A characterization of the Hermitian variety in finite 3-dimensional projective spaces

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Abstract

A combinatorial characterization of a non–singular Hermitian variety of the finite 3-dimensional projective space via its intersection numbers with respect to lines and planes is given.

Keywords: projective space; intersection number; hermitian variety

1 Introduction

Ever since the celebrated theorem of B. Segre [10] on $(q+1)$–arcs of $\mathrm{PG}(2,q)$, q odd, people have tried to characterize classical objects of finite projective geometry via their combinatorial properties. Intersection numbers with respect to the subspaces of a given dimension of the d–dimensional projective space $\mathrm{PG}(d,q)$ of order q have been used frequently for this purpose.

Let us fix some notation. Let $\mathbb{P} = \mathrm{PG}(d,q)$ and let $m_1, \ldots, m_s$ be s integers such that $0 \leqslant m_1 < \ldots < m_s$. For any integer h, $1 \leqslant h \leqslant r-1$, let $\mathcal{P}_h$ denote the family of all h–dimensional subspaces of $\mathbb{P}$. A subset $\mathcal{K}$ of points of $\mathbb{P}$ has *class* $[m_1, \ldots, m_s]_h$ for some s if $|\mathcal{K} \cap \pi| \in \{m_1, \ldots, m_s\}$ for any $\pi \in \mathcal{P}_h$. Moreover, if for every $m_j \in \{m_1, \ldots, m_s\}$ there is at least one subspace $\pi \in \mathcal{P}_h$ such that $|\mathcal{K} \cap \pi| = m_j$ the set $\mathcal{K}$ is of *type* $(m_1, \ldots, m_s)_h$. In this case, the non–negative integers $m_1, \ldots, m_s$ are the *intersection numbers* of $\mathcal{K}$ (with respect to $\mathcal{P}_h$). If $h = 1$ or $h = 2$, we speak of the line–type or plane–type, respectively.

A wide literature is devoted to the theory of sets of a given type, some of which is listed in the references. The interest in studying such sets, in particular for the case of two

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intersection numbers with respect to hyperplanes is motivated in part by their connection with coding theory (cf e.g. [4, 7, 15]).

A non–singular hermitian variety of $\mathrm{PG}(3, q^2)$ has size $(q^3 + 1)(q^2 + 1)$ [6]. Any line intersects the variety in either 1, or $q + 1$, or $q^2 + 1$ points. A plane intersects in either $q^3 + 1$ or $q^3 + q^2 + 1$ points.

In this paper, we will give a new combinatorial characterization of a non–singular hermitian variety of the finite 3-dimensional projective space. This result generalizes an earlier result due to Schillewaert and Thas [8].

Theorem I *Let $\mathcal{K}$ be a set of $k = m(q + 1)$ points of $\mathrm{PG}(3, q)$, for some integer m. Assume that the line type of $\mathcal{K}$ is $(1, s + 1, q + 1)_1$, for some integer s with $1 \leqslant s \leqslant q - 1$. Assume further that the plane type is $(m, h)_2$ for some integer h, then $q = s^2$ and $\mathcal{K}$ is a hermitian surface of $\mathrm{PG}(3, s^2)$.*

As is costumary in the literature, a *j–line* is a line intersecting $\mathcal{K}$ in exactly j points. A *j–plane* is a plane intersecting $\mathcal{K}$ in exactly j points. For simplicity, a 1–line is called a *tangent* line.

1.1 Basic equations of k–sets of type $(m, h)_2$ in $\mathrm{PG}(3, q)$

De Finis [3] studied combinatorial properties of sets in $\mathrm{PG}(3, q)$. Assume that $\mathcal{K}$ is a set of k points in $\mathrm{PG}(3, q)$ with plane type $(m, h)_2$, for two distinct integers m and h. Then k is a solution to the equation

$$k^2(q + 1) - k[(h + m)(q^2 + q + 1) - q^2] + hm(q + 1)(q^2 + 1) = 0. \quad (1)$$

As a preparation for the arguments in the remainder of this paper, we outline a proof of this result:
Let c_j be the number of j–planes of $\mathcal{K}$. Double counting gives

$$c_m + c_h = (q^2 + 1)(q + 1)$$
$$mc_m + hc_h = k(q^2 + q + 1)$$
$$m(m - 1)c_m + h(h - 1)c_h = k(k - 1)(q + 1)$$

from which the quadratic equation for k follows.

We observe the following:

Observation 1: If $h = m + q$ the quadratic equation (1) has the two solutions

$$k = m(q + 1) \text{ and } k = \frac{(q^2 + 1)(q + m)}{q + 1}.$$

Observation 2: If $k = m(q + 1)$ the quadratic equation (1) implies that $h = m + q$.

The present paper is concerned with only the first case of the two cases occuring in (1) when $h = m + q$.

If $\mathcal{K}$ is a k–set of $\mathrm{PG}(3,q)$ with intersection numbers m and $m+q$ with respect to planes, it follows from the basic equations above that $k=m(q+1)$ or $k=(q^2+1)(q+m)/(q+1)$. Hence, in $\mathrm{PG}(3,q^2)$, from $m=q^3+1$ it follows that $k=(q^3+1)(q^2+1)=m(q^2+1)$ and so Theorem I generalizes Theorem 4.1 of [8].

Let us end this section with some remarks. In $\mathrm{PG}(3,q)$ a set of line type $(n)_1$ is either the empty set or the whole space, and a set of line type $(m,q+1)_1$ is either $\mathrm{PG}(3,q)$ less a point ($m=q$) or a plane ($m=1$) (cf e.g. [13]). If $\mathcal{K}$ is a set of points of $\mathrm{PG}(3,q)$ of line type $(1,n)_1$ then by results in [13] $n=q+1$ and so $\mathcal{K}$ is a plane. Thus, to study subsets of $\mathrm{PG}(3,q)$ of class $[1,a,q+1]_1$ means to study sets of line type $(1,a,q+1)_1$.

2 The proof

Throughout this section, $\mathcal{K}$ is a set of points of $\mathrm{PG}(3,q)$ of size $k=m(q+1)$ with line type $(1,s+1,q+1)_1$ and with plane type $(m,h)_2$, $s\geqslant 1$. It follows from Observation 2 that $h=m+q$.

Lemma 1. *If ℓ is a $(q+1)$–secant line then all the planes containing ℓ are h–planes.*

Proof. Let α denote the number of m–planes through ℓ. Counting k via the planes on ℓ gives

$$m(q+1)=k=q+1+\alpha(m-q-1)+(q+1-\alpha)(m-1)$$

so

$$m(q+1)=m(q+1)-\alpha q$$

from which it follows that $\alpha=0$. □

Corollary 2. *There are no $(q+1)$–lines contained in m–planes.*

Lemma 3. *Any m–plane contains at least one tangent line.*

Proof. Recall that every line intersects $\mathcal{K}$ in either $1, s+1$, or $q+1$ points for some s with $1\leqslant s\leqslant q-1$. Assume that there is an m–plane π containing no tangent line. If π contains a $(q+1)$–line, then by Lemma 1, π is an h–plane. This contradiction shows that all lines of π are $(s+1)$–secant lines. Let p be a point of $\mathcal{K}\cap\pi$. Counting m via the lines on p gives $m=1+(q+1)s=sq+s+1$. Counting the incident point–line pairs (p,ℓ), $p\in\pi\cap\mathcal{K}$ gives

$$(sq+s+1)(q+1)=m(q+1)=(q^2+q+1)(s+1)$$

and so $s=q$, which is a contradiction. Hence any m–plane contains at least one tangent line. □

Lemma 4. $m\leqslant sq+1$.

Proof. Let π be an m–plane and ℓ be a line of π tangent to $\mathcal{K} \cap \pi$ at the point p. Let x be the number of tangent lines on p. Counting points of $\mathcal{K} \cap \pi$ via the lines on p gives

$$m = 1 + (q + 1 - x)s,$$

since $x \geqslant 1$ it follows that $m \leqslant sq + 1$. □

Lemma 5. $m = sq + 1$.

Proof. Assume that $m \leqslant sq$. Thus, $h = m + q \leqslant (s + 1)q$. Assume that there exists a h–plane π containing two or more $(q + 1)$–secant lines. Let x be a point of π not in $\mathcal{K}$. All lines of π on x, except possibly for one, intersect $\mathcal{K} \cap \pi$ in at least two and hence in at least $s + 1$ points, so $h \geqslant 1 + q \cdot (s + 1)$, a contradiction. Hence, every h–plane contains at most one $(q + 1)$–line. Let ℓ be a $(q + 1)$–line, let π be an h–plane through ℓ and let x be a point of $\mathcal{K} \cap \pi$ outside ℓ. Counting points of $\mathcal{K} \cap \pi$ via the lines of π passing through x gives $h = 1 + (q + 1)s = sq + s + 1$.

Let α be an m–plane and $\mathcal{K}' = \mathcal{K} \cap \alpha$. The set $\mathcal{K}'$ has size m and is of line type $(1, s + 1)_1$. Let p be a point of $\mathcal{K}'$, and x be the number of tangent lines on p. Counting points of $\mathcal{K} \cap \alpha$ via the lines on p gives $m = 1 + (q + 1 - x)s$, so

$$sq + s + 1 - q = h - q = m = 1 + sq + s - xs$$

that is,

$$x = \frac{q}{s}.$$

Let b_1 and b_{s+1} denote the number of tangent lines and $(s+1)$–lines of α, respectively. Then

$$m\frac{q}{s} = b_1,$$

that is, $mq = sb_1$.

On the other hand, $b_1 + b_{s+1} = q^2 + q + 1$ and $b_1 + (s + 1)b_{s+1} = m(q + 1)$. These last two equations imply that

$$sb_1 = (s + 1)(q^2 + q + 1) - m(q + 1).$$

Therefore,

$$mq = (s + 1)(q^2 + q + 1) - m(q + 1)$$

and so

$$2q + 1 = \frac{(s + 1)(q^2 + q + 1)}{m} = \frac{(s + 1)(q^2 + q + 1)}{sq + s + 1 - q}.$$

Thus,

$$2q = \frac{sq^2 + q^2 + 2q}{sq + s + 1 - q}$$

$$sq^2 + 2sq - 3q^2 = 0$$

from which it follows that $s = 2$ and $q = 4$. Therefore, $m = 7, h = 11$ and $k = 35$. Let b_i, $i \in \{1, 3, 5\}$, denote the number of i–lines of P$G(3, 4)$. The usual counting arguments give:

$$\begin{cases} b_1 + & b_3 + & b_5 & = & (q^2+1)(q^2+q+1) = 357 \\ b_1 + & 3b_3 + & 5b_5 & = & k \cdot (q^2+q+1) = 735 \\ & 6b_3 + & 10b_5 & = & k(k-1) = 1190 \end{cases},$$

so, subtracting the first equation from the second one and dividing the third equation above by 2 give

$$b_3 + 2b_5 = 189 \text{ and } 3b_3 + 5b_5 = 595$$

which is a contradiction. □

Lemma 6. $q = s^2$.

Proof. Let π be an m–plane, p be a point of $\pi \cap \mathcal{K}$, and let x be the number of tangent lines of π passing through p. Counting points of $\mathcal{K} \cap \pi$ via the lines on p gives

$$sq + 1 = m = 1 + (q + 1 - x)s$$

and so $x = 1$. Thus, every point of π in $\mathcal{K}$ is on exactly one tangent line. So the numbers b_1 of tangents and b_{s+1} of $(s+1)$–lines of π are $b_1 = m = sq+1$ and $b_{s+1} = q^2+q+1-m$, respectively. Counting the incident point–line pairs (p, ℓ) of π, $p \in \mathcal{K}$, ℓ a $(s + 1)$–secant line gives

$$(sq + 1)q = (s + 1)b_{s+1},$$

from which it follows that

$$b_{s+1} = q^2 - \frac{q(q-1)}{s+1}.$$

Hence,

$$q^2 - \frac{q(q-1)}{s+1} = b_{s+1} = q^2 + q + 1 - sq - 1$$

$$(s-1)q = \frac{q(q-1)}{s+1}$$

$$s^2 - 1 = q - 1.$$

Thus, $m = s^3 + 1$, $h = s^3 + s^2 + 1$, $k = (s^3 + 1)(s^2 + 1)$ and each line intersects $\mathcal{K}$ in 1, $s+1$ or s^2+1 points. Hence, $\mathcal{K}$ is a $k_{n,3,q}$ set in PG$(3, q)$ (cf [6]) with $k = (q\sqrt{q}+1)(q+1)$, $n = s + 1$ and $q = s^2$. So, since $\mathcal{K}$ contains no plane, $n \neq 1, q$, any point of $\mathcal{K}$ is on at least one $(s + 1)$–secant line and for $q = 4$ no 13–plane contains three 5–lines forming a triangle and $k = 45$, it follows by the results in ([6], Section 19.5 Theorem 19.5.13) that $\mathcal{K}$ is a Hermitian variety of PG$(3, s^2)$. □

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Corrigendum added March 29 2019

A step in the proof of Theorem I is Lemma 5, which by contradiction gives that $m = sq + 1$. In the proof of Lemma 5, the author first proves that if $m \leqslant sq$ then $\mathcal{K}$ is a hypothetical set of points of $\mathrm{PG}(3, 4)$ of size 35, intersected by any plane in $m = 7$ or $h = 11$ points and by every line in 1, 3 or 5 points. Then using the usual incidence equations, with one of them written in a wrong way, he proves that such a set cannot exist, believing wrongly to have obtained the final contradiction. Indeed, in the last part of the proof of Lemma 5 the following system of linear equations and argument are used.

Let b_i, $i \in \{1, 3, 5\}$, denote the number of i–lines[1]*. The usual point–line incidence counting arguments give:*

$$\begin{cases} b_1 + \quad b_3 + \quad b_5 & = (q^2+1)(q^2+q+1) = 357 \\ b_1 + \; 3b_3 + \; 5b_5 & = k \cdot (q^2+q+1) = 735 \\ \qquad 6b_3 + 10b_5 & = k(k-1) = 1190 \end{cases}$$

so, subtracting the first equation from the second one and dividing the third equation above by 2 give

$$b_3 + 2b_5 = 189 \quad \textit{and} \quad 3b_3 + 5b_5 = 595$$

which is a contradiction.

The mistake is that the third equation should be $6b_3 + 20b_5 = 1190$, and so the above argument does not work.

Now, let us consider the previous system with the correct third equation. Thus, $b_5 = 7$ and $b_1 = b_3 = 175$.

Let ℓ and ℓ' two 5–lines and assume that they intersect each other in a point p. The plane π containing ℓ and ℓ' is an h–plane since has at least 9 points. Let x be a point of $\ell \setminus \{p\}$, the lines on x in π and different from ℓ have at least three points in $\mathcal{K} \cap \pi$ and so $\mathcal{K}$ intersects π in at least $5 + 4 \cdot 2 = 13 > 11 = h$ points, a contradiction.

Hence the seven 5–lines are pairwise skew and so they form a partial spread of $\mathrm{PG}(3, 4)$. These lines partition the set of points of $\mathcal{K}$, and since $\mathcal{K}$ is of line type $(1, 3, 5)_1$ it follows that there is no line skew to all of them. Therefore such a partial spread is maximal. But this is a contradiction, since the number of lines of a maximal partial spread in $\mathrm{PG}(3, q)$ is at least $2q$ (cf [2, 1]), and so in this case it should be $7 \geqslant 2q = 8$. Hence, the case $s = 2$, $q = 4$, $m = 7, h = 11$ and $k = 35$ cannot occur and so Lemma 5 is valid.

Let us end, by recalling that in [3] Hirschfeld and Hubaut gave the complete list of sets of line–type $(1, 3, 5)_1$ in $\mathrm{PG}(3, 4)$ and that list contains no set of size 35, so one may obtain the validity of Lemma 5 also via that result.

Additional correction to text: line 7, Section 1: '$1 \leqslant h \leqslant r - 1$' should be '$1 \leqslant h \leqslant d - 1$'.

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[1] An *i–line* (or *i–plane*) is a line (plane) intersecting $\mathcal{K}$ in exactly i–points.

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