

# Wieland drift for triangular fully packed loop configurations

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## Abstract

Triangular fully packed loop configurations (TFPLs) emerged as auxiliary objects in the study of fully packed loop configurations on a square (FPLs) corresponding to link patterns with a large number of nested arches. Wieland gyration, on the other hand, was invented to show the rotational invariance of the numbers  $A_\pi$  of FPLs corresponding to a given link pattern  $\pi$ . The focus of this article is the definition and study of Wieland drift on TFPLs. We show that the repeated application of this operation eventually leads to a configuration that is left invariant. We also provide a characterization of such stable configurations. Finally we apply Wieland drift to the study of TFPL configurations, in particular giving new and simple proofs of several results.

**Keywords:** Triangular fully packed loop configurations, Wieland gyration

## 1 Introduction

Triangular fully packed loop configurations (TFPLs) first appeared in the study of ordinary fully packed loop configurations (FPLs). There they were used to show that the number of FPLs corresponding to a given link pattern with  $m$  nested arches is a polynomial in  $m$ , see [3]. It soon turned out that TFPLs possess a number of nice properties, which made them worthy objects of study by themselves. For instance, they can be seen as a generalized model for Littlewood–Richardson coefficients, thereby establishing an unexpected link to algebra. This was first proved in [6] by a convoluted argument and later

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in [4] in a direct combinatorial manner and in a more general setting. Other combinatorial aspects of TFPLs, many of them still conjectural, are studied in [5, 8].

In 2000 Wieland [9] invented the operation on FPLs which bears his name. The *Wieland gyration* was used to prove the rotational invariance of the numbers  $A_\pi$  of FPLs corresponding to a given link pattern  $\pi$ . It was later heavily used by Cantini and Sportiello [2] to prove the Razumov–Stroganov conjecture. It also came up in connection with TFPLs already in [5] following work of [8].

The main contribution of this article is the explicit definition of Wieland drift for TFPLs together with a detailed study of some of its properties.

While the usual Wieland gyration of FPLs is an involution, our *left-Wieland drift* WL acting on TFPLs is not. By a finiteness argument, the sequence  $(WL^m(f))_{m \geq 0}$  is eventually periodic. In Theorem 15, it will be shown that the length of the period is always one, which means one always reaches a TFPL which is invariant under left-Wieland drift. In fact, if  $N$  is the size of  $f$ , then less than  $2N$  iterations of WL will suffice to obtain such *stable* configurations. A key step in the proof of Theorem 15 is to classify these stable TFPLs. It turns out that this depends solely on the occurrence of a certain type of edges called *drifters*: this is the content of Theorem 16. These results also hold for *right-Wieland drift*.

Now to each TFPL are assigned three binary words  $u$ ,  $v$  and  $w$  that encode its boundary conditions. Such binary words  $\sigma$  are naturally associated with Young diagrams  $\lambda(\sigma)$ , and by the results of [6, 4] TFPLs with boundary  $(u, v; w)$  such that  $|\lambda(u)| + |\lambda(v)| = |\lambda(w)|$  are enumerated by the Littlewood-Richardson-coefficient  $c_{u,v}^w$ . We will show that such TFPLs are stable. In general, the boundary  $(u, v; w)$  of a TFPL has to satisfy  $|\lambda(u)| + |\lambda(v)| \leq |\lambda(w)|$ : this was proved first in [8] using Wieland gyration and a certain degree argument, and later in a combinatorial fashion in [4]. Here we will use left- and right-Wieland drifts to give a simple proof of this inequality.

The paper is divided as follows. In Section 2 we recall the definitions of FPLs and TFPLs as well as elementary properties of binary words and Young diagrams. Section 3 contains the definition of our main construction – the left-Wieland drift acting on TFPLs (based on Wieland’s original definition). It is introduced in Definition 7 and we give its first properties, culminating in Theorem 13. We can then state the theorems about stability of TFPLs, namely Theorems 15 and 16, which are proved in Section 4. Finally, Section 5 contains applications of Wieland drift to enumerative questions concerning TFPLs.

## 2 Definitions and elementary properties

In this section we recall the definitions of FPLs and TFPLs, and the binary words attached to the *boundary* of a TFPL with the necessary conditions they must satisfy.

## 2.1 Fully packed loop configurations

Fully packed loop configurations first came up in statistical physics; they are an alternative representation of *six-vertex model* configurations which are in one-to-one correspondence with *square-ice* configurations, see for example [1] and [9]. Furthermore, they are in bijection with alternating sign matrices and other combinatorial configurations, cf. [7].

We start with the graph  $G_n$ , which is defined as the square grid with  $n^2$  vertices together with  $4n$  *external edges*. The  $(n + 1)^2$  unit squares of this grid, including external cells that have two or three surrounding edges only, are said to be the *cells* of  $G_n$ . They are partitioned into odd and even cells in a chessboard manner where by convention the cells on the main Northwest-Southeast diagonal are odd. In Figure 1, the graph  $G_8$  together with its odd and even cells is depicted.

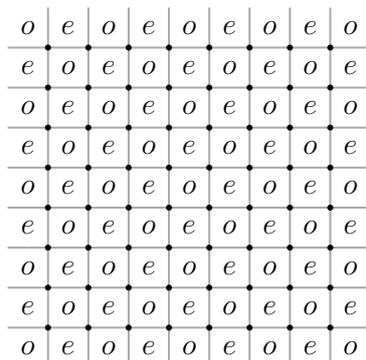


Figure 1: The graph  $G_8$  with its odd and even cells.

**Definition 1.** A *fully packed loop configuration* (FPL) of size  $n$  is a subgraph  $F$  of  $G_n$  satisfying that

1. each vertex of  $G_n$  is incident to two edges of  $F$ , and
2. precisely every other external edge belongs to  $F$ .

Given an FPL  $F$  a *cell of  $F$*  is defined as a cell of  $G_n$  together with those of its surrounding edges that belong to  $F$ . An example of an FPL is given in Figure 2. In a natural way, every FPL defines a non-crossing matching of the occupied external edges – its so-called *link pattern* – by matching those which are joined by a path.

In the course of the study of FPLs corresponding to fixed link patterns with a sufficiently large number of *nested arches*, TFPLs first occurred: such FPLs admit a combinatorial decomposition, in which TFPLs naturally arise. This combinatorial decomposition first came up in the course of the proof in [3] of a conjecture in [10] stating that if we introduce  $m$  nested arches in a fixed link pattern  $\pi$  then the number of FPLs corresponding to this link pattern is a polynomial function in  $m$  as  $m$  varies.

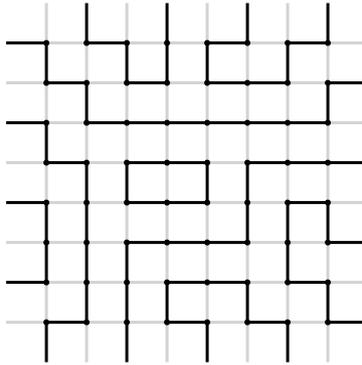


Figure 2: An FPL of size 8.

## 2.2 Triangular fully packed loop configurations

To give the definition of triangular fully packed loop configurations we need the following graph:

**Definition 2** (The graph  $G^N$ ). Let  $N$  be a positive integer. The graph  $G^N$  is defined as the induced subgraph of the square grid made up of  $N$  consecutive centered rows of  $3, 5, \dots, 2N + 1$  vertices from top to bottom together with  $2N + 1$  vertical *external* edges incident to the  $2N + 1$  bottom vertices.

In the following, let  $\mathcal{L}^N = \{L_1, L_2, \dots, L_N\}$  (*resp.*  $\mathcal{R}^N = \{R_1, R_2, \dots, R_N\}$ ) be the set made up of the leftmost (*resp.* rightmost) vertices of the  $N$  rows of  $G^N$ , where the vertices are numbered from left to right. Furthermore the  $N(N + 1)$  unit squares of  $G^N$ , including external unit squares that have three surrounding edges only, are said to be the *cells* of  $G^N$ . They are partitioned into *odd* and *even* cells in a chessboard manner where by convention the top left cell of  $G^N$  is odd. In Figure 3 the graph  $G^7$  together with its odd and even cells is pictured.

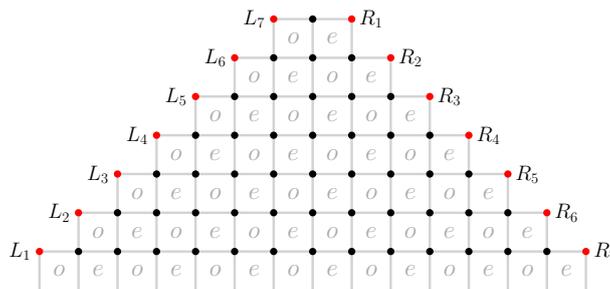


Figure 3: The graph  $G^7$  with its odd and even cells.

**Definition 3** ([4]). Let  $N$  be a positive integer. A *triangular fully packed loop configuration* (TFPL) of size  $N$  is a subgraph  $f$  of  $G^N$  such that:

- (i) Every other external edge starting with the second one belongs to  $f$ .



(ii) the set of Young diagrams fitting in the rectangle with  $k$  rows and  $\ell$  columns.

In Figure 5 an example for the bijection between binary words and Young diagrams is given. The Young diagram corresponding to a binary word  $\sigma$  is denoted by  $\lambda(\sigma)$ . Furthermore,  $\lambda(\tau) \subseteq \lambda(\sigma)$  means that the Young diagram  $\lambda(\tau)$  is included in the Young diagram  $\lambda(\sigma)$  and  $|\lambda(\sigma)|$  denotes the number of cells of the Young diagram  $\lambda(\sigma)$ . Note that  $|\lambda(\sigma)|$  coincides with the number of inversions of the binary word  $\sigma$ .

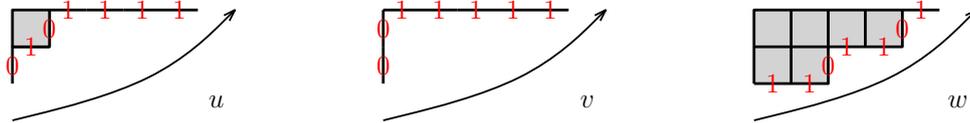


Figure 5: The Young diagrams which correspond to the boundary  $(0101111, 0011111; 1101101)$  of the TFPL in Figure 4.

**Theorem 6.** *[[3, 8, 4]] In order for a TFPL configuration with boundary  $(u, v; w)$  to exist, the following must be satisfied:*

$$|u|_0 = |v|_0 = |w|_0, \tag{1}$$

$$\lambda(u) \subseteq \lambda(w) \text{ and } \lambda(v) \subseteq \lambda(w), \tag{2}$$

$$|\lambda(u)| + |\lambda(v)| \leq |\lambda(w)|. \tag{3}$$

Conditions (1) and (2) are reasonably easy to prove. In Section 5, we will provide a new proof of Condition (3) using Wieland drift on TFPLs.

To end this section we need certain skew shapes which play an important role in the context of left- and right-Wieland drift. A skew shape is said to be a *horizontal strip* (*resp. a vertical strip*) if each of its columns (*resp. rows*) contains at most one cell. Examples are given in Figure 6.

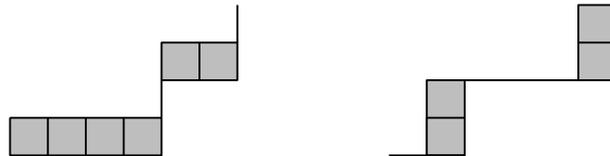


Figure 6: The horizontal strip  $\lambda(1111001100)/\lambda(0111100110)$  and the vertical strip  $\lambda(1100111100)/\lambda(1001111001)$ .

Consider two binary words  $\sigma$  and  $\tau$  satisfying  $|\sigma|_1 = |\tau|_1$  and  $|\sigma|_0 = |\tau|_0$ . Then the skew shape  $\lambda(\tau)/\lambda(\sigma)$  is a horizontal strip (*resp. a vertical strip*) if and only if for each  $j \in \{1, \dots, |\sigma|_1\}$  (*resp.  $j \in \{1, \dots, |\sigma|_0\}$* ) the following holds: *If  $\sigma_i$  is the  $j$ -th one (resp. zero) in  $\sigma$ , then  $\tau_{i-1}$  or  $\tau_i$  (resp.  $\tau_i$  or  $\tau_{i+1}$ ) is the  $j$ -th one (resp. zero) in  $\tau$ .*

In the following, if the skew shaped Young diagram  $\lambda(\tau)/\lambda(\sigma)$  is a horizontal strip (*resp. a vertical strip*), we will write  $\sigma \xrightarrow{h} \tau$  (*resp.  $\sigma \xrightarrow{v} \tau$* ).

### 3 Wieland drift for TFPLs

In this subsection the definitions of left- and right-Wieland drift for TFPLs are given and some first properties are derived. The starting point is the definition of Wieland gyration for FPLs. It is composed of local operations on all *active* cells of an FPL: the active cells of an FPL can be chosen to be either all its odd cells or all its even cells.

Let  $F$  be an FPL and  $c$  be an active cell of  $F$ . Then we must distinguish two cases, namely whether  $c$  contains precisely two edges of  $F$  on opposite sides or not. If this is the case then Wieland gyration  $W$  leaves  $c$  invariant. Otherwise, the effect of  $W$  on  $c$  is that edges and non-edges of  $F$  are exchanged. In Figure 7 the action of  $W$  on an active cell is illustrated. The result of applying  $W$  to each active cell of  $F$  is said to be the image of  $F$  under Wieland gyration and is denoted by  $W(F)$ .

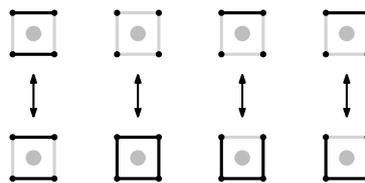


Figure 7: Up to rotation, the action of  $W$  on the active cells of an FPL.

In Figure 8 the image of the FPL depicted in Figure 2 under Wieland gyration with the odd cells being active is pictured.

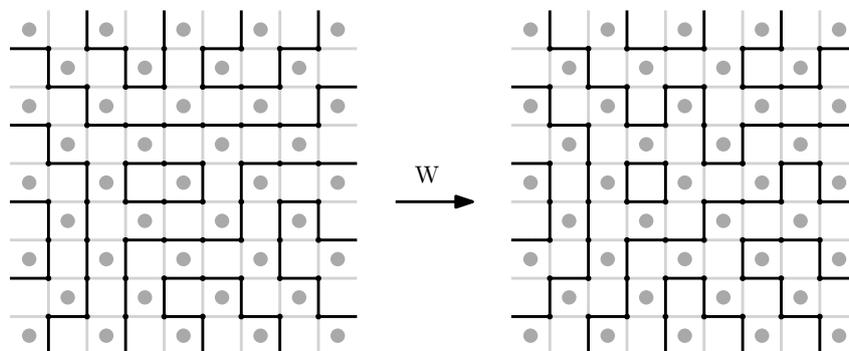


Figure 8: The image of the FPL depicted in Figure 2 under Wieland gyration with the odd cells being active.

Wieland drift as it will be defined for TFPLs is based on the operation  $W$ . Active cells of a TFPL can be chosen as either all its odd cells or all its even cells. Choosing all odd cells as active cells will lead to what will be defined as left-Wieland drift, whereas choosing all even cells as active cells will lead to what will be defined as right-Wieland drift.

**Definition 7** (Left-Wieland drift). Let  $f$  be a triangular fully packed loop configuration with left boundary word  $u$  and let  $u^-$  be a binary word such that  $u^- \xrightarrow{h} u$ . The *image of  $f$  under left-Wieland drift with respect to  $u^-$*  is determined as follows:

1. Insert a vertex  $L'_i$  to the left of  $L_i$  for  $1 \leq i \leq N$ . Then run through the occurrences of ones in  $u^-$ : Let  $\{i_1 < i_2 < \dots < i_{N_1}\} = \{i | u^-_{i} = 1\}$ .
  - (a) If  $u_{i_j}$  is the  $j$ -th one in  $u$ , add a horizontal edge between  $L'_{i_j}$  and  $L_{i_j}$ .
  - (b) If  $u_{i_{j-1}}$  is the  $j$ -th one in  $u$ , add a vertical edge between  $L'_{i_j}$  and  $L_{i_{j-1}}$ .
2. Apply Wieland gyration to each odd cell of  $f$ .
3. Delete all vertices in  $\mathcal{R}^N$  and their incident edges.

After shifting the whole construction one unit to the right, one obtains the desired image  $WL_{u^-}(f)$ .

In the case  $u^- = u$ , we will simply write  $WL(f)$  and speak of the *image of  $f$  under left-Wieland drift*.

In the following, to distinguish between vertices in  $f$  and in  $WL_{u^-}(f)$  the following notation is chosen: when regarding the image under left-Wieland drift with respect to  $u^-$ , we will write  $x'$  for each vertex  $x$  of  $G^N$ .

In Figure 9 the TFPL depicted in Figure 4 with its odd cells marked by gray discs and its image under left-Wieland drift with respect to 0011111 are pictured. It is a TFPL with boundary  $(0011111, 0101111; 1101101)$ . Note that the left boundary of the TFPL pictured in Figure 4 is 0101111 and  $0011111 \xrightarrow{h} 0101111$ . Also, the new right boundary 0101111 and the right boundary 0011111 of the preimage satisfy that  $0011111 \xrightarrow{v} 0101111$ . This turns out to hold in general:

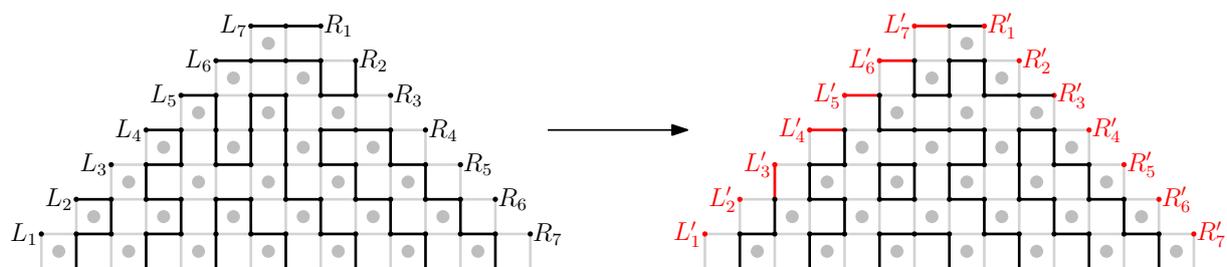


Figure 9: The TFPL depicted in Figure 4 and its image under left-Wieland drift with respect to 0011111.

**Proposition 8.** Let  $f$  be a TFPL with boundary  $(u, v; w)$  and let  $u^-$  be a binary word satisfying  $u^- \xrightarrow{h} u$ . Then  $WL_{u^-}(f)$  is a TFPL with boundary  $(u^-, v^+; w)$  where  $v^+$  is a binary word satisfying  $v \xrightarrow{v} v^+$ .

*Proof.* First we have to check that  $WL_{u^-}(f)$  indeed is a TFPL, that is the four conditions in Definition 3 must be satisfied. By definition the vertices  $L'_1, L'_2, \dots, L'_N$  have degree 0 or 1. For the degree of  $R'_i$  to be 2 in  $WL_{u^-}(f)$  the vertex to the left of  $R_i$  would need to be adjacent both to  $R_{i-1}$  and  $R_i$  in  $f$ , which is excluded since no path in  $f$  joins two vertices in  $\mathcal{R}^N$  by Definition 3(iv). Thus the vertices  $R'_1, R'_2, \dots, R'_N$  have degree 0 or 1 in  $WL_{u^-}(f)$ . All other vertices have degree 2 in  $WL_{u^-}(f)$  since they simply come from the application of  $W$  to cells of  $f$ . Finally let  $f'$  denote the configuration that is obtained before the vertices of  $\mathcal{R}^N$  are deleted. Since Wieland gyration preserves the connectivity of path endpoints in each active cell this is also true in  $f'$ . Thus a path in  $WL_{u^-}(f)$  neither joins two vertices in  $\mathcal{L}^{N'}$  nor two vertices in  $\mathcal{R}^{N'}$  and by that Definition 3(iv) is satisfied.

It remains to check the assertion on the boundary. The left boundary of  $WL_{u^-}(f)$  is  $u^-$  by construction. The right boundary  $v^+$  of  $WL_{u^-}(f)$  satisfies  $v \xrightarrow{v} v^+$  by Proposition 10 below and the characterization of pairs  $\sigma, \sigma^+$  of binary words satisfying  $\sigma \xrightarrow{v} \sigma^+$  at the end of Section 2. Finally, the bottom boundary of  $WL_{u^-}(f)$  is  $w$  because Wieland gyration preserves the connectivity of path endpoints in each active cell.  $\square$

The lemma below treats the effects of left-Wieland drift along the right boundary of a TFPL.

**Lemma 9.** *Let  $f, u^-, v^+$  be as in Proposition 8. Then  $v^+ \neq v$  if and only if there exists a vertex in  $\mathcal{R}^N$  which is incident to a vertical edge of  $f$ .*

*Proof.* We denote by  $x_s$  the vertex to the left of  $R_s$  for all  $1 \leq s \leq N$ .

Let  $f$  be a TFPL with a vertex  $R_j$  incident to a vertical edge and pick  $j$  minimal. Then  $x_j$  is necessarily adjacent both to the vertex to its left and to the vertex below so by Wieland drift  $R'_j$  is of degree 0 in  $WL_{u^-}(f)$ . Since  $R_j$  is of degree 1 this shows  $v \neq v^+$ .

Conversely, suppose that  $v^+ \neq v$ . By Proposition 8 there exists necessarily a  $j \in \{1, 2, \dots, N-1\}$  such that  $v_j = 0$  and  $v_j^+ = 1$ .  $R'_j$  is of degree 0 in  $WL_{u^-}(f)$ , so  $x_j$  is adjacent in  $f$  both to the vertex to its left and to the vertex below it. Since  $R_j$  is of degree 1 it is necessarily incident to a vertical edge.  $\square$

As a byproduct of the previous proof one can in fact precisely describe the right boundary  $v^+$  as follows:

**Proposition 10.** *Conserve the hypotheses of Lemma 9. For each  $i$  such that  $R_i$  is adjacent to a horizontal edge (resp. a vertical edge) then  $v_i^+ = 0$  (resp.  $v_{i+1}^+ = 0$ ). All other values  $v_j^+$ 's are equal to 1.*

**Right-Wieland drift.** In the definition of left-Wieland drift the active cells are all odd cells of a TFPL. When selecting all even cells of a TFPL as active cells *right-Wieland drift* is obtained. It depends on a binary word  $v^-$  satisfying  $v^- \xrightarrow{v} v$  that encodes what happens along the right boundary of a TFPL with right boundary  $v$  and is denoted by  $WR_{v^-}$  respectively  $WR$  if  $v^- = v$ . It is defined in an obvious way as the symmetric version of left-Wieland drift. We shall simply illustrate it with an example in Figure 10.

There are immediate symmetrical versions of Propositions 8 and 10 for  $WR$  which we record:

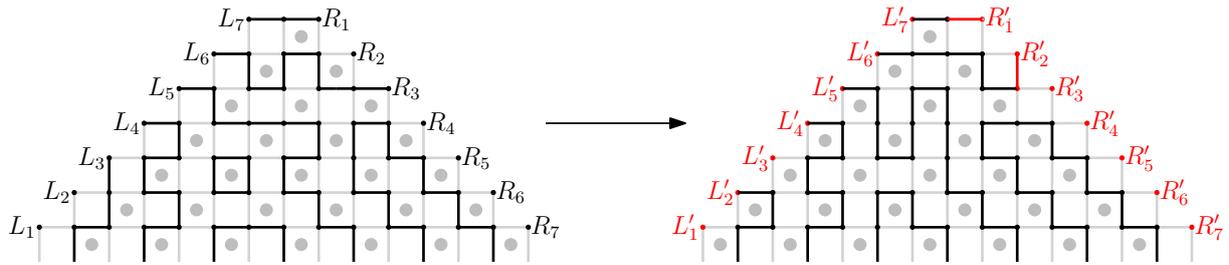


Figure 10: A TFPL and its image under right-Wieland drift with respect to 0011111.

**Proposition 11.** *The image of a TFPL with boundary  $(u, v; w)$  under right-Wieland drift with respect to  $v^-$  is a TFPL with boundary  $(u^+, v^-; w)$  where  $u^+$  is a binary word satisfying  $u \xrightarrow{h} u^+$ .*

**Proposition 12.** *Keep the notations of the previous proposition. For each index  $i$  such that  $L_i$  is adjacent to a horizontal edge (resp. a vertical edge), there holds  $u_i^+ = 1$  (resp.  $u_{i-1}^+ = 1$ ). All other values  $u_j^+$ 's are equal to 0.*

Given a TFPL with right boundary  $v$ , the effect of left-Wieland drift along the right boundary of the TFPL is inverted by right-Wieland drift with respect to  $v$ . On the other hand, given a TFPL with left boundary  $u$  the effect of right-Wieland drift along the left boundary is inverted by left-Wieland drift with respect to  $u$ . Since Wieland gyration is an involution on each cell, it follows:

**Theorem 13.** 1. *Let  $f$  be a TFPL with boundary  $(u^+, v; w)$  and  $u$  be a binary word such that  $u \xrightarrow{h} u^+$ . Then*

$$\text{WR}_v(\text{WL}_u(f)) = f.$$

2. *Let  $f$  be a TFPL with boundary  $(u, v^+; w)$  and  $v$  be a binary word such that  $v \xrightarrow{v} v^+$ . Then*

$$\text{WL}_u(\text{WR}_v(f)) = f.$$

*Remark 14.* It is perhaps useful to point out that  $\text{WR}(\text{WL}(f)) \neq f$  in general. Indeed by Lemma 9 equality will hold precisely when all vertices  $R_i$  of degree one are adjacent to horizontal edges.

In Section 4 we will study the behaviour of TFPLs under iterated applications of WL. In Figure 11 an example of a TFPL to which left-Wieland drift is repeatedly applied is depicted: one checks that the last TFPL in the sequence is invariant under left-Wieland drift. In the following, a TFPL that is invariant under left-Wieland drift is said to be *stable*.

Given a TFPL  $f$ , the sequence  $(\text{WL}^m(f))_{m \geq 0}$  is eventually periodic since there are only finitely many TFPLs of a fixed size. The length of this period is in fact always 1.

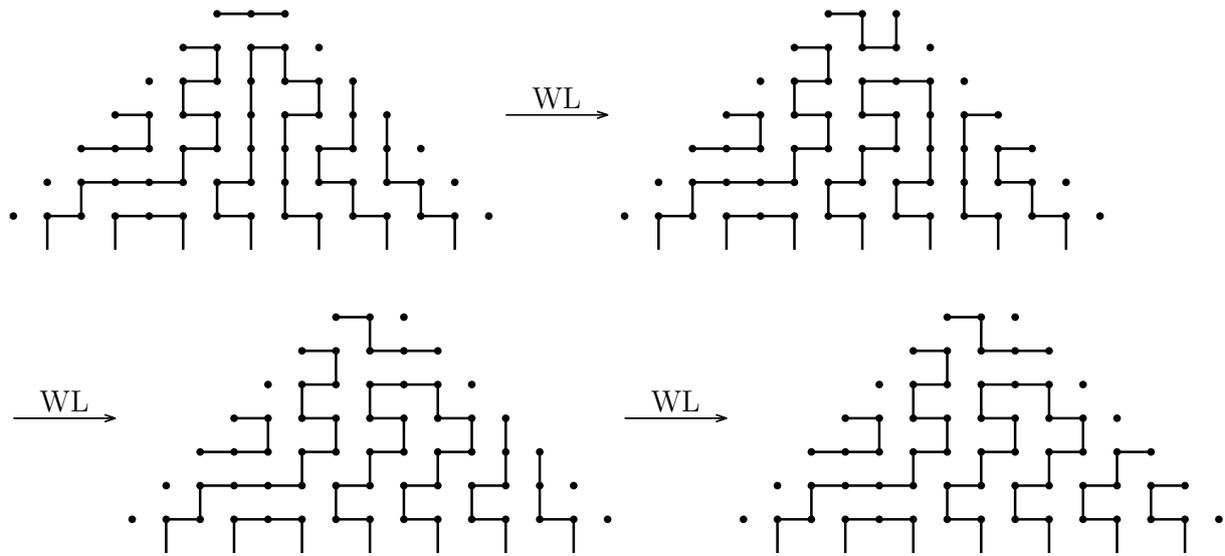


Figure 11: A TFPL to which left-Wieland drift is repeatedly applied.

**Theorem 15.** *Let  $f$  be a TFPL of size  $N$ . Then  $WL^{2N-1}(f)$  is stable, so that the following holds for all  $m \geq 2N - 1$  :*

$$WL^m(f) = WL^{2N-1}(f).$$

*The same holds for right-Wieland drift.*

For that purpose it is necessary to characterize TFPLs that are invariant under left-Wieland drift. Note that a TFPL is invariant under left-Wieland drift if and only if it is invariant under right-Wieland drift by Theorem 13.

## 4 Stable TFPLs

From now on the vertices of  $G^N$  are partitioned into odd and even vertices in a chessboard manner such that by convention the vertices in  $\mathcal{L}^N$  are odd. In our pictures the odd vertices are depicted by circles and the even vertices by squares. An example of a TFPL where the partition of its vertices into odd and even vertices is indicated is depicted in Figure 12.

It will be proved that stable TFPLs can be characterized as follows:

**Theorem 16.** *A TFPL is stable if and only if it contains no edge of the form  $\begin{array}{c} \bullet \\ \vdots \\ \blacksquare \end{array}$ .*

The TFPL depicted in Figure 12 is stable by Theorem 16.

**Definition 17.** An edge of the form  $\begin{array}{c} \bullet \\ \vdots \\ \blacksquare \end{array}$  is called a *drifter*.

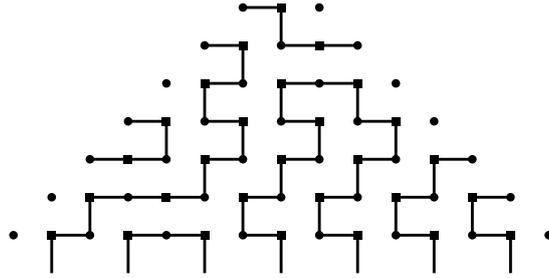


Figure 12: The bottom right TFPL configuration of size 7 in Figure 11 with its odd (*resp.* even) vertices illustrated by circles (*resp.* squares).

In the following, the possible interior cells of a TFPL play an important role in the proofs. For convenience, notations for the 16 odd and 16 even cells of a TFPL are fixed. In Figure 13, the chosen notation can be seen.

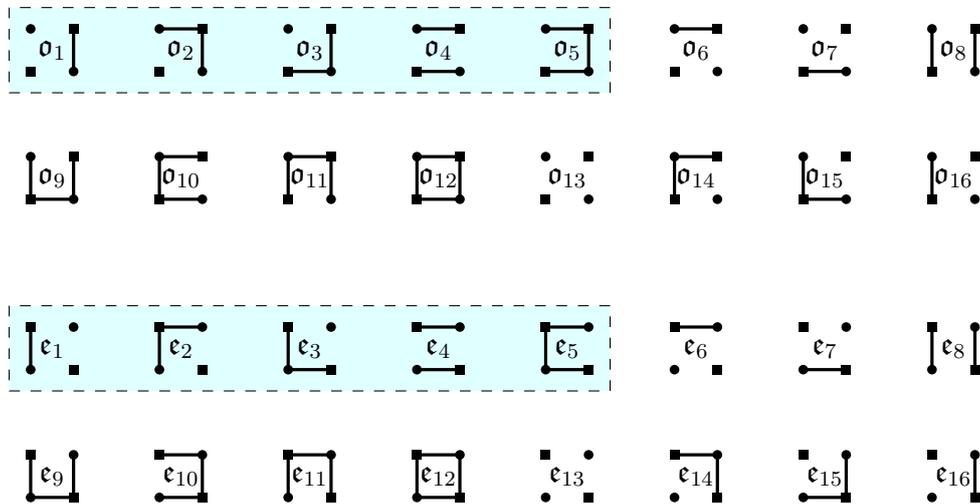


Figure 13: The notations for the 16 odd and 16 even cells of a TFPL, with emphasis on the subsets  $\mathcal{O} = \{o_1, o_2, o_3, o_4, o_5\}$  and  $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ .

#### 4.1 Characterization of stable TFPLs

To prove Theorem 16 we will begin by showing that a TFPL containing a drifter is not stable.

**Proposition 18.** *Let  $f$  be a TFPL that contains a drifter. Then  $WL(f) \neq f$ .*

*Proof.* If  $f$  contains a drifter incident to a vertex in  $\mathcal{R}^N$ , then by Lemma 9 we know that the right boundaries of  $f$  and  $WL(f)$  are different, so necessarily  $WL(f) \neq f$ .

We can now assume that no vertex in  $\mathcal{R}^N$  is incident to a drifter. Let  $\iota$  be a drifter in  $f$  with maximal  $x$ -coordinate and consider the odd cell  $o$  in  $f$  that contains  $\iota$ . Let  $x$

be the top right vertex of  $o$  and  $y$  be the bottom right vertex of  $o$ . By the choice of  $\iota$  the vertices  $x$  and  $y$  are not incident to a drifter.



Therefore,  $o \in \{\mathfrak{o}_8, \mathfrak{o}_9, \mathfrak{o}_{10}, \mathfrak{o}_{11}, \mathfrak{o}_{12}\}$ . If  $o$  is of the form  $\mathfrak{o}_8$  or  $\mathfrak{o}_{10}$  the vertex to the right of  $x'$  is incident to a drifter in  $WL(f)$ . In that case,  $WL(f) \neq f$  because the vertex to the right of  $x$  in  $f$  is not incident to a drifter by assumption. If  $o$  is of the form  $\mathfrak{o}_9, \mathfrak{o}_{11}$  or  $\mathfrak{o}_{12}$ , the vertices  $x'$  and  $y'$  are not adjacent in  $WL(f)$ . Thus,  $WL(f) \neq f$  because  $x$  and  $y$  are adjacent in  $f$ .  $\square$

To prove that a TFPL without a drifter is indeed stable we need to determine the types of cells which may occur. Define  $\mathfrak{D} = \{\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3, \mathfrak{o}_4, \mathfrak{o}_5\}$  and  $\mathfrak{E} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_5\}$ .

**Lemma 19.** *If  $f$  is a TFPL without drifters then all interior odd cells belong to  $\mathfrak{D}$ , while all of its interior even cells belong to  $\mathfrak{E}$ .*

*Proof.* Let  $f$  be a TFPL without a drifter, and  $o$  be one of its interior odd cells. Since  $o$  has no drifter it can only belong to  $\mathfrak{D}$  or have one of the types  $\mathfrak{o}_6, \mathfrak{o}_7$  or  $\mathfrak{o}_{13}$ . But in types  $\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{13}$ , at least one of the right vertices of  $o$  would be incident to a drifter, which is excluded.

The case of even cells is entirely analogous.  $\square$

Furthermore, in a TFPL with no drifter each odd cell has a uniquely determined even cell to its right.

**Lemma 20.** *Let  $f$  be a TFPL without drifters,  $o$  an odd cell of  $f$  and  $e$  the even cell of  $f$  to the right of  $o$ . If  $o$  and  $e$  are interior, then they can only occur as part of one of the following pairs:*



On the other hand, if  $o$  or  $e$  contains an external edge, then  $o$  and  $e$  can only occur as part of one of the following pairs:



*Proof.* Here, only the case when  $o$  is an interior odd cell and  $o = \mathfrak{o}_1$  is considered, the other cases being similar. Obviously, the cell  $e$  cannot equal  $\mathfrak{e}_4$ . But it cannot equal  $\mathfrak{e}_1, \mathfrak{e}_2$  or  $\mathfrak{e}_3$  either, since otherwise one of the right vertices of  $o$  would be incident to a drifter. The only remaining possibility is that  $e$  is of type  $\mathfrak{e}_5$  by Lemma 19.  $\square$

We can now complete the proof of Theorem 16 by showing that a TFPL without drifters is invariant under left-Wieland drift.

**Proposition 21.** *If  $f$  is a TFPL without drifters then  $WL(f) = f$ .*

*Proof.* Let  $o$  be an odd cell of  $f$  and  $e$  be the even cell to its right. By Lemma 20,  $e$  is uniquely determined by  $o$ . The crucial observation is that  $e$  coincides with the image of  $o$  under Wieland gyration. Thus each even cell of  $f$  and its corresponding even cell of  $WL(f)$  coincide. By definition all edges and non-edges of  $f$  incident to a vertex in  $\mathcal{L}^N$  are preserved by left-Wieland drift. In summary,  $WL(f) = f$ .  $\square$

## 4.2 TFPLs are eventually stable under Wieland drift

In this subsection we will prove Theorem 15. The idea of the proof is the following: when applying left-Wieland drift to a TFPL, the drifters of the TFPL are globally moved to the right. Thus after a finite number of applications of left-Wieland drift all drifters eventually disappear through the right boundary. As a consequence of Theorem 16 a stable TFPL is then obtained.

In a TFPL of size  $N$ , there are  $2N + 1$  columns of vertices which we label from left to right by 1 to  $2N + 1$ .

**Proposition 22.** *Let  $f$  be a TFPL of size  $N$  that contains a drifter in the  $n$ -th column but no drifter in the columns  $1, \dots, n - 1$  to its left. Then  $WL(f)$  contains no drifter in any of the columns  $1, \dots, n$ .*

*Proof.* First of all, notice that by the definition of left-Wieland drift there is no vertex of  $\mathcal{L}^{N'}$  incident to a drifter in  $WL(f)$ . By definition of  $WL$  the occurrence of a drifter in an even cell  $e'$  of  $WL(f)$  depends solely on the odd cell to the left of the corresponding even cell  $e$  in  $f$ . By assumption, no odd cell of  $f$  occurring to the left of the  $(n - 1)$ -st column has a vertex incident to a drifter. It follows from the proof of Lemma 19 that all these odd cells belong to  $\mathfrak{D}$ . This entails that all even cells of  $WL(f)$  to the left of the  $n$ -th column belong to  $\mathfrak{E}$  and thus do not contain a drifter. Since these even cells cover all vertical edges in the columns  $1, \dots, n$  which have an odd vertex as their top vertex and an even vertex as their bottom vertex, the proof is complete.  $\square$

*Proof of Theorem 15.* By immediate induction on the result of Proposition 22 we know that the configuration  $WL^{2N+1-n}(f)$  contains no drifter. Thus by Theorem 16 it is stable under  $WL$ , so

$$WL^m(f) = WL^{2N+1-n}(f) \tag{4}$$

for all  $m \geq 2N + 1 - n$ . Since the first column of vertices of a TFPL consists only of the vertex  $L_1$ , we have  $n \geq 2$ , which proves the theorem.  $\square$

## 5 Applications of Wieland drift on TFPLs

### 5.1 Some linear relations

The following was conjectured for Dyck words in [8] and proved in [5] using Wieland gyration on FPLs.

**Proposition 23.** *Let  $u, v$  and  $w$  be binary words. Then*

$$\sum_{u^+ : u \xrightarrow{h} u^+} t_{u^+, v}^w = \sum_{v^+ : v \xrightarrow{v} v^+} t_{u, v^+}^w.$$

*Proof.* Indeed the function  $WL_u(\cdot)$  acts on all TFPLs with boundary  $(u^+, v; w)$  while  $WR_v(\cdot)$  acts on TFPLs with boundary  $(u, v^-; w)$ . By Theorem 13 these functions are inverses of one another, so the result is obtained by taking cardinalities.  $\square$

## 5.2 The inequality in Theorem 6

This states that  $|\lambda(u)| + |\lambda(v)| \leq |\lambda(w)|$  always holds for the boundaries  $(u, v; w)$  of TFPLs. It was given in [8, Lemma 3.7] in the Dyck word case. Later, another proof in connection with TFPLs together with an orientation of the edges was given in [4]. More precisely, it was shown there that in an oriented TFPL with boundary  $(u, v; w)$ , the quantity  $|\lambda(w)| - |\lambda(u)| - |\lambda(v)|$  counts occurrences of certain local patterns in the TFPL.

We now give an independent proof based on the properties of Wieland drift; the idea for this proof comes from the original one by Thapper, which can be seen as relying on Wieland gyration on FPLs in an indirect way.

*Proof of Theorem 6(3).* Let  $f$  be a TFPL with boundary  $(u, v; w)$ . The proof is done by induction on  $|\lambda(u)|$ . In the case when  $|\lambda(u)| = 0$  we have  $\lambda(v) \subseteq \lambda(w)$  by Theorem 6(2), which implies  $|\lambda(v)| \leq |\lambda(w)|$ .

Assume now  $|\lambda(u)| \geq 1$ . By removing a corner of  $\lambda(u)$ , there exists a Young diagram  $\lambda(u^-) \subseteq \lambda(u)$  with one cell less than  $\lambda(u)$ . In particular  $\lambda(u)/\lambda(u^-)$  is a horizontal strip.

We first want to prove that there exists an  $i > 0$  such that  $WL_{u^-}^i(f)$  has right boundary  $v^+ \neq v$ . Assume the contrary, that is the right boundary of  $WL_{u^-}^i(f)$  is  $v$  for all  $i > 0$ . Since there are only a finite number of TFPLs with boundary  $(u^-, v; w)$  there exist integers  $i_0, p > 0$  such that

$$WL_{u^-}^{i_0+p}(f) = WL_{u^-}^{i_0}(f).$$

We can then apply  $WR_v^{i_0}$  to both sides of the identity and by Theorem 13 we obtain  $WL_{u^-}^p(f) = f$ . But these configurations have left boundaries  $u, u^-$  respectively and we assumed  $u^- \neq u$ , which is a contradiction.

Hence, let  $i$  be a positive integer such that  $WL_{u^-}^i(f)$  has boundary  $(u^-, v^+; w)$  where  $v^+ \neq v$ . By Proposition 8 we have  $\lambda(v) \subsetneq \lambda(v^+)$  and therefore  $|\lambda(v)| + 1 \leq |\lambda(v^+)|$ . Applying the induction hypothesis to  $WL_{u^-}^i(f)$  completes the proof:

$$|\lambda(u)| + |\lambda(v)| = |\lambda(u^-)| + 1 + |\lambda(v)| \leq |\lambda(u^-)| + |\lambda(v^+)| \leq |\lambda(w)|.$$

$\square$

### 5.3 Excesses 0, 1 and beyond

For a TFPL with boundary  $(u, v; w)$ , the nonnegative integer  $|\lambda(w)| - |\lambda(u)| - |\lambda(v)|$  is called the *excess* of  $f$ .

**Proposition 24.** *If a TFPL has excess 0, then it is stable.*

*Proof.* It is a consequence of [4, Proposition 5.2] that TFPLs of excess 0 do not contain drifters, so we can conclude with Theorem 16.  $\square$

These TFPLs are known to be counted by Littlewood–Richardson coefficients [6, 4] as recalled in the introduction.

In [4], configurations of excess 1 were also studied in some detail and enumerated. The authors defined a number of *moves* on such (oriented) configurations in order to transform them and ultimately reach a configuration of excess 0. It turns out that these complicated moves are essentially equivalent to a simple application of WL, at least when the configuration is not stable.

Therefore stable configurations should first be studied and enumerated and then other configurations may be related to them through Wieland drift in order to find, for instance, linear relations between their cardinalities. The feasibility of such an approach is in particular supported by Theorem 15.

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