Conditions for the parameters of the block graph of quasi-symmetric designs*

Rajendra M. Pawale
Department of Mathematics
University of Mumbai
Vidyanagari, Mumbai-400 098, India
rmpawale@yahoo.co.in

Mohan S. Shrikhande
Mathematics Department
Central Michigan University
Mount Pleasant, MI, 48859, U.S.A.
Mohan.Shrikhande@cmich.edu

Shubhada M. Nyayate
Department of Mathematics
Dnyanasadhana College
Thane-400 604, India
nyayate.shubhada@gmail.com

Submitted: Dec 10, 2013; Accepted: Jan 25, 2015; Published: Feb 16, 2015
Mathematics Subject Classifications: 05B05; 05B30; 05B25

Abstract

A quasi-symmetric design (QSD) is a 2-(v, k, λ) design with intersection numbers x and y with $x < y$. The block graph of such a design is formed on its blocks with two distinct blocks being adjacent if they intersect in y points. It is well known that the block graph of a QSD is a strongly regular graph (SRG) with parameters (b, a, c, d) with smallest eigenvalue $-m = -\frac{k-x}{y-x}$.

The classification result of SRGs with smallest eigenvalue $-m$, is used to prove that for a fixed pair $(\lambda \geqslant 2, m \geqslant 2)$, there are only finitely many QSDs. This gives partial support towards Marshall Hall Jr.'s conjecture, that for a fixed $\lambda \geqslant 2$, there exist finitely many symmetric (v, k, λ)-designs.

We classify QSDs with $m = 2$ and characterize QSDs whose block graph is the complete multipartite graph with s classes of size 3. We rule out the possibility of a QSD whose block graph is the Latin square graph $LS_m(n)$ or complement of $LS_m(n)$, for $m = 3, 4$.

SRGs with no triangles have long been studied and are of current research interest. The characterization of QSDs with triangle-free block graph for $x = 1$ and $y = x+1$ is obtained and the non-existence of such designs with $x = 0$ or $\lambda > 2(x+2)$ or if it is a 3-design is proven. The computer algebra system Mathematica is used to find parameters of QSDs with triangle-free block graph for $2 \leqslant m \leqslant 100$. We also give the parameters of QSDs whose block graph parameters are (b, a, c, d) listed in Brouwer's table of SRGs.

Keywords: Quasi-symmetric design, Strongly regular graph, Block graph

*Dedicated to the memory of Damaraju Raghavarao (1938-2013).

1 Introduction

A 2-(v,k,λ) design $\mathbf{D}$ (with b blocks and r blocks through a given point) is called *quasi-symmetric* if the sizes of the intersection of two distinct blocks take only two values x and y, with $(0 \leqslant x < y < k)$. We can create a graph $\Gamma = \Gamma(\mathbf{D})$, called the *block graph* of $\mathbf{D}$, by joining two distinct blocks if they have the larger intersection number y. This graph is a *strongly regular graph* with parameters (b,a,c,d). Here, as is customary, b denotes the number of vertices of Γ, a the degree of Γ, c (resp. d), the number of vertices both adjacent (resp. non-adjacent) to any two adjacent (resp. non-adjacent) vertices. In addition, to avoid trivial cases, we assume that a strongly regular graph is neither the trivial graph nor the complete graph. Furthermore, the adjacency matrix of Γ has smallest eigenvalue $-m$, where $m = \frac{k-x}{y-x}$. From now on, we use QSD(resp. SRG) to denote a quasi-symmetric design (resp. strongly regular graph).

It is possible to create a series of equations among the parameters of the QSD. In addition, we also have conditions on the parameters of the associated SRG. These conditions are summarized in lemmas 1 through 5 (in section 2). Lemma 5 expresses these conditions involving thirteen variables $a,b,c,d,m,n,v,r,k,\lambda,x,y,z$, where $z = y-x$, and $n = \frac{r-\lambda}{y-x}$. Since the parameter m is an integer involving the design parameters, we can take advantage of the fact that $m \geqslant 2$ and the results of Seidel [32] and Neumaier[23] for SRGs with least eigenvalue $-m$. This gives nine equations (Lemma 5), which are necessary conditions for the existence of a QSD. Mathematica is used in section 2 to rewrite these conditions into a sequence of necessary conditions for the existence of a QSD.

In this paper, we consider mostly proper QSDs, i.e. designs in which both the intersection numbers occur. If this condition is relaxed, and we allow at most two intersection numbers, then the family of proper and improper QSDs will include *symmetric* designs. Finite affine planes provide an infinite class of proper QSDs with $\lambda = 1$, while finite projective planes give an infinite class of improper QSDs. So if $\lambda = 1$, there are infinitely many proper or improper QSDs. There is a famous open conjecture of Marshall Hall, Jr. [15], Hall's Conjecture: For a fixed $\lambda \geqslant 2$, there are only finitely many symmetric designs (i.e. improper QSDs), whose 'λ-value' is the given λ. For a fixed $\lambda \geqslant 2$, let G_λ denote the class of all QSDs, proper or improper(i.e. symmetric designs). In a personal communication to the authors of [30], N.M. Singhi [40] made the following conjecture, Singhi's Conjecture: For a fixed $\lambda \geqslant 2$, G_λ is finite. In ([30], Corollary 4.2), the equivalence of Hall's and Singhi's conjectures was shown. In partial support of these conjectures, we prove in section 3, that for a fixed pair $(\lambda \geqslant 2, m \geqslant 2)$ there are only finitely many QSDs.

From a QSD $\mathbf{D}$, we get the associated SRG $\Gamma(\mathbf{D})$. So a natural question is, given a SRG with the right parameters to be the block graph of a QSD, when is it in fact, the block graph of a QSD? In section 4, we prove some results concerning this question. The papers [39], [13], [14], and [11] contain some results on this problem. The first of these papers is phrased in the language of two class partially balanced designs (whose duals are SRGs) and uses Hasse-Minkowski theory, while the paper [11] relies on the equivalent theory of quadratic forms. According to [11]: "The question which strongly regular graphs are block graphs of quasi-symmetric designs is a difficult one and there is no chance for a general answer".

We prove, that if the smallest eigenvalue of Γ is -2 (i.e., $k = 2y - x$), then $\mathbf{D}$ is either a Hadamard 3-(v, k, λ) design or a *pair* design (i.e. a 2-$(v, k, 1)$ design, whose blocks are all unordered pairs of points) or the 2-$(6, 3, 2)$ design or complement of one of these designs. It is proved in ([13], Theorem 3.4) that the only QSD with the complete s-partite block graph with parameters $(2s, 2s-2, 2s-4, 2s-2), s \geqslant 2$ is the Hadamard 3-design. We prove that if $\mathbf{D}$ is a QSD whose block graph is the complete multipartite graphs with s classes of size 3, with parameters $(3s, 3(s-1), 3(s-2), 3(s-1))$, then $\mathbf{D}$ is a 2-$(9(1+2u), 6(1+2u), 5+12u)$ design with intersection numbers $3(1+2u)$ and $4(1+2u)$ or complement of this design (with $z = 2u + 1$).

In ([13], Theorem 3.5), the authors ruled out the possibility of a QSD whose block graph is the Latin square graph $LS_2(n)$ or its complement. We extend this result to QSDs whose block graph is the Latin square graph $LS_m(n)$ or complement of $LS_m(n)$, for $m = 3$ and 4.

In section 5, we consider QSDs, whose block graph, and **not** the complement of its block graph is triangle-free, as was assumed in [1]. The block graph Γ of a QSD $\mathbf{D}$ is triangle-free if and only if $\overline{\Gamma}$ has no co-cliques of size three. We rule out the possibility of a QSD whose block graph is triangle-free under each of the following conditions: $x = 0; \lambda > 2(x+2)$; or if it is a 3-design, or $(k-x)/(y-x) = 3, 4$ and 5. As a consequence, there does not exist a QSD with $x = 0$ and having a triangle-free block graph. This shows that the analogue of the conjecture in [20] (by requiring Γ, and not $\overline{\Gamma}$ to be triangle-free) is true. We also characterize QSDs with $k = 2y - x$ or $y = x + 1$ or $x = 1$ having a triangle-free block graph in terms of the 2-$(6, 3, 2)$ design with intersection numbers 1 and 2. Next we assume $x > 0$. We prove that a 2-$(56, 16, 6)$ design and its complement are the only QSDs having triangle-free block graph and $(k-x)/(y-x) = 6$. For higher values of $(k-x)/(y-x)$, we obtain the feasible parameters of QSDs with triangle-free block graph using Mathematica. Table 4 lists the feasible parameters of QSDs with triangle-free block graph for $(k-x)/(y-x) \leqslant 100$. We also provide other tables concerning QSDs, using Brouwer's table [6] of SRGs.

Section 5 concerns QSDs whose block graph has no triangles. SRGs with no triangles (i.e. $c = 0$), are called *triangle-free* and are of current interest (see [3], [4], [5], [9], [10], [19], [37]). Other than the trivial graph and complete bipartite graph $K_{3,3}$ there are only seven known triangle-free SRGs ([12], namely the 5-cycle graph, the Petersen graph, the Clebsch graph, the Hoffman-Singleton graph, the Gewirtz graph, the Higman-Sims graph, and a $(77, 16, 0, 4)$ subgraph of Higman-Sims graph). Out of these seven graphs only two graphs are block graphs of QSDs. The $(77, 16, 0, 4)$ graph is the block graph of 2-$(56, 16, 6)$ design and the Petersen graph is the block graph of a 2-$(6, 3, 2)$ design. Whether there is an eighth triangle-free SRG is a problem mentioned in [9] and [12], which is still open.

In this paper there are a large number of equations involving many variables. We used Mathematica, [41] to derive these. The reader also can use WxMaxima, [42] or any available CAS to handle the tedious calculations, which is available free on the web, if Mathematica is not available. Moreover, many background results are used and several new results are obtained. To make the paper more readable, we denote a result used without proof by *Result* and cite the reference. The results obtained in the paper are denoted by Theorems, Propositions, etc. In addition, we have provided four Appendices at the end of this paper. In Appendix 1, we give the Mathematica code used to find feasible parameters of QSDs for given parameters (b, a, c, d) of its associated SRG, using Brouwer's tables [6]. In Appendix 2, we give the Mathematica code to find feasible parameters of QSDs with triangle-free block graph for $2 \leqslant m \leqslant 100$. In Appendix 3, we give some results about QSDs having a block graph Γ, such that $\bar{\Gamma}$, the complement of Γ has no triangles.

In last Appendix 4, we obtain a result connecting SRGs, 2-distance sets in binary Hamming spaces, and QSDs. This was motivated by a recent paper of Ionin [18].

The main theorems obtained in this paper are the following:

Theorem 20. For a fixed pair (c, m) or (z, m) or $(\lambda \geqslant 2, m \geqslant 2)$, there exist only finitely many QSDs.

Theorem 21. If $\mathbf{D}$ is a QSD with $k = 2y - x$, then $\mathbf{D}$ is either a pair design or 2-$(6, 3, 2)$ design or Hadamard 3-design or complement of one of these designs.

Theorem 24. Let $\mathbf{D}$ be a QSD whose block graph is the complete multipartite graphs with s classes of size 3, with parameters $(3s, 3(s-1), 3(s-2), 3(s-1))$, then $\mathbf{D}$ is a 2-$(9(1+2u), 6(1+2u), 5+12u)$ design with intersection numbers $3(1+2u)$ and $4(1+2u)$ (with $z = 2u+1$) or the complement of this design.

Theorem 29. There is no QSD whose block graph is either a Latin square graph $LS_s(n); n \geqslant s$ or its complement for $s = 3, 4$ or a conference graph.

Theorem 32. Let $\mathbf{D}$ be a QSD with triangle-free block graph and $m = \frac{k-x}{y-x}$.

1. If $m = 2$, then $\mathbf{D}$ is a 2-$(6, 3, 2)$ design with $x = 1, y = 2$.
2. $m \neq 3, 4, 5$
3. If $m = 6$, then $\mathbf{D}$ is either a 2-$(56, 16, 6)$ design or its complement.

Theorem 38. Let $\mathbf{D}$ be a QSD with triangle-free block graph. If $x = 1$, or $y = x + 1$, then $\mathbf{D}$ is a design with parameters $v = 6, k = 3, \lambda = 2, r = 5, b = 10$ and $x = 1, y = 2$.

Theorem 39. There does not exist a QSD with triangle-free block graph, if either $x = 0$, or $\lambda > 2(x + 2)$, or if it is a 3-design.

2 Preliminaries

In this section, we give some preliminary results needed. The reader can refer to [2] and [37] or the cited references for details.

Lemma 1 ([2], [10], [37]). *Let* $\mathbf{D}$ *be a* $(v, b, r, k, \lambda; x, y)$ *QSD. Then the following relations hold:*

1. $vr = bk$ *and* $\lambda(v-1) = r(k-1)$;
2. $k(r-1)(x+y-1) - xy(b-1) = k(k-1)(\lambda-1)$;

Lemma 2 ([31],[37],[38]). *Let* Γ *be a connected SRG* (b, a, c, d), *Then* Γ *has three distinct eigenvalues* $\theta_0 = a$ *with multiplicity 1,* θ_1, θ_2, *where* θ_1, θ_2 *are the roots of the quadratic equation*

$$\rho^2 - (c-d)\rho - (a-d) = 0,$$

the multiplicities f *and* g *are positive integers given by*

$$f, g = \frac{1}{2}\left(b - 1 \pm \frac{(b-1)(c-d) + 2a}{\sqrt{(c-d)^2 + 4(a-d)}}\right).$$

and

$$c = a + \theta_1 + \theta_2 + \theta_1\theta_2, d = a + \theta_1\theta_2.$$

Lemma 3 ([37],Theorem 3.8). *Let* $\mathbf{D}$ *be a* $(v, b, r, k, \lambda; x, y)$ *QSD. Form the block graph* Γ *of* $\mathbf{D}$ *by taking as vertices the blocks of* $\mathbf{D}$, *where two distinct vertices are adjacent whenever the corresponding blocks intersect in* y *points. Assume* Γ *is connected. Then,* Γ *is a SRG with parameters* (b, a, c, d), *where the eigenvalues of* Γ *are given by* $a = \theta_0 = \frac{k(r-1)+(1-b)x}{y-x}, \theta_1 = \frac{r-\lambda-k+x}{y-x}$ *and* $\theta_2 = \frac{-(k-x)}{y-x}$.

Remark 4. We record for future use, some consequences of the above lemma:

$$r(kr - r + \lambda) - bk\lambda = 0. \tag{1}$$

The eigenvalues $\theta_0, \theta_1, \theta_2$ are integers, with $\theta_0 > 0, \theta_1 \geqslant 0$, and $\theta_2 < 0$;

$$a = \frac{k(r-1) + (1-b)x}{y-x}; \tag{2}$$

$$c = \frac{(x-k+r-\lambda)(x-k)}{(y-x)^2} + \frac{x-k}{y-x} + \frac{x-k+r-\lambda}{y-x} + \frac{k(r-1)+(1-b)x}{y-x}; \tag{3}$$

$$d = \frac{k(r-1)+(1-b)x}{y-x} + \frac{(x-k)(-k+r+x-\lambda)}{(y-x)^2}; \tag{4}$$

1. $y - x$ divides both $r - \lambda$ and $k - x$, so we take $y = z + x, k = mz + x$ and $r = nz + \lambda$, for positive integers m and n assuming $m \leqslant n$.

2. If $\lambda > 1$, then $\lambda \geqslant x+1$.

3. The block graph of $\mathbf{D}$ and block graph of $\overline{\mathbf{D}}$, the complement of the design $\mathbf{D}$, are isomorphic.

The following result giving necessary conditions on the block graph of a QSD is essentially the well known Integrality Condition for the parameters of a SRG ([17], Theorem 7.2.4, page 220).

Lemma 5. *Let D be a $(v,b,r,k,\lambda;x,y)$ QSD and Γ be the (b,a,c,d) SRG block graph of D. Then,*

1. $x = -\dfrac{z\left(-(m-1)(m+a)+m\left(\left(m^2-2m+a-c\right)z+(m-1)\,\lambda\right)\right)}{-b(m-1)+\left(m^2-2m+a-c\right)z+(m-1)\lambda}$;

2. $n = \dfrac{m^2-2m+a-c}{m-1}$;

3. $m^2-(d-c)m-(a-d)=0$;

4. $m = \dfrac{1}{2}\left(d-c+\sqrt{(d-c)^2+4(a-d)}\right)$;

5. $c-d=n-2m$ *and* $a-d=m(n-m)$;

6. *If* $d=c$, *then* $m=\sqrt{a-d}$ *and* $n=2m$;

7. $m-1$ *divides* $a-c-1$;

8. $\sqrt{(d-c)^2+4(a-d)}$ *is a perfect square and* $\left(d-c+\sqrt{(d-c)^2+4(a-d)}\right)$ *is an even integer;*

9. *If* $x \neq 0$ *then*
$$A_1x^2+B_1x+C_1=0, \tag{5}$$
where
$$\begin{aligned}
A_1 &= -\left(d\,(c-d+2\,m)\right);\\
B_1 &= \left(2\,d-c+(c-d-1)\;m+m^2\right)\,\left((m-1)d+(1+c-d)\;m^2+m^3\right)\\
&\quad -2\,d\,m\;(c-d+2\,m)\;z;\\
C_1 &= d\,z\;\left((m-1)d+(1+c-d)\;m^2+m^3+\left((d-c)\;m^2-2\,m^3\right)\;z\right),
\end{aligned}$$

where $z=y-x, m=\frac{k-x}{y-x}, n=\frac{r-\lambda}{y-x}$.

Proof. Substitute $y = z + x, k = mz + x, r = nz + \lambda$ in equations (2) and (3) and observe that $a - c = 2\,m - m^2 - n + m\,n$, which gives an expression for n as given in (2). Substitute this value of n in the equation (2) to get an expression for x as given in (1).

Use Mathematica to obtain the equation given in (3). Note that m is a positive root of this quadratic. The cases 4-8 follows from above and/or equations (2)-(4).

Use expression (3) to get $b = \frac{mnz^2 + \left(m^2 + (-n+\lambda-3)m - c + n(x+1)\right)z + x\lambda}{x}$. Substitute this expression for b and $n = 2m + c - d$ in (2) of Lemma 1 to get

$$\lambda = \frac{(2\,d - c + (c - d - 1)\ m + m^2)\ x + (c - d + 2\,m)\ x^2 + (d + ((c - d)\ m + 2\,m^2)\ x)\ z}{m\ (x + m\,z)}.$$

Now from the equation (1), we get (5). □

Lemma 6 ([31], Theorem 2.6)**.** *For a fixed value of the block size k, there exist only finitely many QSDs with $y \geqslant 2$.*

Lemma 7 ([26], Theorem 3.2)**.** *Let $\mathbf{D}$ be a proper QSD with the standard parameter set $(v, b, r, k, \lambda; x, y)$ with $x \neq 0$ and $z = y - x = 1$. Then $\mathbf{D}$ is a design with parameters given in (1) or (2) as follows or $\mathbf{D}$ is a complement of one of the design in (1).*

1. $v = (1 + m)(2 + m)/2, b = (2 + m)(3 + m)/2, r = m + 3, k = m + 1,$ $\lambda = 2, x = 1, y = 2$ *and* $m = 2, 3, \ldots.$
2. $v = 5, b = 10, r = 6, k = 3, \lambda = 3$ *and* $x = 1, y = 2.$

We will need later the following classification results about SRGs.

Result 8 ([7], Theorem 9.2.1; [10], Theorem 4.14)**.** Let Γ be a SRG with smallest eigenvalue -2. Then, Γ is one of

1. the complete n-partite graph, with parameters $(2n, 2n - 2, 2n - 4, 2n - 2), n \geqslant 2$,
2. the lattice graph, with parameters $(n^2, 2(n - 1), n - 2, 2), n \geqslant 3$,
3. the Shrikhande graph, with parameters $(16, 6, 2, 2)$,
4. the triangular graph, with parameters $(n(n - 1)/2, 2(n - 2), n - 2, 4), n \geqslant 5$,
5. one of the three Chang graphs, with parameters $(28, 12, 6, 4)$,
6. the Petersen graph, with parameters $(10, 3, 0, 1)$,
7. the Clebsch graph, with parameters $(16, 10, 6, 6)$,
8. the Schläfli graph, with parameters $(27, 16, 10, 8)$.

Result 9 ([23], Theorem 5.1)**.** Let Γ be a SRG with smallest eigenvalue $-m, m \geqslant 2$ integral. Then, Γ is one of

1. the complete multipartite graphs with s classes of size m, with parameters, $(ms, m(s-1), m(s-2), m(s-1))$,

2. the Latin square graphs $LS_m(n)$, with parameters $(n^2, m(n-1), n+m^2-3m, m(m-1))$,

3. the Steiner graphs $S_m(n)$, with parameters $(\frac{(m+n(m-1))(n+1)}{m}, mn, n+m^2-2m, m^2)$,

4. finitely many other graphs.

The following are known characterizations of QSDs with specific strongly regular block graphs.

Result 10 ([13], Theorem 3.4)**.** The only QSD whose block graph is the complete n-partite graph with parameters $(2n, 2n-2, 2n-4, 2n-2), n \geqslant 2$ is the Hadamard 3-design.

Result 11 ([13], Theorem 3.5)**.** There is no QSD whose block graph is the lattice graph with parameters $(n^2, 2(n-1), n-2, 2), n \geqslant 3$ or its complement.

Result 12 ([11], Proposition 1.1)**.** The block graph of a 2-(v, k, λ) QSD is a triangular graph, if and only if $\mathbf{D}$ is a *pair design* (i.e. a 2-$(v, 2, 1)$ design, whose blocks are all unordered pairs of points) or its complement.

Lemma 13. *Let* $\mathbf{D}$ *be a QSD whose block graph is the complete multipartite graph with* s *classes of size* m*, with the parameters* $(ms, m(s-1), m(s-2), m(s-1))$*. Then*

$$mx^2 + \left(zm^2 + (z-\lambda-1)m + \lambda\right)x + mz(mz + (1-m)\lambda - 1) = 0. \qquad (6)$$

Proof. As $d = c+m$ from Lemma 5 (5), we get $n = m$. Solve the equation $b = a+m$ for b to get $b = \frac{m^2z^2+mxz+m\lambda z+x\lambda}{x+z}$. Now use the equation (1) to get the equation (6). ☐

Lemma 14. *Let* $\mathbf{D}$ *be a QSD whose block graph is the Latin square graph* $LS_m(n)$*. Then*

$$n\,x^2 - n\,x\,(1-m+m\,n-2\,m\,z) + m\,z\,(-1+m-m\,n+m\,n\,z) = 0. \qquad (7)$$

Proof. Solve $d = m(m-1)$ for λ to get $\lambda = \frac{b\,x+m\,n\,z-n\,x\,z-m\,n\,z^2}{x+m\,z}$. Take $b = n^2$ and use equation (1), to get the equation (7). ☐

Lemma 15. *Let* $\mathbf{D}$ *be a QSD whose block graph is the Steiner graph* $S_m(n)$*. Then,*

$$x^2 + m^2\,(-1+z)\,z + x\,(-m+n-m\,n+2\,m\,z) = 0. \qquad (8)$$

Proof. Solve $d = m^2$ for λ to get $\lambda = \frac{b\,x+(m+m\,n-n\,x)\,z-m\,n\,z^2}{x+m\,z}$. Take $b = \frac{(1+n)\,(m-n+m\,n)}{m}$ and use equation (1) to get the equation (8). ☐

Lemma 16. *Let* $\mathbf{D}$ *be a QSD, with the non-zero intersection numbers* x *and* y *such that block graph of* $\mathbf{D}$ *is triangle-free. Then,*

$$A_2\lambda^2 + B_2\lambda + C_2 = 0, \tag{9}$$

where

$$\begin{aligned} A_2 &= (x+mz)\left(xm^3 + \left(z^2 - z - 3x\right)m^2 + (3x + (2x+1)z)m + x^2 - 2x - z\right), \\ B_2 &= -2x^2zm^4 - x\left(2z^3 - 3z^2 + (1-9x)z + 2x^2 - x\right)m^3 \\ &\quad + \left(9x^3 - 9zx^2 - 6x^2 - 2z^3 + \left(-6x^2 - 7x + 1\right)z^2\right)m^2 \\ &\quad + \left(-12x^3 + 9x^2 + z^2 + \left(-6x^3 - 2x^2 + 6x\right)z\right)m - x(2x-1)\left(x^2 - 2x - z\right), \\ C_2 &= mx(mx - 3x - 2z)^2(x + mz - 1) \end{aligned}$$

and $z = y - x, m = (k-x)/(y-x)$.

Proof. Substitute $y = z + x, k = mz + x, r = nz + \lambda$ and $c = 0$ in equation (3) to obtain the following expression for b:

$$b = \frac{zm^2 + \left(n\left(z^2 - z\right) - 3z\right)m + n(xz + z) + (x + mz)\lambda}{x}. \tag{10}$$

Use (2) of Lemma 1 to get the following expression for n:

$$n = \frac{m(mx - 3x - 2z + (x+mz)\lambda)}{x^2 - 2x - z + m(zx + x)}. \tag{11}$$

Finally substitute values of k, r and b in the equation (1) to get the equation (9). □

3 Finiteness results in support of Hall's conjecture

In this section, we prove a finiteness result in support of Singhi's conjecture, which is equivalent to Hall's conjecture. The main result of this section is that for a fixed pair $(\lambda \geqslant 2, m \geqslant 2)$, there exist only finitely many QSDs.

Theorem 17. *For a fixed pair* (c, m)*, there exist only finitely many QSDs.*

Proof. Suppose $x = 0$. Substitute $x = 0, k = my, n = 2m + c - d, \lambda = \frac{c + 3m - m^2 - n + mn - mny}{m}$ and $r = ny + \lambda$ in (2) of Lemma 1 to get $m^3 + (c - d + 1)m^2 + d(m-1) - (c - d + 2m)ym^2 = 0$. As $a - d \geqslant 0$ from (3), Lemma 5, we have $d - c \leqslant m$. Hence for a fixed pair (c, m), y has finitely many choices.

Now suppose $x > 0$. If $d = 0$, then from equation (5) we get $(m-1)m^2(c+m)(c+m+1)x = 0$ which is a contradiction. Hence $d > 0$. The discriminant of quadratic (5) in x is

$$\Delta = ((m-1)d + (1 + c - d)\ m^2 + m^3)\Delta_1,$$

where

$$\Delta_1 = \left(2\,d - c + (-1 + c - d)\; m + m^2\right)^2\, \left((m-1)d + (1 + c - d)\; m^2 + m^3\right) \\ - 4\,d\; (m-1)\; (c - d + 2\,m)\; \left(d + (c-d)\; m + m^2\right)\; z.$$

As before we have $d - c \leqslant m$. Observe that the coefficient of z in Δ is negative. Hence for a fixed pair (c, m), z is bounded by a function of (c, m). Hence x takes only finitely many values. Now use Lemma 6 to complete the proof. □

Theorem 18. *For a fixed pair (z, m), there exist only finitely many QSDs.*

Proof. In view of Neumaier's Result 9, it is enough to show that for a fixed pair (z, m), there exist only finitely many QSDs whose block graphs are either the complete multipartite graphs with u classes of size m, with parameters $(mu, m(u-1), m(u-2), m(u-1))$ or Latin square graphs $LS_m(n)$ or Steiner graphs $S_m(n)$.

Suppose $\mathbf{D}$ is a QSD whose block graph is the complete multipartite graph with s classes of size m, with the parameters $(ms, m(s-1), m(s-2), m(s-1))$. Then the equation (6) holds. Observe that $\Delta = m^2(mz - z - 1)^2 + (m-1)^2\lambda^2 + 2(m-1)m(mz - z + 1)\lambda$ is the discriminant of the quadratic (6) in x. With $\theta = m(mz - z + 1) + (m-1)\lambda$ get $\Delta - \theta^2 = -4(m-1)m^2 z$ and $\Delta - (\theta - 1)^2 = (1 - 2m)\,(2zm^2 - 2zm - 1) + 2(m-1)\lambda$. As Δ must be a perfect square, $\lambda < \frac{(2m-1)\left(2zm^2 - 2zm - 1\right)}{2(m-1)}$. Hence for a fixed pair (z, m), λ takes finitely many values. As x satisfies the equation (6), x takes finitely many vales. As $k = mz + x$ by the Lemma 6, for a fixed pair (z, m), there exist only finitely many QSDs whose block graphs are the complete multipartite graph with u classes of size m.

Suppose $\mathbf{D}$ is a QSD whose block graph is the Latin square graph $LS_m(n)$. Then the equation (7) holds. The discriminant of the quadratic (7) in x is

$$\Delta_1 = m^2\, n^4 - 4\; (m-1)\; m\, n\, z - 2\, m\, n^3\; (m - 1 + 2\, m\, z) + n^2\; \left((m-1)^2 - 4\, m\, z + 8\, m^2\, z\right).$$

With $\theta_1 = m\, n^2 + n\; (1 - m - 2\, m\, z) - 2\; (m\, z\, (z-1))$ get

$$\Delta_1 - \theta_1^2 = -4\, m^2\, (z-1)^2\, z^2 - 4\, m\, n\, z^2\; (-1 - m + 2\, m\, z)$$

and

$$\Delta_1 - (\theta_1 - 1)^2 \\ = 2\, m\, n^2 - \left(1 - 2\, m\, z + 2\, m\, z^2\right)^2 - 2\, n\; \left(m - 1 + 2\, m\, z - 2\, m\; (1 + m)\; z^2 + 4\, m^2\, z^3\right).$$

Observe that $(\theta_1 - 1)^2 < \Delta_1 < \theta_1^2$ for sufficiently large n with respect to m and z. Hence n is bounded by a function of (z, m). Hence for a fixed pair (z, m), x takes only finitely many values. As before, for a fixed pair (z, m), there exist only finitely many QSDs whose block graphs are Latin square graphs.

Let $\mathbf{D}$ be a QSD whose block graph is the Steiner graph $S_m(n)$. Then the equation (8) holds. The discriminant of quadratic (8) in x is $\Delta_2 = m^2 + (m-1)^2\, n^2 - 2\; (m-1)\; m\, n\; (2\, z - 1)$. With $\theta_2 = m + (m-1)\; n - 2\, m\, z$ get $\Delta_2 - \theta_2^2 = -4\, m^2\; (z-1)\; z$

and $\Delta_2 - (\theta_2 - 1)^2 = 2\,(m+1)\,n - (1+2\,m\,z)\,(1-2\,m+2\,m\,z)$. Observe that $(\theta_2-1)^2 < \Delta_2 < \theta_2^2$ for sufficiently large n with respect to m and $z > 1$. For $z = 1$, observe that either $x = 0$ or $x = nm - m - n$ and for $x = 0$ get $k = m$, $\lambda = 1$. Observe that designs corresponding to $x = nm - m - n$ are complements of designs with $x = 0$. Hence n is bounded by a function of (z, m). Hence for a fixed pair (m, z), x takes only finitely many values. As before for a fixed pair (z, m), there exist only finitely many QSDs whose block graphs are Steiner graphs. □

Theorem 19. *For a fixed pair $(\lambda \geqslant 2, m \geqslant 2)$, there exist only finitely many QSDs.*

Proof. For a fixed $\lambda > 1$, by (2) of Remark 4, x is bounded by λ. In view of the Theorem 18, we show that for a fixed triple (λ, x, m), z takes only finitely many values. As before, in view of the Result 9, we need to consider only three cases of QSDs, namely QSDs whose block graphs are the complete multipartite graphs with u classes of size m, with parameters $(mu, m(u-1), m(u-2), m(u-1))$ and the Latin square graphs $LS_m(n)$ and the Steiner graphs $S_m(n)$.

If $\mathbf{D}$ is a QSD whose block graph is the complete multipartite graph, then parameters of $\mathbf{D}$ satisfy the equation (6). Hence z has finitely many choices.

Suppose $\mathbf{D}$ is a QSD whose block graph is the Latin square graph $LS_m(n)$. If $x = 0$, then from the equation (7) get $m(n(z-1)+1) - 1 = 0$, which is impossible. Assume $x > 0$, write equation (7) as a quadratic in n and find its discriminant Δ.

$$mxn^2 + \left(\left(z - z^2\right)m^2 + (-2zx - x)m - x^2 + x\right)n - (m-1)mz = 0.$$

$$\begin{aligned}\Delta \;=\;& m^4z^4 - 2m^3(m-2x)z^3 + m^2\left(m^2 + 6x^2 + (-2m-2)x\right)z^2 \\ & +2mx\left(m^2 - m + 2x^2 + (m-2)x\right)z + x^2(m+x-1)^2.\end{aligned}$$

With $\theta = m^2z^2 - m(m-2x)z + x(m+x-1)$ observe that $\Delta - \theta^2 = 4(m-1)m^2xz$ and $\Delta - (\theta+1)^2 = -2x^2 - 2mx + 2x - 2m^2z^2 + 2m\,(2xm^2 - 2xm + m - 2x)\,z - 1$. This implies $\theta^2 < \Delta < (\theta+1)^2$ for sufficiently large z in terms of x and m. Hence z is bounded by some function of (x, m).

Let $\mathbf{D}$ be a QSD whose block graph is the Steiner graph $S_m(n)$. As before take $b = \frac{(n+1)(nm+m-n)}{m}$ and $\lambda = \frac{b\,x + (m+m\,n-n\,x)\,z - m\,n\,z^2}{x+m\,z}$. If $x = 0$, then from the equation (8) get $m^2(z-1)z = 0$, which implies $z = 1$. Observe that for these values $\lambda = 1$. Assume $x > 0$ and from the equation (8) get $n = \frac{z^2m^2 - zm^2 - xm + 2xzm + x^2}{(m-1)x}$. Substitute these values in the expression of λ to get $\lambda = \frac{(x+mz-1)(x+mz)}{(m-1)m}$.

Hence z is bounded by a function of (λ, m, x). □

We summarize the above results as:

Theorem 20. *For a fixed pair (c, m) or (z, m) or $(\lambda \geqslant 2, m \geqslant 2)$, there exist only finitely many QSDs.*

4 Block graphs of QSDs

In this section, we prove some results about which SRGs could be block graphs of QSDs. In the next theorem, we classify QSDs with $m = 2$, using the classification Result 8.

Theorem 21. *If* $\mathbf{D}$ *is a QSD with* $k = 2y - x$*, then* $\mathbf{D}$ *is either a pair design or* 2-$(6, 3, 2)$ *design or Hadamard 3-design or complement of one of these designs.*

Proof. Observe that, $\mathbf{D}$ is a QSD with $k = 2y - x$ if and only if the smallest eigenvalue of block graph Γ of $\mathbf{D}$ is -2. Hence Γ is one of the graphs listed in Result 8. If block graph of $\mathbf{D}$ is complete n-partite graph with parameters $(2n, 2n-2, 2n-4, 2n-2), n \geqslant 2$, then by Result 10, D is the Hadamard 3-design. To rule out the possibilities of QSD whose block graph is the lattice graph, we use Result 11. If block graph of $\mathbf{D}$ is the triangular graph, then by Theorem 12, $\mathbf{D}$ is a pair design or its complement.

In remaining cases from the fact that discriminant of the quadratic (5) is non-negative we get $z = 1$. If block graph of $\mathbf{D}$ is the Shrikhande graph, with parameters $(16, 6, 2, 2)$ or the Schläfli graph, with parameters $(27, 16, 10, 8)$ or the Clebsch graph, with parameters $(16, 10, 6, 6)$, then the discriminant of the same quadratic (5) is not a perfect square. If the block graph of $\mathbf{D}$ is one of the three Chang graphs, with parameters $(28, 12, 6, 4)$, then using the same equation (5) we get, $x = 0$ or $x = 4$. If $x = 0$, then $\lambda = 1$ and $\mathbf{D}$ is a 2-(8,2,1) design. The design associated with $x = 4$ is the complement of 2-$(8, 2, 1)$ design. If block graph of $\mathbf{D}$ is the Petersen graph, with parameters $(10, 3, 0, 1)$, then again using the same equation we get $x = 1$. Calculate the remaining parameters of $\mathbf{D}$, to see that $\mathbf{D}$ is a 2-(6,3,2) design. □

Remark 22. Observe that 2-$(9, 5, 10)$ and 2-$(17, 5, 10)$ QSDs with intersection numbers $x = 1$ and $y = 3$ and 2-$(14, 6, 15)$ design with intersection numbers $x = 2$ and $y = 4$ do not arise in the above classification, since the first two are ruled out by Calderbank's inequality (see [28]) and the third is ruled out using [8].

In the following proposition, we obtain the parameters of QSDs whose block graph is the Steiner graph $S_m(n)$ in terms of functions of (x, z, m).

Proposition 23. *Let* $\mathbf{D}$ *be a QSD whose block graph is the Steiner graph* $S_m(n)$*. If* $x = 0$*, then* $z = \lambda = 1, k = m, v = m - n + mn$*, and if* $x \neq 0$*, then* $v = \frac{(z-1)zm^2+2xzm+x^2}{x}$*,* $k = mz + x$ *and* $\lambda = \frac{(x+mz-1)(x+mz)}{(m-1)m}$*.*

Proof. Let $\mathbf{D}$ be a QSD whose block graph is the Steiner graph $S_m(n)$. As before take $b = \frac{(n+1)(nm+m-n)}{m}$ and $\lambda = \frac{bx+(m+mn-nx)z-mnz^2}{x+mz}$. If $x = 0$, then from the equation (8) get $m^2(z-1)z = 0$, which implies $z = 1$. Observe that for these values $\lambda = 1$ and $v = m - n + mn$.

Assume $x > 0$ and from the equation (8) get $n = \frac{z^2m^2-zm^2-xm+2xzm+x^2}{(m-1)x}$. Substitute these values in the expression of λ to get $\lambda = \frac{(x+mz-1)(x+mz)}{(m-1)m}$. Obtain v using the expression $vr = bk$. □

The SRGs with eigenvalues a, 0 and $-m$ (equivalently $a = d$) are characterized in [13] in terms of the complete multipartite graphs with s classes of size m, with parameters $(ms, m(s-1), m(s-2), m(s-1))$ with $s \geqslant 2$. In [13], Theorem 3.4, they proved that the only QSD with the complete s-partite block graph with parameters $(2s, 2s-2, 2s-4, 2s-2)$, $s \geqslant 2$ is the Hadamard 3-design. In view of this, we give the following characterization.

Theorem 24. *Let* $\mathbf{D}$ *be a QSD whose block graph is the complete multipartite graph with* s *classes of size* 3, *with parameters* $(3s, 3(s-1), 3(s-2), 3(s-1))$. *Then,* $\mathbf{D}$ *is a* 2-$(9(1+2u), 6(1+2u), 5+12u)$ *design with intersection numbers* $3(1+2u)$ *and* $4(1+2u)$ *(with* $z = 2u+1$*) or the complement of this design.*

Proof. Substitute $m = 3$ in the equation (6) to get

$$3x^2 + (12z - 2\lambda - 3)x + 3z(3z - 2\lambda - 1) = 0. \tag{12}$$

Observe that $\Delta = 9(2z-1)^2 + 4\lambda^2 + 12(2z+1)\lambda$ is the discriminant of the quadratic (12) in x. Take $\theta = 3(2z+1) + 2\lambda$ and observe that $\Delta - \theta^2 = -72z$ and $\Delta - (\theta - 6)^2 = 24\lambda$. As Δ is a perfect square get $\Delta = (\theta - t)^2$, for $1 \leqslant t \leqslant 5$, which gives $\lambda = \frac{t^2 - 6t - 12zt + 72z}{4t}$. As λ is an integer, t is an even integer. For $t = 2$, get $\lambda = 6z - 1$. Substitute $\lambda = 6z - 1$ in the equation (12) and get $x = 3z$. Observe that $b = \frac{3}{2}(9z - 1)$. Hence z is an odd integer. Take $z = 2u + 1$, for a non-negative integer u. Finally observe that $x = 3(1+2u), y = 4(1+2u), v = 9(1+2u), b = 3(4+9u), r = 2(4+9u), k = 6(1+2u), \lambda = 5 + 12u$.

For $t = 4$, using the equation (12), observe that $x(-2 + 3x + 9z) = 0$. Hence $x = 0$. As $\lambda = \frac{1}{2}(3z - 1)$, z is an odd integer. Take $z = 2u+1$, for a non-negative integer u and get $b = 3(4+9u), v = 9(1+2u), r = 4 + 9u, k = 3(1+2u), \lambda = 1 + 3u$.

Observe that these designs are complements of each other. □

In ([13], Theorem 3.5), they ruled out the possibility of a QSD whose block graph is the Latin square graph $LS_2(n)$ or its complement. We rule out in Theorems 25–26, the possibility of QSD whose block graph is the Latin square graph $LS_m(n)$ or its complement, for $m = 3, 4$.

Theorem 25. *There is no QSD whose block graph is the Latin square graph* $LS_m(n); n \geqslant m$, *for* $m = 3$ *and* 4.

Proof. Suppose $\mathbf{D}$ is a QSD whose block graph is the Latin square graph $LS_m(n)$, for $n \geqslant m$. Note that $x > 0$. We show that the equation (7) does not have integer solutions for $m = 3$ and 4.

From the equation (7) it is clear that n divides $m(m-1)z$. We substitute $z = nt/(m(m-1))$, for a positive integer t, in the equation (7) and observe that

$$\Delta = m^2\ (t - x + m\,x)\ \left(\left(m^2 - 4\,t\right)\ t + m\ \left(-m + m^2 - 4\,t\right)\ x\right)$$

is the discriminant of this quadratic in z, which is negative if $m^2 - 4\,t \leqslant 0$. Hence $t < m^2/4$.

For $m = 3$, $t \leqslant 2$. If $m = 3, t = 1$ then $\Delta = 9(1 + 2x)(5 + 6x)$, which is not a perfect square as $1+2x$ and $5+6x$ are relatively prime and $5+6x$ is not a perfect square. Observe that If $m = 3, t = 2$ then $\Delta < 0$.

For $m = 4$, $t \leqslant 3$. If $m = 4, t = 1$ then $\Delta = 64(1 + 3x)(3 + 8x)$, which is not a perfect square, as $1 + 3x$, $3 + 8x$ are relatively prime and $3 + 8x$ is not a perfect square.

If $t = 2$ then $\Delta = 256(1 + x)(2 + 3x)$, which is again not a perfect square, as $1 + x$, $2 + 3x$ are relatively prime and $2 + 3x$ is not a perfect square.

If $t = 3$ then $\Delta = 576(1 + x)$. Take $x = s^2 - 1$, for $s \geqslant 2$ and observe that $z = \frac{1}{4}(s^2 - s + 1)$ or $z = \frac{1}{4}(s^2 + s + 1)$. In both the cases z is not an integer. □

Theorem 26. *There is no QSD whose block graph is the complement of a Latin square graph $LS_s(n); n \geqslant s$, for $s = 3, 4$.*

Proof. Suppose $\mathbf{D}'$ is a QSD whose block graph is the complement of $LS_3(n)$ with parameters are $(n^2, (n-2)(n-1), n^2 - 6n + 10, (n-3)(n-2))$.

Take $b = n^2$ and solve $d = (n-3)(n-2)$ for λ to get

$$\lambda = \frac{(x+z)n^2 - 5zn - mnz^2 + (n(m-x) - m^2 + m + 6)\,z}{x + mz}.$$

Solve $a = (n-2)(n-1)$ for n to get $n = m + 2$. Now use the equation (1) to get

$$(m+2)x\left(m^2 + (1-2z)m - x + 1\right) - mz\left((z-1)m^2 + (2z-1)m - 1\right) = 0. \tag{13}$$

From the equation (13) observe that $m + 2$ divides $6z$. Substitute $z = (m+2)t/6$, for a positive integer t, in the equation (13) and get

$$\begin{aligned} 36x^2 \quad &+ \quad 12\left(tm^2 - 3m^2 + 2tm - 3m - 3\right)x \\ &+ \quad mt\left(tm^3 + 4tm^2 - 6m^2 + 4tm - 6m - 6\right) = 0. \end{aligned} \tag{14}$$

We show that the equation (14) does not have integer solutions. Observe that

$$\Delta_1 = -2^4 3^3\left(m^2 + m + 1\right)\left(2\,t\,m\,(m+1) - 3m^2 - 3m - 3\right)$$

is the discriminant of the quadratic (14) in x, which is negative for $t > 1$. Take $t = 1$ and observe that $\Delta_1 = 2^4 3^3 (m^2 + m + 1)(m^2 + m + 3)$. As Δ_1 is a perfect square of an integer and $(m^2 + m + 1)$, $(m^2 + m + 3)$ are relatively prime, at least one of these two must be a perfect square. Observe that $(2m+1)^2 < 4(m^2 + m + 1) < (2m+2)^2$ and $(2m+1)^2 < 4(m^2 + m + 3) < (2m+2)^2$ for $m > 2$. For $m = 2$ it can be easily seen that the equation (14) does not have integer solutions.

Now, suppose $\mathbf{D}'$ is a QSD whose block graph is the complement of $LS_4(n)$ whose parameters are $(n^2, (n-3)(n-1), n^2 - 8n + 18, (n-3)(n-4))$ for $n \geqslant 4$.

Take $b = n^2$ and solve $d = (n-3)(n-4)$ for λ to get

$$\lambda = \frac{xn^2 - mnz^2 + ((1+n-m)m + n^2 - 7n - nx + 12)\,z}{x + mz}.$$

Solve $a = (n-3)(n-1)$ for n to get $n = m+3$. Now use the equation (1) to get

$$(m+3)x^2 - (m+3)\left((m+1)^2 - 2mz\right)x + mz\left(zm^2 + 3zm - (m+1)^2\right) = 0. \qquad (15)$$

From the equation (15) observe that $m+3$ divides $12z$. Substitute $z = (m+3)t/12$, for a positive integer t, in the equation (15) and get

$$144x^2 + 24\left(m(m+3)t - 6(m+1)^2\right)x + mt\left(m(m+3)^2t - 12(m+1)^2\right) = 0 \qquad (16)$$

We show that the equation (16) does not have integer solutions. Observe that

$$\Delta_1 = -2^8 3^3 (m+1)^2\left(tm^2 - 3m^2 + 2tm - 6m - 3\right)$$

is the discriminant of the quadratic (16) in x, which is negative for $t > 3$.

Suppose $t = 1$, from the equation (16) get

$$m^4 - 6m^3 - 120xm^2 - 15m^2 - 216xm - 12m + 144x^2 - 144x = 0$$

and observe that 3 divides m. Put $m = 3s$ in the above equation and get

$$16x^2 - 8\left(15s^2 + 9s + 2\right)x + s\left(9s^3 - 18s^2 - 15s - 4\right) = 0.$$

Observe that $\Delta = 256(3s+1)^2\left(6s^2 + 4s + 1\right)$ is the discriminant of this quadratic in x, which implies $6s^2 + 4s + 1$ is a perfect square. Hence s must be an even integer, but then z is not an integer.

Suppose $t = 2$. Put $m = 3(-1 + 2z)$ in the equation (15) and observe that $8(3z - 1)^2\left(6z^2 - 4z + 1\right)$ is the discriminate of the quadratic (15) in x. Hence $2\left(6z^2 - 4z + 1\right)$ is a perfect square, which implies 2 is a factor of $(6z^2 - 4z + 1)$, a contradiction.

Suppose $t = 3$, from the equation (16), $x = \frac{1}{4}\left(m^2 + 3m + 4\right)$ or $x = \frac{1}{4}\left(m^2 - m\right)$. We observe that for both these values of x, λ is not an integer, as $m \geqslant 2$. □

Remark 27. We believe that using similar calculations as given in proofs of above theorems and using number theory, for higher values of m, the parametric classification of QSDs with block graph, the complete multipartite graph with s classes of size m or Latin square graph $LS_m(n)$ or complement of $LS_m(n)$ can be obtained.

The complement $\bar{G}$ of a SRG G with parameters (b, a, c, d) is also a SRG with parameters $(b, b-a-1, b-2-2a+d, b-2a+c)$. If a SRG G is isomorphic to it's complement $\bar{G}$, then $b = 2a+1$ and $c = d-1$. The class of self-complementary SRGs are called *conference* graphs. Conference graphs is a class of SRGs with parameters $(4d+1, 2d, d-1, d)$. Examples of conference graphs such as Paley graphs are well known (See [7], [10], [12]).

We next rule out the possibility of a QSD whose block graph is a conference graph, using elementary arguments.

Proposition 28. *There is no QSD whose block graph is a conference graph, with parameters $(4d+1, 2d, d-1, d)$.*

Proof. Substitute $y = x+z, k = mz + x$ and $r = nz + \lambda$ in the equations (2), (3) and (4). Solve the equation $c = d-1$ for n to get $n = 2m-1$. Substitute $n = 2m-1$ and solve the equation $b = 2a+1$ for b to get $b = \frac{z-2\,m\,z-2\,x\,z+4\,m\,x\,z-2\,m\,z^2+4\,m^2\,z^2+2\,x\,\lambda+2\,m\,z\,\lambda}{2\,x+z}$. Substitute these values of b and n in the equation (1) and solve for λ to get $\lambda = \frac{x-2\,m\,x-2\,x^2+4\,m\,x^2-m\,z-2\,x\,z+4\,m\,x\,z-z^2+2\,m\,z^2}{(2m-1)x+(m-1)z}$. Observe that as $m \geqslant 2$ and $z \geqslant 1$, $(2m-1)x + (m-1)z \neq 0$. Now equation (1) to get $P = 0$, where

$$P = (1-2m)x\left(2m^2 - 2zm - 2m - x + 1\right) + mz\left(2zm^2 - 2m^2 - zm + 2m - 1\right).$$

Write $4P = \left(-4\left(-z^2+z+2x\right)m^2 + 2(4zx+4x+z)m + 4(x-1)x - z\right)(2m-1) - z$. We observe that $2m-1$ is a factor of z. Substitute $z = (2m-1)t$, for positive integer t in $P = 0$ and get

$$x^2 + \left(2(2t-1)m^2 + 2(1-t)m - 1\right)x + mt\left(4tm^3 - 2(2t+1)m^2 + (t+2)m - 1\right) = 0.$$

Observe that $\left(-2m^2+2m-1\right)\left(2(m-1)m(4t-1)-1\right)$ is the discriminant of this quadratic in x, which is negative, as $m \geqslant 2$. This is a contradiction as the equation $P = 0$ has integer roots. □

We summarize above results in the following Theorem:

Theorem 29. *There is no QSD whose block graph is either a Latin square graph $LS_s(n)$ $(n \geqslant s)$ or its complement for $s = 3, 4$ or a conference graph.*

Let Γ be a (b, a, c, d) SRG. To find feasible parameters of a QSD whose block graph is Γ, we work through the following steps:

1. m is obtained using (4) of the Lemma 5 and then n by (2).

2. As $x \geqslant 0$, (1) of the Lemma 5 implies either

$$b(m-1) < \left(m^2 - 2m + a - c\right)z + (m-1)\lambda < \frac{(m-1)(m+a)}{m}$$

or

$$\frac{(m-1)(m+a)}{m} < \left(m^2 - 2m + a - c\right)z + (m-1)\lambda < b(m-1).$$

Here $b > a$ and $m > 1$, rules out the first possibility.
This gives us $z \leqslant \dfrac{(b-1)(m-1)}{m^2-2m+a-c}$ and $\lambda \leqslant \dfrac{b(m-1) - (m^2-2m+a-c)\,z}{m-1}$.

3. Now for each m, z and λ; x is calculated from (1) of the Lemma 5.

4. k, r and y are calculated using x, z, m, n and λ, then v and b using (1) of the Lemma 1.

For each set of parameters of a SRG other than a triangular graph or complement of a triangular graph or conference graph or Latin square graph $LS_m(n)$ for $m = 2, 3, 4$, or complement of $LS_m(n)$ for $m = 2, 3, 4$ or SRGs with smallest eigenvalue -2 listed in Table 11.12 of Brouwer [6], we execute the code given in the Appendix 1 in Mathematica to find the parameters of QSDs associated with it. As block graphs of a QSD and its complement are isomorphic we list parameters of QSDs with $v \geqslant 2k$. We list in Table 1 parameters of QSDs whose block graphs are known. Table 2 contains the parameters of SRGs which are not block graphs of QSDs. Table 3 is the list of parameters of SRGs and associated parameters of QSDs where existence of both is unknown. There are two points to be noted, Table 2 is the largest and there are different QSDs having same block graph.

Table1: Feasible parameters of QSDs related to known SRGs

Strongly Regular Graph				Quasi-symmetric Design					
b	a	c	d	v	k	λ	x	y	$\exists$
26	15	8	9	13	3	1	0	1	Yes
35	18	9	9	15	3	1	0	1	Yes
50	28	15	16	25	4	1	0	1	Yes
56	45	36	36	21	6	4	0	2	Yes
57	24	11	9	19	3	1	0	1	Yes
63	30	13	15	36	16	12	6	8	Yes
63	32	16	16	28	4	1	0	1	Yes
					12	11	4	6	Yes
70	27	12	9	21	3	1	0	1	Yes
					9	12	3	5	No
77	16	0	4	56	16	6	4	6	Yes
77	60	47	45	22	6	5	0	2	Yes
82	45	24	25	41	5	1	0	1	Yes
85	54	33	36	51	6	1	0	1	Yes
					15	7	3	5	No
99	48	22	24	55	15	7	3	5	?
99	50	25	25	45	5	1	0	1	Yes
100	33	14	9	25	3	1	0	1	Yes
111	44	19	16	37	4	1	0	1	Yes
117	36	15	9	27	3	1	0	1	Yes
120	77	52	44	21	7	12	1	3	Yes
122	66	35	36	61	6	1	0	1	?
126	65	28	39	105	40	18	14	16	?
130	48	20	16	40	4	1	0	1	Yes
143	72	36	36	66	6	1	0	1	Yes
					30	29	12	15	Yes
143	70	33	35	78	36	30	15	18	Yes

(Continued on next page)

Table 1 – continued

Strongly Regular Graph				Quasi-symmetric Design					
b	a	c	d	v	k	λ	x	y	$\exists$
153	120	91	105	136	40	13	10	12	?
155	42	17	9	31	3	1	0	1	Yes
					7	7	1	3	Yes
170	91	48	49	85	7	1	0	1	Yes
176	45	18	9	33	3	1	0	1	Yes
					15	35	6	9	?
176	105	68	54	22	7	16	1	3	Yes
190	84	38	36	76	6	1	0	1	Yes
183	70	29	25	61	5	1	0	1	Yes
					21	21	6	9	?
195	98	49	49	91	7	1	0	1	Yes
					28	18	7	10	No
196	60	23	16	49	4	1	0	1	Yes
					9	6	1	3	Yes
204	140	94	100	136	10	1	0	1	?
208	132	81	88	144	45	20	12	15	?
208	75	30	25	65	5	1	0	1	Yes
					20	19	5	8	?
216	90	39	36	81	6	1	0	1	?
220	135	78	90	176	36	9	6	8	?
220	84	38	28	45	9	8	1	3	Yes
					18	34	6	9	?
221	64	24	16	52	4	1	0	1	Yes
					16	20	4	7	No
222	51	20	9	37	3	1	0	1	Yes
226	120	63	64	113	8	1	0	1	Yes
231	30	9	3	56	16	18	4	8	?
247	54	21	9	39	3	1	0	1	yes
					12	22	3	6	?
253	140	87	65	23	7	21	1	3	Yes
273	102	41	36	91	6	1	0	1	Yes
					40	52	16	20	?

Remark 30. Corresponding to SRG (69,48,32,36), which is unknown, we are getting two feasible parameter sets of QSD's namely 2-(46,6,1) (with $x = 0, y = 1, m = 6$), which does not exist (see [16]) and 2-(46,16,8)(with $x = 4, y = 6, m = 6$), existence of which is not known (See [35]). Corresponding to SRG (69,20,7,5) we are getting QSD parameters 2-(24,8,7)(with $x = 2, y = 4, m = 3$), which does not exist(See [35]). Observe that SRG (69,48,32,36) is the Steiner graph $S_6(8)$ and SRG (69,20,7,5) is the complement $S_6(8)$.

b	a	c	d	b	a	c	d	b	a	c	d	b	a	c	d
16	5	0	2	119	54	21	27	162	105	72	60	225	112	55	56
26	10	3	4	119	64	36	32	165	36	3	9	226	105	48	49
27	10	1	5	120	51	18	24	165	128	100	96	231	200	172	180
35	16	6	8	120	68	40	36	169	60	23	20	243	22	1	2
40	12	2	4	120	56	28	24	169	108	67	72	243	220	199	200
40	27	18	18	120	63	30	36	169	72	31	30	243	110	37	60
45	12	3	3	120	42	8	18	169	99	53	56	243	132	81	60
45	32	22	24	121	50	21	20	170	78	35	36	247	192	146	160
50	7	0	1	121	70	39	42	175	30	5	5	253	112	36	60
50	42	35	36	122	55	24	25	175	144	118	120	256	51	2	12
50	21	8	9	125	28	3	7	175	72	20	36	256	204	164	156
56	10	0	2	125	96	74	72	175	102	65	51	256	68	12	20
57	32	16	20	125	52	15	26	176	40	12	8	256	187	138	132
64	18	2	6	125	72	45	36	176	135	102	108	256	75	26	20
64	27	10	12	126	25	8	4	176	130	93	104	256	180	124	132
64	36	20	20	126	100	78	84	176	49	12	14	256	85	24	30
70	42	23	28	126	45	12	18	176	126	90	90	256	170	114	110
81	20	1	6	126	80	52	48	176	70	18	34	256	90	34	30
81	60	45	42	126	50	13	24	176	85	48	34	256	165	104	110
81	30	9	12	126	75	48	39	176	90	38	54	256	102	38	42
81	50	31	30	126	60	33	24	183	112	66	72	256	153	92	90
82	36	15	16	130	81	48	54	190	105	56	60	256	105	44	42
85	20	3	5	135	64	28	32	195	96	46	48	256	150	86	90
85	64	48	48	135	70	37	35	196	135	90	99	256	119	54	56
96	19	2	4	136	60	24	28	196	65	24	20	256	136	72	72
96	76	60	60	136	75	42	40	196	130	84	90	256	120	56	56
96	20	4	4	136	63	30	28	196	91	42	42	256	135	70	72
96	75	58	60	136	72	36	40	196	104	54	56	273	170	103	110
100	22	0	6	144	39	6	12	209	100	45	50	275	112	30	56
100	77	60	56	144	104	76	72	209	108	57	54	275	162	105	81
100	66	41	48	144	55	22	20	210	99	48	45	276	135	78	54
100	44	18	20	144	88	52	56	210	110	55	60	276	140	58	84
100	55	30	30	144	65	28	30	221	156	107	117	279	128	52	64
100	45	20	20	144	78	42	42	222	170	122	140	279	150	85	75
100	54	28	30	144	66	30	30	225	70	25	20	280	36	8	4
105	72	51	45	144	77	40	42	225	154	103	110	280	243	210	216
111	66	37	42	155	112	78	88	225	84	33	30	280	117	44	52
112	30	2	10	156	30	4	6	225	140	85	90	280	162	96	90
112	81	60	54	156	125	100	100	225	98	43	42	280	135	70	60
117	80	52	60	162	56	10	24	225	126	69	72	280	144	68	80

Table 2: Strongly regular graphs not associated with QSDs

Strongly Regular Graph				Quasi-symmetric Design				
b	a	c	d	v	k	λ	x	y
69	48	32	36	46	6	1	0	1
					16	8	4	6
85	14	3	2	35	14	13	5	8
133	24	5	4	57	9	3	1	3
136	105	80	84	85	15	4	1	3
148	63	22	30	112	28	9	6	8
148	84	50	44	37	9	8	1	3
205	96	50	40	41	9	9	1	3
216	90	39	36	81	6	1	0	1
259	42	5	7	148	36	15	8	12
236	180	135	144	177	12	1	0	1
					45	15	9	12
261	52	11	10	117	13	3	1	3
					52	51	22	27
261	208	165	168	145	65	52	25	30
261	176	112	132	232	56	15	12	14
265	96	32	36	160	64	42	24	28
265	168	107	105	106	42	41	14	18
266	45	0	9	210	45	12	9	12
266	220	183	176	57	12	11	0	3

Table 3: List of feasible parameters of SRGs and associated feasible parameters of QSDs where existence of both is unknown.

5 Quasi-symmetric designs with triangle-free block graph

We now consider QSDs, whose block graph, and **not** the complement of its block graph is triangle-free, as was assumed in [1].

Proposition 31. *There does not exist a QSD with $x = 0$ and having triangle-free block graph.*

Proof. Substitute $x = 0, c = 0, k = my$ and $r = ny + \lambda$ in the equation (3) to get $m^2 + ((y-1)n + \lambda - 3)m + n = 0$. Since $n \geqslant m \geqslant 2$, this is impossible as y and λ are positive. □

In the rest of the section we assume $x > 0$.

Theorem 32. *Let* $\mathbf{D}$ *be a QSD with triangle-free block graph and* $m = \frac{k-x}{y-x}$.

1. *If* $m = 2$*, then* $\mathbf{D}$ *is a* 2-$(6, 3, 2)$ *design with* $x = 1, y = 2$.
2. $m \neq 3, 4, 5$

3. *If* $m = 6$*, then* $\mathbf{D}$ *is either a* 2-$(56, 16, 6)$ *design or its complement.*

Proof. We substitute $m = 2$ and $c = 0$ in the equation (5). As $d - c \leqslant m$, $d = 1, 2$. Using the discriminant observe that the only feasible value for z is 1. Further for $z = 1$ get $x = 1$. Now use Lemma 7 to complete the case with $m = 2$.

As before we substitute $m = 3$ and $c = 0$ in the equation (5). Take $d = 1, 2, 3$ and observe that the discriminant is either negative or not a perfect square.

Using similar calculations the possibilities of a QSD with $m = 4$ and having triangle-free block graph can be ruled out. For $m = 5$, $d = 1, 2, 3, 4$. Observe that the discriminant of equation (5) is a perfect square only for $d = 2$ and $z = 3$. Further observe that for these values of d and z, $x = 3$ but r is not an integer.

For $m = 6$, $0 < d \leqslant 6$. Observe that $d = 4$ and $z = 2$ are only possible values as the discriminant of equation (5) is a perfect square of a positive integer. For these values of d and z get $x = 28$ or 4. Observe that for $x = 4$, $\mathbf{D}$ is the 2-(56,16,6) design and design associated with $x = 28$ is the complement of 2-(56,16,6). □

Remark 33. The parameter set 2-(56,16,6) ($x = 4, y = 6$) is one of the 13 sets among the 73 admissible parameter sets of exceptional QSDs on up to 70 points for which a design is known to exist(See [35], [22]). The block graph of 2-(56,16,6) design is the triangle-free SRG (77,16,0,4) with $m = 6$.

For higher values of m, we obtain the feasible parameters of QSDs with triangle-free block graph using Mathematica. Table 4 lists the feasible parameters of QSDs with triangle-free block graph for $m \leqslant 100$ obtained by executing the code given in the Appendix 2 in Mathematica.

Proposition 34. *Let* $\mathbf{D}$ *be a QSD with triangle-free block graph. If* $y = x + 1$*, then* $\mathbf{D}$ *is a design with parameters* $v = 6, k = 3, \lambda = 2, r = 5, b = 10$ *and* $x = 1, y = 2$*.*

Proof. If $\mathbf{D}$ is a QSD with $y = x + 1$, then by Lemma 7, up to complementation, $x = 1$ and either $\lambda = 2$ or $\lambda = 3$ and $m = 2$. For $\lambda = 2$ use equation (9) get $m = 2$. Observe that $z = 1, x = 1, \lambda = 3$ and $m = 2$, do not satisfy equation (9). □

Proposition 35. *Let* $\mathbf{D}$ *be a QSD with triangle-free block graph. Then* $\lambda \leqslant 2(x + 2)$*.*

Proof. In view of the above theorems, we take $z = 1 + t$ and $m = 3 + s$ for non-negative integers s, t. We observe $A > 0, B^2 - 4AC - (2A(x + 2))^2 \leqslant 0$ and $-B - 2A(x + 2) \leqslant 0$. Use the larger root of equation (9) to get $\lambda \leqslant 2(x + 2)$. □

Proposition 36. *Let* $\mathbf{D}$ *be a QSD with triangle-free block graph. If* $x = 1$*, then* $\mathbf{D}$ *is a design with parameters* $v = 6, k = 3, \lambda = 2, r = 5, b = 10$ *and* $x = 1, y = 2$*.*

Proof. From Proposition 35, we get $\lambda \leqslant 6$. Substitute $z = 1 + t, m = 3 + s$ for positive integers s, t in equation (9). For $2 \leqslant \lambda \leqslant 6$, observe $A\lambda^2 + B\lambda + C > 0$ except for $\lambda = 2$. For $\lambda = 2$ we get $(-t - 1)s^2 + (2t^2 - 2t - 6)\, s + 6t^2 + 5t - 5 = 0$. The discriminant of this quadratic in s is $t^4 + 4t^3 + 6t^2 + 6t + 4$ which lies strictly between $(t^2 + 2t + 1)^2$ and $(t^2 + 2t + 2)^2$. Hence equation (9) does not have integer solutions. Therefore either $y = x + 1$ or $m \leqslant 3$. Use Theorem 32 and Proposition 34 to complete the proof. □

	Strongly Regular Graph				Quasi-symmetric Design					
m	b	a	c	d	v	k	λ	x	y	$\exists$
2	10	3	0	1	6	3	2	1	2	Yes
6	77	16	0	4	56	16	6	4	6	Yes
12	266	45	0	9	210	45	12	9	12	?
20	667	96	0	16	552	96	20	16	20	?
25	15170	385	0	10	9472	1408	335	208	256	?
30	154253	784	0	4	82622	32046	23205	12426	13080	?
30	1394	175	0	25	1190	175	30	25	30	?
42	2585	288	0	36	2256	288	42	36	42	?
56	4402	441	0	49	3906	441	56	49	56	?
						1386	554	490	506	?
72	7031	640	0	64	6320	640	72	64	72	?
77	101386	2205	0	49	74329	33460	20545	15057	15296	?
79	1719070	5071	0	15	949661	162734	50479	27881	29588	?
90	10682	891	0	81	9702	891	90	81	90	?
92	3076075	7008	0	16	1684476	265356	76335	41796	44226	?
						496692	267449	146448	150255	?
						762048	629552	344736	349272	?

Table 4: Feasible parameters of QSDs with triangle-free block graphs

A quasi-symmetric 3-design is a QSD in which any 3-tuple of points occur in λ_3 blocks. Let $\mathbf{D} = (X, \beta)$ be a QS 3-design with intersection numbers $x, y (0 < x < y)$ and p in X. Suppose $\mathbf{D}_p = \{B \setminus \{p\} : B \in \beta, p \in B\}$ and $\mathbf{D}^p = \{C : C \in \beta, p \notin C\}$, then $\mathbf{D}_p$ is a 2-$(v-1, k-1, \lambda_3)$ design with intersection numbers $x-1, y-1$ and $\mathbf{D}^p$ is a 2-$(v-1, k, \lambda-\lambda_3)$ design with intersection numbers x, y. Designs $\mathbf{D}_p$ and $\mathbf{D}^p$ are respectively called the *derived* and the *residual* designs of $\mathbf{D}$. We also have $\lambda_3(v-2) = \lambda(k-2)$.

In [24], the following result is proved: $\mathbf{D}$ is a 3-(v, k, λ) QSD such that complement of a block graph of $\mathbf{D}$ is triangle-free if and only if $\mathbf{D}$ is a Hadamard 3-design, or $\mathbf{D}$ is a 3-$((\lambda+2)(\lambda^2+4\lambda+2)+1, \lambda^2+3\lambda+2, \lambda)$, $\lambda = 1, 2, \ldots$ or $\mathbf{D}$ is a complement of one of these designs. These are two type of designs out of three possibilities obtained in the classification of quasi-symmetric 3-designs with an intersection number zero([10], Theorem 1.35). In view of this result we prove the following.

Proposition 37. *There does not exist a quasi-symmetric 3-design with a triangle-free block graph.*

Proof. In view of Proposition 31, let $\mathbf{D}$ be a QS 3-design with non-zero intersection numbers and a triangle-free block graph. Take $c = 0$ and substitute $y = z + x$ and $k = mz + x$ in equation (3) to get

$$-bx + \left(m^2 - 3m\right)z + r(zm - m + x + 1) + (m-1)\lambda = 0. \tag{17}$$

As the residual of $\mathbf{D}$ is a proper QSD, substitute parameters of the residual of $\mathbf{D}$ in equation (17) to get

$$(r-b)x + \left(m^2 - 3m\right)z + (zm - m + x + 1)(r - \lambda) + (m-1)\left(\lambda - \lambda_3\right) = 0. \tag{18}$$

Subtract equation (18) from equation (17) to get

$$-rx + (zm - m + x + 1)\lambda + (m-1)\lambda_3 = 0. \tag{19}$$

Similarly, substitute parameters of the derived design of $\mathbf{D}$ in equation (17) to get

$$-r(x-1) + \left(m^2 - 3m\right)z + (zm - m + x)\lambda + (m-1)\lambda_3 = 0 \tag{20}$$

Subtract equation (20) from equation (19) to get $-zm^2 + 3zm - r + \lambda = 0$.

Now take $r = nz + \lambda$ to obtain $m^2 - 3m + n = 0$. This gives $m = n = 2$. Taking $m = n = 2$ solve $c = 0$ for b to get $b = \frac{4z^2+(2x-4)z+(x+2z)\lambda}{x}$. Taking these values solve the equation 1 for λ to get $\lambda = \frac{x^2+2zx+x+z}{x+2z}$. Substitute these values in the equation (1) to get $x^2 + 4zx - 3x + 4z^2 - 3z = 0$. Observe that the discriminant of this quadratic in x is negative, which is a contradiction. □

We summarize the above results as:

Theorem 38. *Let* $\mathbf{D}$ *be a QSD with triangle-free graph. If* $x = 1$*, or* $y = x + 1$*, then* $\mathbf{D}$ *is a design with parameters* $v = 6, k = 3, \lambda = 2, r = 5, b = 10$ *and* $x = 1, y = 2$.

Theorem 39. *There does not exist a QSD with triangle-free graph, if either* $x = 0$*, or* $\lambda > 2(x+2)$*, or if it is a 3-design.*

Remark 40. We believe that there is no QSD with triangle-free block graph other than 2-(6,3,2), 2-(56,16,6) designs and their complements. Perhaps, the first step towards proving this is to obtain all feasible parameters of QSDs with triangle-free block graph, which seems difficult. One may have to develop new tools in design theory and number theory.

Appendix 1

We give below Mathematica code to find feasible parameters of QSDs for given parameters (b, a, c, d) of a SRG.

$$m := \frac{1}{2}\left(-c + d + \sqrt{c^2 - 2dc + d^2 + 4a - 4d}\right)$$
$$n := \frac{m^2 - 2m + a - c}{m - 1}$$
$$Z := \frac{(b-1)(m-1)}{m^2 - 2m + a - c}$$

$T := \dfrac{b(m-1)-(m^2-2m+a-c)\,z}{m-1}$
$Ex := (k-1)xyr^2 - \big(k^2(x+y-1)-xy\big)\,\lambda r + k\,\big((\lambda-1)k^2+(x+y-\lambda)k-xy\big)\,\lambda$
$L := (v-1)\lambda-(k-1)r$
$k := mz+x$
$y := z+x$
$r := nz+\lambda$
$v := bk/r$
$x := -\dfrac{z\,(-(m-1)(m+a)+m\,((m^2-2m+a-c)\,z+(m-1)\,\lambda))}{-b(m-1)+(m^2-2m+a-c)\,z+(m-1)\lambda}$
$P := -b(m-1)+\big(m^2-2m+a-c\big)\,z+(m-1)\lambda$
For $[z=1,\ z<Z,$
  For $[\ \lambda=1,\ \lambda<T,$
    If [IntegerQ[m] && IntegerQ[x] && $v \geqslant 2k$ && $Ex == 0$ && IntegerQ[v] &&
      $L == 0$ && $P \neq 0$,
     Print[“$x =$ ”, x, “,$y =$ ”,y, “,$v =$ ”,v, “, $k =$ ”,k, “, $\lambda =$ ”,λ, “,$m =$ ”,m, “,$n =$ ”,n]];
$\lambda = ++$]; $z++$]

Appendix 2

We give below Mathematica code to find feasible parameters of QSDs with triangle-free block graph for $2 \leqslant m \leqslant 100$.

$$a := \frac{-(b\,x)-m\,z+n\,x\,z+m\,n\,z^2+x\,\lambda+m\,z\,\lambda}{z}$$
$$c := \frac{-(b\,x)-3\,m\,z+m^2\,z+n\,z-m\,n\,z+n\,x\,z+m\,n\,z^2+x\,\lambda+m\,z\,\lambda}{z}$$
$$d := \frac{-(b\,x)-m\,z+m^2\,z-m\,n\,z+n\,x\,z+m\,n\,z^2+x\,\lambda+m\,z\,\lambda}{z}$$
$$k := mz+x$$
$$r := nz+\lambda$$
$$y := z+x$$
$$t := \frac{-\Big((-2\,d+m+d\,m-m^2)^2\ (d-d\,m-m^2+d\,m^2-m^3)\Big)}{4\,d\ (d-2\,m)\ (-1+m)\ (-d+d\,m-m^2)}$$
$$\Delta_1 := \big((1+d)\ m-m^2-2\,d\big)^2\ \big(d-d\,m+(d-1)\ m^2-m^3\big) + 4\,d\ (d-2\,m)\ (m-1)\ \big(d\,m-m^2-d\big)\ z$$
$$\Delta := \sqrt{(d-d\,m+(d-1)\ m^2-m^3)\,\Delta_1}$$

$$x := \frac{2\,d^2 - d\,m - 3\,d^2\,m + 3\,d^2\,m^2 + m^3 - 3\,d\,m^3 - d^2\,m^3 + 2\,d\,m^4 - m^5}{2\,(d^2 - 2\,d\,m)} + \frac{4\,d\,m^2\,z - 2\,d^2\,m\,z + \Delta}{2\,(d^2 - 2\,d\,m)}$$

$$b := \frac{-3\,m\,z + m^2\,z + n\,z - m\,n\,z + n\,x\,z + m\,n\,z^2 + x\,\lambda + m\,z\,\lambda}{x}$$

$$\lambda := \frac{2\,d\,x - m\,x - d\,m\,x + m^2\,x - d\,x^2 + 2\,m\,x^2 + d\,z - d\,m\,x\,z + 2\,m^2\,x\,z}{m\,(x + m\,z)}$$

$n := 2m - d$

$v := bk/r$

For $[m = 2,\ m < 101,$

For $[d = 1,\ d < m + 1,$

For $[z = 1,\ z < t + 1,$

If [IntegerQ[Δ] && IntegerQ[λ] && IntegerQ[v],

Print[m, “ ”, b, “ ”, a, “ ”, c, “ ”, d, “ ”, v, “ ”, k, “ ”, λ, “ ”, x, “ ”, y]]; $z++$];

$d++$]; $m++$]

Appendix 3: QSDs with $\overline{\Gamma}$ triangle-free

A QSD is called triangle-free if the complement of its block graph is triangle-free. Triangle-free QSDs with $x = 0$ were first studied in [1] and then in [20]. Such QSDs were also referred to as *triangle-free designs*. Such designs were first studied in [1] and [20]. Some later references are [21], [24], [25], [26], [27], [29], [33]. It was shown in [1], that for a QSD $\mathbf{D}$ with intersection numbers $x = 0$ and y, the block size k satisfies $2y \leqslant k \leqslant y(y+1)$. Furthermore, the lower bound is attained for Hadamard 3-designs and the upper bound is associated with the extension problem of symmetric designs (see [10]). Following [34], we call $\mathbf{D}$ *exceptional*, if $2y < k < y(y+1)$. The paper [20], contains the conjecture: There are only finitely many exceptional triangle-free QSDs with $x = 0$. A computer search carried out in [1] supports this conjecture.
The paper [1] contains the following results about such designs:

1. $2y \leqslant k \leqslant y(y+1)$.
2. $k = 2y$ if and only if $\mathbf{D}$ is a Hadamard 3-design.
3. $k = y(y+1)$ if and only if $\mathbf{D}$ is Witt design 3-design or its residual.

Equivalent to the above results are the following observations given in [27]:

1. $2y - 1 \leqslant \lambda \leqslant y^2 + y - 1$.
2. $\lambda = 2y - 1$ if and only if $k = 2y$ if and only if $\mathbf{D}$ is a Hadamard 3-design.

3. $\lambda = y^2 + y - 1$ or $\lambda = y^2$ if and only if $k = y(y + 1)$.

A triangle-free QSD is called exceptional if $x = 0$ and $2y < k < y(y + 1)$ (equivalently $2 < m < y + 1$). In [20], there is a conjecture that there are finitely many exceptional triangle-free QSDs. In support of this conjecture, it is shown in [20] that the block size k of an exceptional design $\mathbf{D}$ is a prime power if and only if $\mathbf{D}$ is a Hadamard 3-design and $k = 2^n, n \geqslant 2$. Thus exceptional triangle-free designs do not exist for k an odd prime power. Also is shown that for a fixed value of m or y or λ with $m \geqslant 3$ and $y \geqslant 2$, there are finitely many such QSDs.

Triangle free QSDs with $x \neq 0$ was also first investigated in [33]. The classification of triangle-free QS 3-designs is given in [24]. Further investigation of triangle-free QSDs with non-zero intersection numbers are carried out in [21], [25], [26], [27], [29]. Considerable evidence is presented in support of the conjecture that the only triangle-free QSDs with $x > 0$ are the complements of QSDs with $x = 0$. Two results proved are summarized below:

Result 41 ([21], [25], [26], [27], [29])**.** Let $\mathbf{D}$ be a triangle-free QSD with non-zero intersection numbers. Then the following hold:

1. If $k = 2y - x$ or $\lambda = y + 1$ or $y = x + 1$, then $\mathbf{D}$ is a trivial design with parameters $v = 5, b = 10, r = 6, k = 3, \lambda = 3$ and $x = 1, y = 2$.

2. $\lambda \geqslant 2y - x - 3$ and $k \leqslant y(y - x) + x$.

3. $\lambda \neq y - 1$, $\lambda \neq y$, $\lambda \neq 2y$, $k \neq 3y - 2x$, $y \neq x + 2$ and $y \neq x + 3$.

4. If $v \geqslant 2k$, then $x \leqslant z^2 + z$.

Result 42 ([26], [27], [29])**.** For fixed $\lambda \geqslant 1$ or $z \geqslant 1$ or a fixed pair $(x, m > 3)$, there exist finitely many triangle-free QSDs with non-zero intersection numbers.

Appendix 4: SRGs, spherical 2-distance sets, and QSDs

The recent paper of Ionin [18] deals with the connections between SRGs, 2-distance sets in binary Hamming spaces, and QSDs. We give the relevant definitions and results from this paper.

Definition 43. The binary Hamming space H_n is the set of all n-tuples $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where each a_i is 0 or 1. The Hamming distance $h(\mathbf{a}, \mathbf{b})$, between $\mathbf{a}$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in H_n$ is the number of coordinates $a_i \neq b_i$. A set $X \subset H_n$ is called a 2-distance set if $h(\mathbf{a}, \mathbf{b})$ takes only two distinct values h_1 and h_2 for any two different elements $\mathbf{a}, \mathbf{b} \in X$. A *sphere* with center $\mathbf{c} \in H_n$ and integer radius k, $1 \leqslant k \leqslant n-1$, is the set of all $\mathbf{x} \in H_n$ at distance k from $\mathbf{c}$. Any subset of a sphere (of radius k) called a *spherical set* (of radius k).

Definition 44. Let $X = \{\mathbf{x_1}, \mathbf{x_1}, \ldots, \mathbf{x_b}\}$ be a 2-distance set of cardinality b in H_n and let $h_1 < h_2$ be the nonzero distances in X. Form a graph Γ_X with vertex set X, where two distinct $\mathbf{x_i}, \mathbf{x_j}$ are adjacent if and only if $h(\mathbf{x_i}, \mathbf{x_j}) = h_1$. Let $\mathbf{x_i} = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $B_i = \{j \in [n] : x_{ij} = 1\}$, for $i = 1, 2, \ldots, n$, where $[n] = \{1, 2, \ldots, n\}$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. Define the incidence structure $\mathbf{D_X} = ([n], \mathcal{B})$.

Definition 45. Any spherical 2-distance set $X \subset H_n$ is said to *represent a graph* Γ in H_n if Γ is isomorphic to Γ_X. The smallest n for which such a set X exists is called the *binary spherical representation* number of Γ and is denoted by $bsr(\Gamma)$.

The following two results are in [18]:

Result 46. ([18])

1. Every simple graph Γ, except null graphs and complete graphs, admits a spherical representation in H_n, if n is sufficiently large.
2. If Γ is a non-complete regular graph with $e \geqslant 1$ edges, then $bsr(\Gamma) \leqslant e$.
3. If Γ is a connected SRG of order n, then $bsr(\Gamma) \leqslant n$.
4. A graph Γ can be represented in H_n by a spherical 2-distance sets of radius 2 if and only if Γ is isomorphic to the line graph of a graph of order n.

The next result from [18] gives a characterization of SRGs which are block graphs of QSDs in terms of $bsr(\Gamma)$.

Result 47. [18] Let Γ be a connected SRG of order b, and let f be the multiplicity of the least eigenvalue of Γ. Then $bsr(\Gamma) \geqslant b - f$ and equality holds if and only if Γ is the block graph of a QS 2-design.

The next corollary is a consequence of the above result and Theorem 21.

Corollary 48.

1. *If Γ is the Shrikhande graph with parameters $(16, 6, 2, 2)$, then $bsr(\Gamma) > 7$.*
2. *If Γ is the Clebsch graph with parameters $(16, 10, 6, 6)$, then $bsr(\Gamma) > 6$.*
3. *If Γ is the Schläfli graph with parameters $(27, 16, 10, 8)$, then $bsr(\Gamma) > 7$.*
4. *If Γ is the Latin square graph $LS_m(n)$ for $m = 2, 3, 4$, then $bsr(\Gamma) > 1 + m(n-1)$.*
5. *If Γ is the complement of Latin square graph $LS_m(u)$ with parameters $(u^2, (u-1)(u-m+1), m^2 - 2um + m + u^2 - 2, (-1+m-u)(m-u))$ for $m = 2, 3, 4$, then $bsr(\Gamma) > u^2 - mu + m$.*
6. *If Γ is the SRG with parameters $(4d+1, 2d, d-1, d)$, then $bsr(\Gamma) > 1 + 2d$.*
7. *If Γ is the SRG with parameters $(64, 18, 2, 6)$, then $bsr(\Gamma) > 46$ (See Remark 4.5 [18]).*

Remark 49. The following SRGs cannot be block graphs of QSDs: the Shrikhande graph, with parameters $(16, 6, 2, 2)$ or the Clebsch graph, with parameters $(16, 10, 6, 6)$ (see Example 5.8 and Example 5.9 in [18], Theorem 21). The paper [18] shows the importance finding $bsr(\Gamma)$, which may be difficult, even if all four parameters of Γ are known. But, if all four parameters of Γ are known, then we have Mathematica program which finds feasible parameters of QSDs whose block graph is Γ. For example, can one determine $bsr(S_m(n))$ or $bsr(S_3(n))$?

Acknowledgement:

The authors are grateful for the anonymous referee's helpful suggestions and comments which greatly improved a previous draft of the manuscript submitted.

References

[1] A. Baartmans and M. S. Shrikhande. Designs with no three mutually disjoint blocks. *Discrete Math.*, 40(2-3):129–139, 1982.

[2] T. Beth, D. Jungnickel, and H. Lenz. *Design theory*. Cambridge University Press, Cambridge, 1986.

[3] N. L. Biggs. Families of parameters for SRNT graphs. Research report, 2009. arXiv:0911.2455

[4] N. L. Biggs. Strongly regular graphs with no triangles. Research report, 2009. arXiv:0911.2160

[5] N. L. Biggs. The second subconstituent of some strongly regular graphs. Preprint, 2010. arXiv:1003.0175

[6] A. E. Brouwer. Strongly regular graphs. *In: Handbook of Combinatorial Designs, Second Edition*, Series: Discrete Mathematics and Its Applications, C. Colbourn, J. Dinitz, CRC Press, 2009.

[7] A. E. Brouwer and W. H. Haemers. *Spectra of graphs*. Universitext. Springer, New York, 2012.

[8] A. R. Calderbank and P. Frankl. Binary codes and quasi-symmetric designs. *Discrete Math.*, 83(2-3):201–204, 1990.

[9] P. J. Cameron. The pre-history of the Higman-Sims graph, Peter Cameron's blog, posted 23/11/2011.

[10] P. J. Cameron and J. H. van Lint. *Designs, graphs, codes and their links*, volume 22 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1991.

[11] M. J. Coster and W. H. Haemers. Quasi-symmetric designs related to the triangular graph. *Des. Codes Cryptogr.*, 5(1):27–42, 1995.

[12] C. D. Godsil. Problems in algebraic combinatorics. *Electron. J. Combin.*, 2:Feature 1, approx. 20 pp. 1995.

[13] J.-M. Goethals and J. J. Seidel. Strongly regular graphs derived from combinatorial designs. *Canad. J. Math.*, 22:597–614, 1970.

[14] W. H. Haemers. A nonexistence result for quasi-symmetric designs. *Sankhyā Ser. A*, 54(Special Issue dedicated to the memory of R.C. Bose):189–190, 1992. Combinatorial Mathematics and Applications (Calcutta, 1988).

[15] M. Hall, Jr. *Combinatorial theory.* Wiley Classics Library. John Wiley & Sons Inc., New York, second edition, 1998. A Wiley-Interscience Publication.

[16] S. K. Houghten, L. H. Thiel, J. Janssen, and C. W. H. Lam. There is no (46,6,1) block design. *J. Combin Des.*, 9:60–71, 2001.

[17] Y. J. Ionin and M. S. Shrikhande. *Combinatorics of Symmetric Designs.* Cambridge University Press, Cambridge, 2006.

[18] Y. J. Ionin. Binary representations of regular graphs. *Int. J. Comb.*, pages Art. ID 101928, 15, 2011.

[19] M. H. Klin and A. J. Woldar. Dale Mesner, Higman & Sims, and the strongly regular graph with parameters $(100, 22, 0, 6)$. *Bull. Inst. Combin. Appl.*, 63:13–35, 2011.

[20] N. B. Limaye, S. S. Sane, and M. S. Shrikhande. The structure of triangle-free QSDs. *Discrete Math.*, 64(2-3):199–207, 1987.

[21] V. C. Mavron, T. P. McDonough, and M. S. Shrikhande. On quasi-symmetric designs with intersection difference three. *Des. Codes Cryptogr.*, 63(1):73–86, 2012.

[22] A. Munemasa, V. D. Tonchev. A new quasi-symmetric 2-(56,16,6) design obtained from codes. *Discrete Math.*, 284:231–234, 2004.

[23] A. Neumaier. Strongly regular graphs with smallest eigenvalue $-m$. *Arch. Math. (Basel)*, 33(4):392–400, 1979/80.

[24] R. M. Pawale. Quasi-symmetric 3-designs with triangle-free graph. *Geom. Dedicata*, 37(2):205–210, 1991.

[25] R. M. Pawale. Non-existence of triangle-free quasi-symmetric designs. *Des. Codes Cryptogr.*, 37(2):347–353, 2005.

[26] R. M. Pawale. Quasi-symmetric designs with fixed difference of block intersection numbers. *J. Combin. Des.*, 15(1):49–60, 2007.

[27] R. M. Pawale. A note on triangle-free quasi-symmetric designs. *J. Combin. Des.*, 19(6):422–426, 2011.

[28] R. M. Pawale. Quasi-symmetric designs with the difference of block intersection numbers two. *Des. Codes Cryptogr.*, 58(2):111–121, 2011.

[29] R. M. Pawale. Finiteness results in triangle-free quasi-symmetric designs. *J. Combin. Des.*, 21(4):157–162, 2013.

[30] S. S. Sane and M. S. Shrikhande. Finiteness questions in quasi-symmetric designs. *J. Combin. Theory Ser. A*, 42(2):252–258, 1986.

[31] S. S. Sane and M. S. Shrikhande. Quasi-symmetric $2, 3, 4$-designs. *Combinatorica*, 7(3):291–301, 1987.

[32] J. J. Seidel, Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3. *Linear Algera and Appl. 1*, 281-298, 1968.

[33] M. S. Shrikhande. Designs with triangle-free graph, Proceedings of the seminar on 'Combinatorics and Applications in honour of Prof. S.S. Shrikhande on his 65th birthday', Indian Statistical Institute, December 14-17, 334-339, 1982.

[34] M. S. Shrikhande. On the parameters of a certain exceptional block design. *Utilitas Mathematica*, 26:103–108, 1984.

[35] M. S. Shrikhande. Quasi-symmetric designs, in: C.J. Colbourn, J.F. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, 430–434, 1996.

[36] M. S. Shrikhande. Designs, intersection numbers, and codes. *IMA Volume in Mathematics and its Applications*,(D.K. Ray-Chaudhuri, ed.), 21, Coding Theory and Design Theory:304–318, 1990.

[37] M. S. Shrikhande and S. S. Sane. *Quasi-symmetric designs*, Volume 164 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.

[38] S. S. Shrikhande and Bhagwandas. Duals of incomplete block designs. *J. Indian Statist. Assoc.*, 3:30–37, 1965.

[39] S. S. Shrikhande, D. Raghavarao, and S. K. Tharthare. Non-existence of some unsymmetrical partially balanced incomplete block designs. *Canad. J. Math.*, 15:686–701, 1963.

[40] N. M. Singhi. Personal communication.

[41] Wolfram. Mathematica: A computer algebra system. Version 5., 1988. http://www.wolfram.com/mathematica/

[42] WxMaxima.Version 11.04.0, 2011 http://wxmaxima.soft112.com/