7)(8, 9, 10)$ (1 and 4 in the same 3-cycle of x). By (8)(i) we have $d(t, x) \leqslant 4$. So, since we are aiming to show that $d(t, x) \leqslant 4$, we may suppose $r \geqslant 3$. Now consider the possibility that $r = 3$ and 1 and 4 are in different 3-cycles of x. Then, without loss, $x = (1, *, *)(4, \delta, \epsilon)(\alpha, \beta, \gamma)$ in which case $d(t, x) \leqslant 4$ by (9). Thus, when $r = 3$, we may suppose 1 and 4 are in the same 3-cycle of x. Consequently, as $r \geqslant 3$, we may find two 3-cycles of x, (α, β, γ) and $(\delta, \epsilon, \lambda)$ such that $\{\alpha, \beta, \gamma, \delta, \epsilon, \lambda\} \cap \{1, 4\} = \emptyset$. Now we define ρ_*, σ_* and ξ_* by taking $\rho_* = \sigma_* = \tau_*$ and $\xi_* = (\alpha, \beta, \gamma), (\delta, \epsilon, \lambda)$.

Next we look at the case $s_* = 3$. Then we have $|supp(x) \backslash supp(t)| \geqslant 3$, $|supp(t) \backslash supp(x)| \geqslant 3$ and $|\Delta| = s_* = 3$. Choosing $\alpha_1, \alpha_2, \alpha_3 \in supp(x) \backslash\ supp(t)$, $\beta_1, \beta_2, \beta_3 \in \Delta$ and $\gamma_1, \gamma_2, \gamma_3 \in supp(t) \backslash supp(x))$, we let

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)$$

and

$$\sigma_* = (\gamma_1, \gamma_2, \gamma_3)(\beta_1, \beta_2, \beta_3).$$

Finally we come to $s_* = 4$. So without loss we have $supp(\tau_*) \cap supp(x) = \{1, 2, 4, 5\}$. Suppose, for the moment, that for all 3-cycles (α, β, γ) we have $\{1, 2\} \cap \{\alpha, \beta, \gamma\} \neq \emptyset \neq$

$\{4,5\} \cap \{\alpha,\beta,\gamma\}$. Then it follows that $r = 2$ and, without loss, $x = (1,4,7)(2,5,8)$. But then $d(t,x) \leqslant 4$ by (8)(ii). Thus we may suppose x contains a 3-cycle (α,β,γ) such that $(\alpha,\beta,\gamma) \cap \{1,2\} = \emptyset$, and we can now define ρ_* and σ_*. Since $|\Delta| = s_* = 4$, we have $\beta_1,\beta_2,\beta_3 \in \Delta$. Let $\rho_* = (1,2,3)(\beta_1,\beta_2,\beta_3)$ and $\sigma_* = (\alpha,\beta,\gamma)(\beta_1,\beta_2,\beta_3)$. This completes the case $s_* = 4$ and (10).

Yet another special case must be looked at before doing $r_* = 1$.

(**11**) Let $t = (1,2,3)(4,5,6)$ with $\tau_* = (1,2,3)$. Suppose $x = (1,*,*)(2,*,*) \in X$ with $supp(\tau_*) \cap supp(x) = \{1,2\}$. Then $d(t,x) \leqslant 3$.

Since $\tau_* = (1,2,3)$, $supp(t) \cap supp(x) = \{1,2\}$ or $\{1,2,4,5,6\}$. If $supp(t) \cap supp(x) = \{1,2\}$ and, say $\Omega \backslash (supp(t) \cap supp(x)) = \{11,12\}$, then define $x_1 = (4,5,6)(10,11,12), x_2 = (4,5,6)(\alpha,\beta,\gamma)$ where (α,β,γ) is a 3-cycle not containing 10. While in the other case with, say $\Omega \backslash (supp(t) \cap supp(x)) = \{8,9,10,11,12\}$ we define $x_1 = (8,9,10)(7,11,12), x_2 = (8,9,10)(\alpha,\beta,\gamma)$ where (α,β,γ) is a 3-cycle not containing 7. Hence $d(t,x) \leqslant 3$.

(**12**) $r_* = 1$

So we have either, without loss, $supp(\tau_*) \cap supp(x) = \{1\}$ or $\{2,3\}$. In view of (10), as $r > 1$, either $d(t,x) \leqslant 3$ or we may find a 3-cycle (α,β,γ) of x for which $supp(\tau_*) \cap \{\alpha,\beta,\gamma\} = \emptyset$. In the latter case we define $\rho_* = \sigma_* = \tau_*$ and $\xi_* = (\alpha,\beta,\gamma)$.

(**13**) $r_* = 0$

Just as in (5) we take $\rho_* = 1 = \sigma_*$.
Now let $y = \rho_*\tau_0\rho_1\rho_2\tau_3, z = \sigma_*\sigma_1\sigma_2\tau_0\sigma_4$ and $w = \xi_*\sigma_1\sigma_2\tau_0\sigma_4$ (where w is only defined if in (6), (7), (10), (12), (13) ξ_* is defined). Then $y,z,w \in X$ with (t,y,z,w,x) is a path in $\mathcal{C}(G,X)$. Consequently $d(t,x) \leqslant 4$. Since x was an arbitrary vertex, this shows that Diam $(\mathcal{C}(G,X)) \leqslant 4$ and completes the proof of Theorem 1.3. □

We end this paper with a table containing some calculations on diameters and discs using Magma[10]. Each entry in the table first gives the size of the relevant $\Delta_i(t)$ for the given r and n with the number in brackets being the number of $C_G(t)$-orbits on $\Delta_i(t)$. A blank entry means that $|\Delta_i(t)| = 0$.

	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$
r=1						
$n = 7$	9 (2)	24 (2)	36 (1)	-	-	-
$n = 8$	21 (2)	90 (3)	-	-	-	-
$n = 9$	41 (2)	126 (3)	-	-	-	-
r=2						
$n = 10$	35 (4)	192 (6)	1,008 (10)	2,628 (20)	3,672 (13)	864 (5)
$n = 11$	83 (4)	1,080 (9)	7,560 (23)	9,756 (23)	-	-
$n = 12$	203 (5)	6,300 (16)	28,296 (34)	2,160 (5)	-	-
$n = 13$	563 (5)	25,740 (30)	42,336 (25)	-	-	-
$n = 14$	1,571 (5)	67,140 (48)	51,408 (7)	-	-	-
$n = 15$	4,035 (5)	168,948 (54)	27,216 (1)	-	-	-
$n = 16$	9,363 (5)	310,956 (55)	-	-	-	-
r=3						
$n = 9$	25 (4)	216 (4)	1,512 (11)	486 (6)	-	-
$n = 12$	49 (7)	648 (8)	9,936 (39)	90,990 (139)	327,024 (404)	64,152 (102)
$n = 13$	121 (7)	2,808 (18)	79,488 (85)	724,086 (383)	783,432 (332)	11,664 (3)
$n = 14$	265 (7)	9,936 (23)	390,582 (138)	3,217,806 (564)	865,890 (143)	-
$n = 15$	745 (9)	62,424 (46)	2,414,610 (243)	8,733,420 (594)	-	-
$n = 16$	2,545 (9)	482,760 (90)	17,798,778 (578)	7,341,516 (220)	-	-
$n = 17$	8,089 (9)	3,400,272 (145)	50,175,126 (728)	870,912 (16)	-	-
$n = 18$	24,441 (10)	16,126,398 (210)	92,757,960 (679)	-	-	-

Continued on Next Page…

Table 1 – Continued

	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$
r=4						
$n = 12$	159 (6)	8,532 (20)	193,104 (121)	44,604 (37)	-	-
$n = 15$	367 (11)	37,044 (52)	3,053,160 (682)	81,668,484 (8,294)		
$n = 16$	991 (11)	271,236 (92)	56,926,656 (2,351)	390,829,212 (13,122)	419,904 (12)	-
$n = 17$	2,239 (11)	1,350,612 (112)	487,124,064 (4,539)	1,036,246,284 (12,578)	-	-
r=5						
$n = 15$	751 (8)	154,440 (44)	17,669,304 (783)	27,020,304 (996)	-	-

Table 1: Disc sizes and $C_G(t)$-orbits

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