Leapfrog Constructions: From Continuant Polynomials to Permanents of Matrices

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Abstract

We study noncommutative continuant polynomials via a new leapfrog construction. This needs the introduction of new indeterminates and leads to generalizations of Fibonacci polynomials, Lucas polynomials and other families of polynomials. We relate these polynomials to various topics such as quiver algebras and tilings. Finally, we use permanents to give a broad perspective on the subject.

1 Introduction

The continuants (or continuant polynomials) p_n were introduced in the noncommutative setting in [6] by P.M. Cohn, who used them to describe some groups of invertible matrices and, under suitable hypotheses, to analyze comaximal relations in a ring. See [5] for more details. Continuants also appear in connection with the weak algorithm [5] and more recently they have been used to characterize Euclidean pairs and quasi Euclidean rings [1]. Cohn also calls the construction of continuants *leapfrog construction*.

In this paper, we prefer to present Cohn's construction with a different notation, which we think is more convenient, and which highlights the leapfrog structure of the construction of the polynomials p_n (Section 2). We reinterpret the continuants p_n in term of suitable quiver algebras with two vertices A and B, in which the paths leap alternatively from A to B. Cohn's polynomials p_n are related to the elementary group $E_2(R)$ (equation (8)). Here we find the leapfrog structure again, because the n-th powers of a suitable

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coset $E_2(R)P(0)$ of an element $P(0)$ of $GL_n(R)$ modulo $E_2(R)$ jump alternatively from $E_2(R)$ when n is even to $E_2(R)P(0)$ when n is odd.

In a completely different setting, other polynomials h_n were introduced in [2], to compute the inverse of an isomorphism f in a factor category $\mathcal{A}/\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_n$ from the inverses of the images of the isomorphism f in factor categories $\mathcal{A}/\mathcal{I}_t$ of $\mathcal{A}$ modulo ideals $\mathcal{I}_t$ of $\mathcal{A}$. Since there is a surprising similarity between the structure of Cohn's polynomials p_n and that of the polynomials h_n, we have studied whether both families of polynomials were specializations of a more general class of polynomials. In this investigation, we have met with other classes of noncommutative polynomials (the generalized Fibonacci polynomials f_n, the generalized Lucas polynomials ℓ_n and other "circular" polynomials c_n), which have evident combinatorial interpretations: they parametrize tilings and circular tilings of stripes of squares with dominoes (Section 7). Commutative generalized Fibonacci polynomials appear naturally in Combinatorics [3], related to the Fibonacci sequence, and in Complex Analysis [7], related to generalized continuous fractions. Our noncommutative generalized Fibonacci polynomials are the noncommutative analogue of the generalized Fibonacci polynomials studied in [3].

We have thus found a very natural "hierarchy" of polynomials (see the diagram in Remark 16(4)). Our polynomials turn out to be noncommutative permanents of suitable matrices in noncommutative indeterminates (Theorem 17 and Corollary 18).

The rings and algebras we work with are associative and have an identity element.

2 Two sets of indeterminates in Cohn's continuants p_n

We begin this Section by recalling the definition of the continuants [6], which are noncommutative polynomials with coefficients in the ring $\mathbb{Z}$ of integers. Let $t_1, t_2, t_3, \dots$ be infinitely many noncommutative indeterminates over the ring $\mathbb{Z}$. There is a strictly ascending chain

$$\mathbb{Z}\langle t_1\rangle \subset \mathbb{Z}\langle t_1, t_2\rangle \subset \mathbb{Z}\langle t_1, t_2, t_3\rangle \subset \dots$$

of noncommutative integral domains, where $\mathbb{Z}\langle t_1, \dots, t_n\rangle$ denotes the ring of polynomials in the noncommutative indeterminates $t_1, \dots, t_n$ with coefficients in $\mathbb{Z}$, and the union of this chain is the ring $\mathbb{Z}\langle t_1, t_2, t_3, \dots\rangle$. The continuants $p_n = p_n(t_1, \dots, t_n)$ are defined by the recursion formulae:

$$\begin{array}{l} p_{-1} = 0, \quad p_0 = 1, \text{ and, for } n \geqslant 1, \\ p_n(t_1, \dots, t_n) = p_{n-1}(t_1, \dots, t_{n-1})t_n + p_{n-2}(t_1, \dots, t_{n-2}). \end{array} \tag{1}$$

For every $f \in \mathbb{Z}\langle t_1, t_2, t_3, \dots\rangle$, we will use P.M. Cohn's convenient notation, introduced in [5, p. 147], denoting the 2×2-matrix $\begin{pmatrix} f & 1 \\ 1 & 0 \end{pmatrix}$ by $P(f)$. Then

$$P(t_1)P(t_2)\dots P(t_n) = \begin{pmatrix} p_n(t_1, \dots, t_n) & p_{n-1}(t_1, \dots, t_{n-1}) \\ p_{n-1}(t_2, \dots, t_n) & p_{n-2}(t_2, \dots, t_{n-1}) \end{pmatrix}. \tag{2}$$

Since the inverse of the matrix $P(f)$ is given by $P(0)P(-f)P(0)$, we easily get that the inverse of the matrix in (2) is given by

$$(-1)^n \begin{pmatrix} p_{n-2}(t_{n-1},\ldots,t_2) & -p_{n-1}(t_{n-1},\ldots,t_1) \\ -p_{n-1}(t_n,\ldots,t_2) & p_n(t_n,\ldots,t_1) \end{pmatrix}. \tag{3}$$

This leads to the following well-known relations:

$$p_n(t_1,\ldots,t_n)p_{n-1}(t_{n-1},\ldots,t_2) - p_{n-1}(t_1,\ldots,t_{n-1})p_{n-1}(t_n,\ldots,t_2) = (-1)^n \tag{4}$$

$$p_n(t_1,\ldots,t_n)p_{n-1}(t_{n-1},\ldots,t_1) = p_{n-1}(t_1,\ldots,t_{n-1})p_n(t_n,\ldots,t_1) \tag{5}$$

This last relation is due to Wedderburn [9].

Using the associativity of the product of matrices in (2), we also get, for any $1 \leqslant k \leqslant n$, the formula:

$$\begin{aligned} &p_n(t_1,\ldots,t_n) \\ &= p_k(t_1,\ldots,t_k)p_{n-k}(t_{n-k+1},\ldots,t_n) + p_{k-1}(t_1,\ldots,t_{k-1})p_{n-k-1}(t_{n-k},\ldots,t_n). \end{aligned} \tag{6}$$

For $k=1$, this gives back the equation given in [5, (14), page 148]:

$$p_n(t_1,\ldots,t_n) = t_1 p_{n-1}(t_2,\ldots,t_n) + p_{n-2}(t_3,\ldots,t_n).$$

The continuant polynomial $p_n(t_1,\ldots,t_n)$ can be presented via a leapfrog construction in the following sense. The first term of p_n is $t_1t_2\cdots t_n$. The next terms are obtained by erasing two consecutive indeterminates (the frog leaps over them) from $t_1t_2\cdots t_n$ to get the sum: $t_3t_4\cdots t_n + t_1t_4t_5\cdots t_n + t_1t_2t_5\cdots t_n + \ldots$. As far as the following terms are concerned, we erase 2 pairs of consecutive indeterminates (2 jumps) and get the terms

$$\sum_{1\leqslant i_1<i_2-1\leqslant n} t_1\cdots \widehat{t_{i_1}}\widehat{t_{i_1+1}}\cdots \widehat{t_{i_2}}\widehat{t_{i_2+1}}\cdots t_n.$$

We then continue adding terms corresponding to 3 leaps, 4 leaps, and so on. Finally, we can write

$$p_n(t_1,\ldots,t_n) = \sum_{i_1,i_2,\ldots,i_j} t_1\cdots \widehat{t_{i_1}}\widehat{t_{i_1+1}}\cdots \widehat{t_{i_2}}\widehat{t_{i_2+1}}\cdots \widehat{t_{i_j}}\widehat{t_{i_j+1}}\cdots t_n, \tag{7}$$

where $1 \leqslant j \leqslant \lfloor n/2 \rfloor$ and $i_j + 1 < i_{j+1}$ for every j,

As we have already said in the Introduction, in order to highlight another leapfrog structure of the construction of the polynomials p_n, we find more convenient to denote the indeterminates t_{2n-1} with odd index $2n-1$ by x_n, the indeterminates t_{2n} with even index $2n$ by y_n, and, similarly, the continuants p_{2n} by G_n and the continuants p_{2n-1} by H_n, so that G_n is a polynomial in the indeterminates $x_1, y_1, \dots, x_n, y_n$, and H_n is a polynomial in the indeterminates $x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n$.

In order to convince the reader that our different notation is more expressive than the original one, consider the first continuants p_n. They are

$$\begin{aligned}
&p_0 = 1, \quad p_1 = t_1, \quad p_2 = t_1t_2 + 1, \quad p_3 = t_1t_2t_3 + t_1 + t_3,\\
&p_4 = t_1t_2t_3t_4 + t_1t_2 + t_1t_4 + t_3t_4 + 1,\\
&p_5 = t_1t_2t_3t_4t_5 + t_1t_2t_3 + t_1t_2t_5 + t_1t_4t_5 + t_3t_4t_5 + t_1 + t_3 + t_5,\\
&p_6 = t_1t_2t_3t_4t_5t_6 + t_1t_2t_3t_4 + t_1t_2t_3t_6 + t_1t_2t_5t_6 + t_1t_4t_5t_6 + t_3t_4t_5t_6\\
&\qquad + t_1t_2 + t_1t_4 + t_1t_6 + t_3t_4 + t_3t_6 + t_5t_6 + 1, \dots
\end{aligned}$$

It is rather difficult to recognize the symmetry and the pattern of the polynomials p_n. But notice that the monomials in p_i are of even degree if i is even and of odd degree if i is odd. The same polynomials with the new notation become

$$\begin{aligned}
&G_0 = 1, \quad G_1 = x_1y_1 + 1, \quad G_2 = x_1y_1x_2y_2 + x_1y_1 + x_1y_2 + x_2y_2 + 1,\\
&G_3 = x_1y_1x_2y_2x_3y_3 + x_1y_1x_2y_2 + x_1y_1x_2y_3 + x_1y_1x_3y_3\\
&\qquad + x_1y_2x_3y_3 + x_2y_2x_3y_3 + x_1y_1 + x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_3 + 1
\end{aligned}$$

and

$$\begin{aligned}
&H_0 = 0, \quad H_1 = x_1, \quad H_2 = x_1y_1x_2 + x_1 + x_2,\\
&H_3 = x_1y_1x_2y_2x_3 + x_1y_1x_2 + x_1y_1x_3 + x_1y_2x_3 + x_2y_2x_3 + x_1 + x_2 + x_3.
\end{aligned}$$

The pattern with the notation x_i, y_i, G_i and H_i is much clearer.

Moreover, the ring $\mathbb{Z}\langle t_1, t_2, t_3, \dots\rangle$ is 2-graded, that is, graded over the group $\mathbb{Z}/2\mathbb{Z}$, because every polynomial is the sum of a sum of monomials of even degree and a sum of monomials of odd degree. The polynomials G_n turn out to be homogeneous of degree $\overline{0} \in \mathbb{Z}/2\mathbb{Z}$, and the polynomials H_n turn out to be homogeneous of degree $\overline{1} \in \mathbb{Z}/2\mathbb{Z}$. The reason of this lies in the defining recursion relation $p_n = p_{n-1}t_n + p_{n-2}$. Here the indices n and $n-2$ in p_n and p_{n-2} have the same parity. The index $n-1$ in p_{n-1} has different parity, but the degree of $p_{n-1}t_n$ has the same parity as p_n and p_{n-2}.

From

$$P(x_i)P(y_i) = \begin{pmatrix} x_iy_i + 1 & x_i \\ y_i & 1 \end{pmatrix},$$

it follows that

$$\begin{gathered}
\begin{pmatrix} x_1y_1 + 1 & x_1 \\ y_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} x_ny_n + 1 & x_n \\ y_n & 1 \end{pmatrix} = \\
\begin{pmatrix} G_n(x_1, y_1, \dots, x_n, y_n) & H_n(x_1, y_1 \dots, y_{n-1}, x_n) \\ H_n(y_1, x_2, \dots, x_n, y_n) & G_{n-1}(y_1, x_2, \dots, y_{n-1}, x_n) \end{pmatrix}.
\end{gathered} \tag{8}$$

Since, for any $f \in \mathbb{Z}\langle x_1, y_1, \dots\rangle$, we have

$$\begin{pmatrix} f & 1 \\ 1 & 0 \end{pmatrix}^{-1} = P(0)P(-f)P(0) = \begin{pmatrix} 0 & 1 \\ 1 & -f \end{pmatrix},$$

we easily conclude that the matrix on the right in equation (8) is invertible. Using the remarks about the parity of the degree of the homogeneous polynomials G_n and H_n, we get that the inverse of the matrix on the right in equation (8) is

$$\begin{pmatrix} G_{n-1}(x_n, y_{n-1}, \dots, y_1) & -H_n(x_n, \dots, y_1, x_1) \\ -H_n(y_n, x_n, \dots, y_1) & G_n(y_n, x_n, \dots, y_1, x_1) \end{pmatrix}. \tag{9}$$

From this inverse, the interested reader can easily obtain the equations analogous to the ones in (4) and (5).

The recursion formulae for p_n translate into the following ones between the G_n, H_n, x_n and y_n:

$$G_{n+1} = H_{n+1}y_{n+1} + G_n \quad \text{and} \quad H_{n+1} = G_n x_{n+1} + H_n. \tag{10}$$

Using the associativity of the product of matrices, we easily obtain the following formulas, for every $1 \leqslant k \leqslant n$:

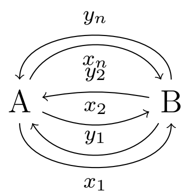
$$\begin{aligned} G_n(x_1, \dots, y_n) = G_k(x_1, \dots, x_{k-1}, y_k)G_{n-k}(x_{k+1}, \dots, y_n) \\ + H_k(x_1, \dots, y_{k-1}, x_k)H_{n-k}(y_{k+1}, x_{k+2} \dots, y_n) \end{aligned} \tag{11}$$

and

$$\begin{aligned} H_n(x_1, y_1, \dots, x_n) = G_k(x_1, \dots, y_k)H_{n-k}(x_k, y - k, \dots, x_n) \\ + H_k(x_1, y_1, \dots, x_k)G_{n-k-1}(y_{n-k}, \dots, y_{n-1}, x_n). \end{aligned} \tag{12}$$

All the polynomials p_n, H_n and G_n are sums of monomials with all the coefficients equal to 1. From the defining relations (10), it is easily seen that each G_n is a sum of monomials of all possible even degrees $\leqslant 2n$ and each H_n is a sum of monomials of all possible odd degrees $\leqslant 2n-1$. Also, the number m_n of monomials in p_n, which is clearly equal to $p_n(1, 1, \dots, 1)$, satisfies the relations $m_0 = 1, m_1 = 1, m_2 = 2, m_n = m_{n-1}+m_{n-2}$, hence $m_n = F_{n+1}$, the $(n+1)$-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.

Now consider the following directed graph (quiver) Γ_n with two vertices A and B:



For each $i = 1, 2, \dots, n$, the directed graph Γ_n has one arrow x_i from A to B, and one arrow y_i from B to A. Thus Γ_n has $2n$ arrows.

Let k be a field. Consider the quiver algebra $k\Gamma_n$ and the ideal I of $k\Gamma_n$ generated by all paths $x_iy_j \colon A \xrightarrow{x_i} B \xrightarrow{y_j} A$ with $i > j$ and all paths $y_ix_j \colon B \xrightarrow{y_i} A \xrightarrow{x_j} B$ with

$i \geqslant j$. The quotient algebra $k\Gamma_n/I$ is a finite dimensional k-algebra, because the longest possible path not in I is the path $x_1y_1x_2y_2\dots x_ny_n$ of length $2n$. In particular, $R := k\Gamma_n/I$ is an artinian ring, so that the Jacobson radical $J(R)$ is a nilpotent ideal that contains all nilpotent elements of R. Moreover, $R/J(R) \cong k \times k$. The algebra $R = R_0 \oplus R_1$ is 2-graded, where R_0 corresponds to the paths of even length and R_1 to the paths of odd length. In particular, the images of the polynomials G_n in R are in R_0 and the images of the polynomials H_n are in R_1. Notice that G_n is a linear combination of paths from A to A (i. e., cycles), and H_n is a linear combination of paths from A to B. The elements x_iy_j with $i \geqslant j$ are nilpotent of index 2, hence they belong to $J(R)$, so $\sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j$ is nilpotent, and $1-\sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j$ is invertible in R. Here, and in the next Proposition, the same symbol denotes both an element of $k\Gamma_n$ and its image in R.

Theorem 1. *In the ring $R = k\Gamma_n/I$, we have that*

$$H_n = \left(1 - \sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j\right)^{-1}\left(\sum_{i=1} x_i\right) \quad \text{and} \quad G_n = \left(1 - \sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j\right)^{-1}$$

for every $n \geqslant 0$.

Proof. We claim that $H_n = G_n\left(\sum_{i=1}^n x_i\right)$ and $G_n\left(1-\sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j\right) = 1$ in R for every $n \geqslant 1$. The proof of the claim is by induction on n. For $n = 1$, we have $G_1x_1 = (x_1y_1+1)x_1 = x_1 = H_1$ and $G_1(1-x_1y_1) = (1+x_1y_1)(1-x_1y_1) = 1$, because $y_1x_1 \in I$. Now assume that the claim is true for n in $R := k\Gamma_n/I$. We will show that the claim is also true for $n+1$ in $R' := k\Gamma_{n+1}/I'$. Using the relations (10), we have

$$\begin{aligned}
H_{n+1} - G_{n+1}\left(\sum_{i=1}^{n+1} x_i\right) &= (G_nx_{n+1} + H_n) - (H_{n+1}y_{n+1} + G_n)\left(\sum_{i=1}^{n+1} x_i\right)\\
&= G_nx_{n+1} + H_n - G_n\left(\sum_{i=1}^{n+1} x_i\right)\\
&= H_n - G_n\left(\sum_{i=1}^{n} x_i\right) = 0.
\end{aligned}$$

This proves the first formula in the claim for $n+1$, i. e., that

$$H_{n+1} = G_{n+1}\left(\sum_{i=1}^{n+1} x_i\right)$$

in R'. But $(G_{n+1}-G_n)\left(\sum_{i=1}^{n+1} x_i\right) = H_{n+1}y_{n+1}\left(\sum_{i=1}^{n+1} x_i\right) = 0$, so that we also have $H_{n+1} = G_n\left(\sum_{i=1}^{n+1} x_i\right)$ in R'. Finally, in $R' := k\Gamma_{n+1}/I'$, we have that

$$\begin{aligned}
&G_{n+1}\left(1-\textstyle\sum_{1\leqslant i\leqslant j\leqslant n+1} x_iy_j\right)\\
&\quad = (H_{n+1}y_{n+1} + G_n)\left(1-\textstyle\sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j - \sum_{i=1}^{n+1} x_iy_{n+1}\right)\\
&\quad = H_{n+1}y_{n+1} + G_n\left(1-\textstyle\sum_{1\leqslant i\leqslant j\leqslant n} x_iy_j\right) - G_n\left(\textstyle\sum_{i=1}^{n+1} x_iy_{n+1}\right)\\
&\quad = 1 + \left(H_{n+1} - G_n\left(\textstyle\sum_{i=1}^{n+1} x_i\right)\right)y_{n+1} = 1.
\end{aligned}$$

This concludes the proof of the claim. The statement of the Proposition for $n \geqslant 1$ follows immediately from the claim. The case $n = 0$ is trivial. $\square$

This theorem again shows that the structure of the H_n's is different from the structure of the G_n's, so that the choice of using different notations for the two families of polynomials is appropriate.

Notice that, in our graph Γ_n with two vertices, the paths leap alternatively from A to B.

3 Continuants and groups of matrices

For any fixed ring R, let $GL_2(R)$ denote the group of all invertible 2×2-matrices with entries in R, that is, $GL_2(R) = U(M_2(R))$, the group of units of the ring $M_2(R)$ of all 2×2-matrices with entries in R. Let $E_2(R)$ be the *elementary group*, that is, the subgroup of $GL_2(R)$ generated by all triangular matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and all triangular matrices $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, where x and y range in R. Notice that the triangular matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ in $GL_2(R)$ generate $E_2(R)$ as a semigroup, because the inverse of the triangular matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$, and similarly for $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$. Since the matrices of the type $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ form an abelian group isomorphic to the additive group of R, and similarly for the matrices $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, an arbitrary element of $E_2(R)$ is a product of finitely many elements of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} xy+1 & x \\ y & 1 \end{pmatrix}.$$

These are exactly the factors on the left in the equation (8). Thus an arbitrary element of $E_2(R)$ is a matrix of the form

$$\begin{pmatrix} G_n(x_1, y_1, \dots, x_n, y_n) & H_n(x_1, y_1 \dots, y_{n-1}, x_n) \\ H_n(y_1, x_2, \dots, x_n, y_n) & G_{n-1}(y_1, x_2, \dots, y_{n-1}, x_n) \end{pmatrix},$$

with $x_1, y_1, \dots, x_n, y_n \in R$.

Let G be the subsemigroup of the multiplicative semigroup $M_2(R)$ generated by all matrices of type

$$P(x) := \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

where x ranges in R. As Cohn proved in [6], the semigroup G, set of all products $P(x_1) \cdots P(x_n)$ with $n \geqslant 1$ and $x_1, \dots, x_n \in R$, is a group, because $P(0)^2$ is the identity of $GL_2(R)$ and $P(x)^{-1} = P(0)P(-x)P(0)$.

Theorem 2. *For any ring R, exactly one of the following two conditions holds:*

(a) *Either $G = E_2(R)$, or*

(b) *The group G is the semidirect product $E_2(R) \rtimes C$ of the group $E_2(R)$ and the cyclic group C of order 2 generated by the involution $P(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

The action of $P(0)$ on $E_2(R)$ is given by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto P(0) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} P(0) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \mapsto P(0) \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} P(0) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Proof. Suppose $G \neq E_2(R)$. Then $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = P(x)P(0)$ and $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = P(0)P(y)$, so that $E_2(R)$ is contained in G. It is easily verified that the action of $P(0)$ on $E_2(R)$, given by conjugation by the involution $P(0)$, is as in the last part of the statement of the Theorem. Since every generator $P(x)$ of G can be written as $P(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} P(0)$, where $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in E_2(R)$, it follows that $E_2(R)$ is a normal subgroup of G and $G = E_2(R)C$. In order to conclude the proof that G is the semidirect product of $E_2(R)$ and C, it remains to notice that $P(0) \notin E_2(R)$, because $G \neq E_2(R)$ and $G = E_2(R)C$. □

Proposition 3. *If R is any ring of characteristic 2, then $G = E_2(R)$.*

Proof. If R has characteristic 2, then, for $x_1 = 0$ and $y_1 = x_2 = y_2 = 1$, we have that

$$P(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x_1y_1 + 1 & x_1 \\ y_1 & 1 \end{pmatrix} \begin{pmatrix} x_2y_2 + 1 & x_2 \\ y_2 & 1 \end{pmatrix} \in E_2(R).$$

Thus $G = E_2(R)$ by Theorem 2. □

In the next proposition, $\operatorname{char}(R)$ denotes the characteristic of the ring R and $\det(A)$ denotes the determinant of the matrix A.

Proposition 4. *Let R be a commutative ring. Then:*

(a) $E_2(R) = \{ A \in G \mid \det(A) = 1 \}$.

(b) $G = E_2(R)$ *if and only if* $\operatorname{char}(R) = 2$.

Proof. As R is a commutative ring, the determinant $\det\colon G \to U(R)$ is a group morphism, whose kernel contains $E_2(R)$. Thus we have three groups $G \supseteq \ker\det \supseteq E_2(R)$. From Theorem 2, we know that $[G : E_2(R)] \leqslant 2$. Thus we have three possible cases:

First case: $G = \ker\det \supset E_2(R)$. In this case, $P(0) \in G = \ker\det$, so that $\det(P(0)) = 1$, that is, $-1 = 1$, hence $\operatorname{char}(R) = 2$. By Proposition 3, it follows that $G = E_2(R)$. This is not true in this first case. Thus this first case can never take place.

Second case: $G \supset \ker\det = E_2(R)$. In this case, $\operatorname{char}(R) \neq 2$ by Proposition 3. Thus (a) and (b) both hold in this second case.

Third case: $G = \ker\det = E_2(R)$. Then (a) is trivially true. Moreover, as in the first case, we have that $P(0) \in \ker\det$, so that $\det(P(0)) = 1$, hence $\operatorname{char}(R) = 2$. Thus (b) also holds. □

Let us go back to the case of R nonnecessarily commutative. When $G = E_2(R) \rtimes C$, we again found a leapfrog structure, because $\begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix} P(0)$ is an element of the coset $E_2(R)P(0)$ of G, so that the products on the left of (2) jump alternatively from elements of $E_2(R)$ when n is even to elements of the coset $E_2(R)P(0)$ when n is odd.

4 A second sequence of noncommutative polynomials h_n

In this Section, we study another sequence of noncommutative polynomials similar to the sequence of continuants p_n considered in the previous two sections. In Section 2, we have preferred to present continuants as polynomials in two infinite sets of indeterminates $x_1, x_2, x_3, \dots$ and $y_1, y_2, y_3, \dots$. The polynomials h_n we will construct now are noncommutative polynomials in the infinite set of indeterminates $x_1, x_2, x_3, \dots$ plus one more indeterminate y. The polynomials h_n have been introduced in [2], in the study of semilocal categories, in order to present a sort of Chinese Remainder Theorem that holds in preadditive categories. Let us recall the definition of those polynomials h_n, adapting the notation to the context of this paper.

In the paper [2], the authors essentially introduce noncommutative polynomials

$$h_n = h_n(x_1, x_2, \dots, x_n, y)$$

with coefficients in $\mathbb{Z}$, $n \geqslant 1$, defined as follows. Let $x_1, x_2, x_3, \dots, y$ be infinitely many noncommutative indeterminates over the ring $\mathbb{Z}$. Let $\mathbb{Z}\langle x_1, x_2, x_3, \dots, x_n, y\rangle$ denote the ring of polynomials in the $n+1$ noncommutative indeterminates x_1, x_2, x_3, …, x_n, y with coefficients in $\mathbb{Z}$. For each $n \geqslant 1$, there is a unique polynomial $h_n = h_n(x_1, x_2, \dots, x_n, y)$ in $\mathbb{Z}\langle x_1, x_2, \dots, x_n, y\rangle$ such that

$$1 + h_n y = (1 + x_1 y)(1 + x_2 y) \dots (1 + x_n y). \tag{13}$$

In fact, such a polynomial h_n exists because the product on the right in the equation (13) is of the form "1+ monomials that terminate with y". Moreover, h_n is the unique polynomial that satisfies the equation (13), because $\mathbb{Z}\langle x_1, x_2, \dots, x_n, y\rangle$ is an integral domain.

Proposition 5. *The polynomials h_n, $n \geqslant 1$, have the following properties:*

(a) $1 + yh_n = (1 + yx_1)(1 + yx_2) \dots (1 + yx_n)$ *for every* $n \geqslant 1$.

(b) $h_1 = x_1$, *and* $h_n = x_n + h_{n-1}(1 + yx_n)$ *for every* $n \geqslant 2$.

(c) *For every* $n \geqslant 1$,

$$h_n = \sum_{1 \leqslant i \leqslant n} x_i + \sum_{1 \leqslant i_1 < i_2 \leqslant n} x_{i_1} y x_{i_2} + \sum_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} x_{i_1} y x_{i_2} y x_{i_3} + \dots + x_1 y x_2 y \dots y x_n.$$

Proof. (a) Multiplying the equation (13) by y on the left, we get that

$$\begin{aligned} y(1 + h_n y) &= y(1 + x_1 y)(1 + x_2 y) \dots (1 + x_n y) \\ &= (y + yx_1 y)(1 + x_2 y) \dots (1 + x_n y) = (1 + yx_1) y (1 + x_2 y) \dots (1 + x_n y) \\ &= (1 + yx_1)(1 + yx_2) y \dots (1 + x_n y) \\ &\vdots \\ &= (1 + yx_1)(1 + yx_2) \dots (1 + yx_n) y. \end{aligned}$$

But

$$y(1 + h_n y) = y + yh_n y = (1 + yh_n) y,$$

so that (a) holds because $\mathbb{Z}\langle x_1, \dots, x_n, y \rangle$ is an integral domain.

(b) Induction on $n \geqslant 1$. From the definition of h_1, we have that $1 + h_1 y = 1 + x_1 y$, so $h_1 = x_1$. As far as an arbitrary $n \geqslant 1$ is concerned, we have, from (13), that

$$\begin{aligned} 1 + h_{n+1} y &= (1 + x_1 y) \dots (1 + x_{n+1} y) = (1 + h_n y)(1 + x_{n+1} y) \\ &= 1 + h_n y + x_{n+1} y + h_n y x_{n+1} y, \end{aligned}$$

from which $h_{n+1} = h_n + x_{n+1} + h_n y x_{n+1} = x_{n+1} + h_n(1 + yx_{n+1})$.

(c) follows from the equation (13). □

Corollary 6. $h_n(x_1, x_2, \dots, x_n, y) = x_1 + (1 + x_1 y) h_{n-1}(x_2, x_3 \dots, x_n, y)$

Proof. From Proposition 5(a), applied to $n + 1$ and n, we have that

$$\begin{aligned} 1 + yh_{n+1} &= (1 + yx_1)(1 + yx_2) \dots (1 + yx_{n+1}) \\ &= (1 + yx_1)(1 + yh_n(x_2, x_3 \dots, x_{n+1}, y)) \\ &= 1 + y(x_1 + (1 + x_1 y) h_n(x_2, x_3 \dots, x_{n+1}, y)). \end{aligned}$$

Now we conclude from the fact that our ring is an integral domain. □

Corollary 6 is the analogue of the formula (14) in [5, p. 148].
The first polynomials h_n are

$$h_1 = x_1, \quad h_2 = x_1 y x_2 + x_1 + x_2,$$
$$h_3 = x_1 y x_2 y x_3 + x_1 y x_2 + x_1 y x_3 + x_2 y x_3 + x_1 + x_2 + x_3.$$

Thus

$$h_1(x_1) = H_1(x_1), \quad h_2(x_1, x_2, y) = H_2(x_1, y, x_2),$$

where H_n denotes the continuant introduced in Section 2, but

$$h_3(x_1, x_2, x_3, y) \neq H_3(x_1, y, x_2, y, x_3).$$

The product decomposition analogous to (2) and (8) for the polynomials h_n is the decomposition

$$\begin{pmatrix} x_1 & 1 \\ 1 & -y \end{pmatrix}\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & -y \end{pmatrix}\cdots\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n & 1 \\ 1 & -y \end{pmatrix} = \begin{pmatrix} h_n & 1 \\ 1 & -y \end{pmatrix}. \tag{14}$$

The proof is by induction on n. The case $n = 1$ is trivial. For the inductive step, it sufficies to check that

$$\begin{pmatrix} h_n & 1 \\ 1 & -y \end{pmatrix}\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n+1} & 1 \\ 1 & -y \end{pmatrix} = \begin{pmatrix} h_{n+1} & 1 \\ 1 & -y \end{pmatrix},$$

and this trivially follows from Proposition 5(b).

As usual, from the associativity of the product of matrices, we get the following formula, true for all $1 \leqslant l \leqslant n$ and relating h_n, h_l and h_{n-l}:

$$h_n = h_l(y h_{n-l}(x_{l+1}, \dots, x_n, y) + 1) + h_{n-l}(x_{l+1}, \dots, x_n, y), \tag{15}$$

where $h_n = h_n(x_1, \dots, x_n, y)$ and $h_l = h_l(x_1, \dots, x_l, y)$.

The properties of these polynomials h_n are very similar to the properties of the continuants studied in the previous sections. For instance, the analogues of the first formulae in [5, p. 148, (16) and (17)] are the equalities

$$h_n(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, y) = h_{n-1}(x_1, \dots, x_{i-1}, \widehat{x_i},, x_{i+1}, \dots, x_n, y)$$

for every $i = 1, 2, \dots, n$. To prove this, set $x_i = 0$ in the defining formula (13). The second formula in [5, p. 148, (16)] is:

Lemma 7. $h_n(1, x_2, \dots, x_n, y) = h_{n-1}(1 + x_2 + y x_2, x_3, \dots, x_n, y)$.

Proof. From Corollary 6, we have that

$$\begin{aligned} &h_{n-1}(1 + x_2 + y x_2, x_3, \dots, x_n, y) \\ &\qquad = 1 + x_2 + y x_2 + (1 + (1 + x_2 + y x_2) y) h_{n-2}(x_3, \dots, x_n, y) \\ &\qquad = 1 + (1 + y) x_2 + (1 + y + x_2 y + y x_2 y) h_{n-2}(x_3, \dots, x_n, y) \\ &\qquad = 1 + (1 + y) x_2 + (1 + y)(1 + x_2 y) h_{n-2}(x_3, \dots, x_n, y) \\ &\qquad = 1 + (1 + y)(x_2 + (1 + x_2 y) h_{n-2}(x_3, \dots, x_n, y)) \\ &\qquad = 1 + (1 + y) h_{n-1}(x_2, \dots, x_n, y) = h_n(1, x_2, \dots, x_n, y) \end{aligned}$$

by Corollary 6 again. □

Similarly, changing the sign of all the indeterminates $x_1, \dots, x_n, y$ in (13), we see that

$$h_n(-x_1, \dots, -x_n, -y) = -h_n(x_1, \dots, x_n, y),$$

which is the analogue of [5, p. 148, (19)].

All the polynomials h_n are sums of monomials with all the coefficients equal to 1. From Proposition 5(b), it is easily seen that each h_n is a sum of monomials of all possible odd degrees $\leqslant 2n-1$. Also, the number of monomials on the right hand side of (13) is 2^n, so that h_n is a sum of $2^n - 1$ monomials, that is, $h_n(1, 1, \dots, 1; 1) = 2^n - 1$.

The results in this Section show that the polynomials h_n have a behavior that is very similar to the behavior of the polynomials H_n defined in Section 2. It is very natural to ask what are the analogues of the polynomials G_n. The answer is that they are the noncommutative polynomials q_n defined by

$$q_0 = 1 \qquad \text{and} \qquad q_n = (1 + x_1 y)(1 + x_2 y) \dots (1 + x_n y).$$

We have that

$$\begin{pmatrix} x_1 & 1 \\ 1 & -y \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & -y \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ 1 & -y \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_n & h_n \\ 0 & 1 \end{pmatrix}. \tag{16}$$

Each q_n is a sum of 2^n distinct monomials, and $q_n(1, 1, \dots, 1; 1) = 2^n$.

Similar results hold for the polynomials q'_n defined by $q'_0 = 1$ and

$$q'_n(x_1, \dots, x_n, y) = (1 + y x_1)(1 + y x_2) \dots (1 + y x_n).$$

Then $q'_n = y h_n + 1$ (Proposition 5(a)). Trivially,

$$q'_n(x_1, \dots, x_n, y) = (1 + y x_1) q'_{n-1}(x_2, \dots, x_n, y).$$

Moreover,

$$\begin{aligned} &\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & 1 \\ 1 & -y \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & -y \end{pmatrix} \cdots \\ &\qquad \cdots \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n & 1 \\ 1 & -y \end{pmatrix} = \begin{pmatrix} q'_n & 0 \\ h_n & 1 \end{pmatrix}. \end{aligned} \tag{17}$$

We collect some formulae for q_n and q'_n. We leave the easy proofs to the reader.

Lemma 8. (a) $q_n(x_1, \dots, x_n, y) = h_n(x_1, \dots, x_n, y) y + 1$
(a') $q'_n(x_1, \dots, x_n, y) = y h_n(x_1, \dots, x_n, y) + 1$.
(b) $h_{n+1}(x_1, \dots, x_{n+1}, y) = q_n(x_1, \dots, x_n, y) x_{n+1} + h_n(x_1, \dots, x_n, y)$.
(b') $h_{n+1} = x_1 q'_n(x_2, \dots, x_n, y) + h_n(x_2, \dots, x_n, y)$.

(c) $q_n(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n,y) = q_{n-1}(x_1,\ldots,x_{i-1},\widehat{x_i},\ldots,x_{n-1},y)$.
(c') $q'_n(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n,y) = q'_{n-1}(x_1,\ldots,x_{i-1},\widehat{x_i},x_{i+1},\ldots,x_n,y)$.
(d) $q_n(-x_1,\ldots,-x_n,-y) = q_n(x_1,\ldots,x_n,y)$.
(d') $q'_n(-x_1,\ldots,-x_n,-y) = q'_n(x_1,\ldots,x_n,y)$.
(e) *For* $1 \leqslant l \leqslant n$, $q_n(x_1,\ldots,x_n,y) = q_l(x_1,\ldots,x_l,y)q_{n-l}(x_{l+1},\ldots,x_n,y)$.
(e') *For* $1 \leqslant l \leqslant n$, $q'_n(x_1,\ldots,x_n,y) = q'_l(x_1,\ldots,x_l,y)q'_{n-l}(x_{l+1},\ldots,x_n,y)$.
(f) *For* $1 \leqslant l \leqslant n$,

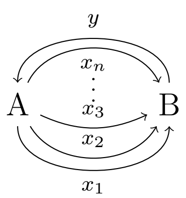
$$h_n(x_1,\ldots,x_n,y) = q_l(x_1,\ldots,x_l,y)h_{n-l}(x_{l+1}(x_{l+1},\ldots,x_n,y) + h_l(x_1,\ldots,x_l,y)$$

(f') *For* $1 \leqslant l \leqslant n$,

$$h_n(x_1,\ldots,x_n,y) = h_l(x_1,\ldots,x_l,y)q'_{n-l}(x_{l+1},\ldots,x_n,y) + h_{n-l}(x_{l+1},\ldots,x_n,y).$$

5 The graph algebra for the polynomials h_n

The polynomials h_n are also elements of a graph algebra with relations $k\Delta_n/I$. The quiver Δ_n is the following:



Thus the directed graph Δ_n has n arrows $x_1,\ldots,x_n$ from A to B and one arrow y from B to A, so that Δ_n has $n+1$ arrows. Let k be a field, consider the quiver algebra $k\Delta_n$ and the ideal I of $k\Delta_n$ generated by all paths x_iyx_j with $i \geqslant j$. The quotient algebra $k\Delta_n/I$ is a finite dimensional k-algebra, because the longest possible path not in I is the path $yx_1yx_2y\ldots x_ny$ of length $2n+1$. In particular, $R := k\Delta_n/I$ is an artinian ring with $R/J(R) \cong k \times k$. The algebra $R = R_0 \oplus R_1$ is 2-graded, where R_0 corresponds to the paths of even length and R_1 to the paths of odd length. In particular, the images of the polynomials h_n in R are all in R_1. The elements x_iy are nilpotent of index 2, hence they belong to $J(R)$, so $\sum_{i=1}^n x_iy$ is nilpotent, and $1-\sum_{i=1}^n x_iy$ is invertible.

Theorem 9. *In the ring* $R = k\Delta_n/I$*, we have that:*
(a) $h_n = \left(1-\sum_{i=1}^n x_iy\right)^{-1}\left(\sum_{i=1}^n x_i\right)$,
(b) $q_n = \left(1-\sum_{i=1}^n x_iy\right)^{-1}$,
(c) $h_n = \left(\sum_{i=1}^n x_i\right)\left(1-\sum_{i=1}^n yx_i\right)^{-1}$,
(d) $q'_n = \left(1-\sum_{i=1}^n yx_i\right)^{-1}$, *and*
(e) $h_n\left(\sum_{i=1}^n yx_i\right) = \left(\sum_{i=1}^n x_iy\right)h_n$, *for every* $n \geqslant 1$.

Proof. (a) It suffices to show that

$$\left(1 - \sum_{i=1}^{n} x_i y\right) h_n = \left(\sum_{i=1}^{n} x_i\right).$$

We do it by induction on $n \geqslant 1$. The case $n = 1$ is easy, because

$$(1 - x_1 y)h_1 = (1 - x_1 y)x_1 = x_1.$$

Suppose our formula true for n. Then

$$\begin{aligned}
&\left(1 - \sum_{i=1}^{n+1} x_i y\right) h_{n+1} \\
&= \left(1 - \sum_{i=1}^{n} x_i y - x_{n+1} y\right) (x_{n+1} + h_n(1 + y x_{n+1})) \\
&= \left(1 - \sum_{i=1}^{n} x_i y\right) x_{n+1} + \left(1 - \sum_{i=1}^{n} x_i y\right) h_n(1 + y x_{n+1}) - x_{n+1} y h_n (1 + y x_{n+1}) \\
&= x_{n+1} - \sum_{i=1}^{n} x_i y x_{n+1} + \left(\sum_{i=1}^{n} x_i\right) (1 + y x_{n+1}) \\
&= \sum_{i=1}^{n+1} x_i,
\end{aligned}$$

which concludes the proof of (a).

(b) We will prove that $q_n \left(1 - \sum_{i=1}^{n} x_i y\right) = 1$ in R by induction on $n \geqslant 1$. The case $n = 1$ is trivial. Suppose the formula true for $n - 1$. Then

$$\begin{aligned}
q_n \left(1 - \textstyle\sum_{i=1}^{n} x_i y\right) &= q_{n-1}(1 + x_n y) \left(1 - \textstyle\sum_{i=1}^{n} x_i y\right) \\
&= q_{n-1} \left(1 + x_n y - \textstyle\sum_{i=1}^{n} x_i y\right) = q_{n-1} \left(1 - \textstyle\sum_{i=1}^{n-1} x_i y\right) = 1
\end{aligned}$$

by the inductive hypothesis.

(e) is easy and left to the reader.

(d) Like for (b), it suffices to show by induction on n that

$$q'_n \left(1 - \sum_{i=1}^{n} y x_i\right) = 1.$$

(c) From (e) and (a), we have that

$$h_n \left(1 - \sum_{i=1}^{n} y x_i\right) = \left(1 - \sum_{i=1}^{n} x_i y\right) h_n = \sum_{i=1}^{n} x_i.$$

Thus (c) also holds. □

Again, in our graph Δ_n with two vertices A and B, the paths leap alternatively from A to B.

Proposition 10. *The matrices* $\begin{pmatrix} x_i & 1 \\ 1 & -y \end{pmatrix}$ $(i = 1, 2, \dots, n)$ *and* $\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}$ *are invertible in the ring* $R = k\Delta_n/I$ *(that is, they are invertible elements of the ring* $M_2(R)$*). Their inverses are*

$$\begin{pmatrix} y - yx_iy & 1 - yx_i \\ 1 - x_iy & -x_i \end{pmatrix} \quad \text{and} \quad P(0)P(-y)P(0)$$

respectively.

6 The generalized Fibonacci polynomials f_n

We now generalize the continuants $p_n(t_1, t_2, \dots, t_n)$ of Section 2 from the case of one sequence of indeterminates t_n to the case of two sequences of indeterminates. The new polynomials we obtain appear in the study of generalized continued fractions, hence we call them *generalized Fibonacci polynomials.*

Our new polynomials f_n are again polynomials with coefficients in $\mathbb{Z}$ and in the noncommutative indeterminates $x_1, x_2, x_3, \dots$ and $y_1, y_2, y_3, \dots$ They are defined by the recursion formulae:

$$\begin{aligned} &f_{-1} = 0, \quad f_0 = 1, \\ &f_n(x_1, \dots, x_n, y_1, \dots, y_n) = f_{n-1}(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})x_n \\ &\qquad\qquad\qquad\qquad + f_{n-2}(x_1, \dots, x_{n-2}, y_1, \dots, y_{n-2})y_n. \end{aligned} \tag{18}$$

Thus, when we specialize all the indeterminates y_i to 1, these polynomials turn out to be the continuants p_n of Section 2 in one countable set of indeterminates, that is, $p_n(t_1, \dots, t_n) = f_n(t_1, \dots, t_n, 1, 1, \dots, 1)$. Also,

$$f_n(x, \dots, x, 1, 1, \dots, 1) = F_n(x),$$

the commutative Fibonacci polynomials, which have received much attention [3, Introduction]. Moreover, $f_n(x, \dots, x, y, \dots, y)$ turns out to be the noncommutative analogue of the generalized Fibonacci polynomial $\{n\}_{x,y}$ studied in [3].

The first of these polynomials f_n are

$$\begin{aligned} &f_0 = 1, \qquad f_1 = x_1, \qquad f_2 = x_1x_2 + y_2, \\ &f_3 = x_1x_2x_3 + x_1y_3 + y_2x_3, \\ &f_4 = x_1x_2x_3x_4 + x_1x_2y_4 + x_1y_3x_4 + y_2x_3x_4 + y_2y_4, \\ &f_5 = x_1x_2x_3x_4x_5 + x_1x_2x_3y_5 + x_1x_2y_4x_5 + x_1y_3x_4x_5 \\ &\qquad + x_1y_3y_5 + y_2x_3x_4x_5 + y_2x_3y_5 + y_2y_4x_5, \dots \end{aligned}$$

The number of monomials in each f_n is the $(n+1)$-th Fibonacci number F_{n+1}.

These polynomials can be built using a leapfrog construction similar to that in Section 2 for continuants polynomials. For f_n, start writing the product $x_1 \dots x_n$ and add all the monomials obtained by replacing all the possible disjoint consecutive pairs x_ix_{i+1} by y_{i+1}.

Notice that the indeterminate y_1 does not appear in any polynomial

$$f_n(x_1, \dots, x_n, y_1, \dots, y_n),$$

that is, every $f_n(x_1, \dots, x_n, y_1, \dots, y_n)$ has degree 0 in the indeterminate y_1. Let us prove by induction on n that

$$f_n(x_1,\ \dots,\ x_n,\ y_1,\ x_1x_2,\ x_2x_3,\ x_3x_4,\ \dots,\ x_{n-1}x_n) = F_{n+1}x_1x_2\dots x_n. \tag{19}$$

For $n = -1$, we have that $f_{-1} = 0 = F_0$. For $n = 0$, we have that $f_0 = 1 = F_1$. Assume the result true for the indices smaller than n. Then

$$\begin{aligned} &f_n(x_1, \dots,\ x_n,\ y_1,\ x_1x_2,\ x_2x_3,\ x_3x_4, \dots,\ x_{n-1}x_n) \\ &\quad = f_{n-1}(x_1, \dots, x_{n-1},\ y_1,\ x_1x_2,\ x_2x_3,\ x_3x_4, \dots,\ x_{n-2}x_{n-1})x_n \\ &\qquad\qquad + f_{n-2}(x_1, \dots, x_{n-2},\ y_1,\ x_1x_2,\ x_2x_3,\ x_3x_4, \dots,\ x_{n-3}x_{n-2})x_{n-1}x_n \\ &\quad = F_nx_1x_2\dots x_{n-1}\cdot x_n + F_{n-1}x_1x_2\dots x_{n-2}\cdot x_{n-1}x_n \\ &\quad = F_{n+1}x_1x_2\dots x_n. \end{aligned}$$

Let us mention another interesting specialization: if one puts $y_i = x_i$, for any $i = 1, \dots, n$ we obtain new polynomials in the variables $x_1, \dots, x_n$:

$$d_n(x_1, \dots, x_n) = f_n(x_1, \dots, x_n, x_1, \dots, x_n).$$

In particular, we have $d_0 = 1$, $d_1(x_1) = x_1$, $d_2(x_1, x_2) = x_1x_2 + x_2$, $d_3(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_3 + x_2x_3$. Specializing further, we get the sequence of natural numbers $D_n = d_n(0, 1, 2, \dots, n-1)$ that gives the number of derangements in the symmetric group S_n. We will see that the polynomials f_n admit a matrix presentation (cf. (20)). This easily leads to a presentation of the elements of the sequence D_n (cf. also [8]).

The polynomials f_n are also homogeneous polynomials of degree n if we give all the indeterminates x_i degree 1 and all the indeterminates y_i degree 2. Thus, if we view $f_n(x_1, \dots, x_n, y_1, \dots, y_n)$ as an element of $k\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$, where k is any commutative ring and $k\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$ is the free k-algebra in the noncommutative indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$, then

$$f_n(\lambda x_1, \dots, \lambda x_n, \lambda^2 y_1, \dots, \lambda^2 y_n) = \lambda^n f_n(x_1, \dots, x_n, y_1, \dots, y_n)$$

for every $\lambda \in k$.

Finally, it is not difficult to see that $f_n(x, \dots, x, y, \dots, y)$ is the sum of all monic monomials of degree n in the free algebra $\mathbb{Z}\langle x, y\rangle$ in the two indeterminates x, y when the indeterminate x is given degree 1 and y is given degree 2. Two other formulae, that essentially appear in [3], are

$$f_n(2, 2, \dots, 2, -1, -1, \dots, -1) = n$$

and, more generally,

$$f_n(x+1, x+1, \dots, x+1, -x, -x, \dots, -x) = 1 + x + x^2 + \dots + x^{n-1}.$$

These new polynomials f_n are also entries of 2×2-matrices, as in the identity (2). Now we have that

$$\begin{pmatrix} x_1 & 1 \\ y_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ y_n & 0 \end{pmatrix} = \begin{pmatrix} f_n(x_1,\dots,x_n,y_1,\dots,y_n) & f_{n-1}(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1}) \\ y_1 f_{n-1}(x_2,\dots,x_n,y_2,\dots,y_n) & y_1 f_{n-2}(x_2,\dots,x_{n-1},y_2,\dots,y_{n-1}) \end{pmatrix}. \tag{20}$$

This is clear for $n = 1$. The general case can be proved by induction, since on writing $f_i = f_i(x_1,\dots,x_i,y_1,\dots,y_i)$, $f'_i = f_i(x_2,\dots,x_{i+1},y_2,\dots,y_{i+1})$, we have that

$$\begin{pmatrix} f_{n-1} & f_{n-2} \\ y_1 f'_{n-2} & y_1 f'_{n-3} \end{pmatrix} \begin{pmatrix} x_n & 1 \\ y_n & 0 \end{pmatrix} = \begin{pmatrix} f_n & f_{n-1} \\ y_1 f'_{n-1} & y_1 f'_{n-2} \end{pmatrix}. \tag{21}$$

Regrouping the first k matrices, for $1 \leqslant k \leqslant n$, in (20), we get that the matrix on the right hand side of this equation is equal to the product

$$\begin{pmatrix} f_k(x_1,\dots,y_k) & f_{k-1}(x_1,\dots,y_{k-1}) \\ y_1 f_{k-1}(x_2,\dots,y_k) & y_1 f_{k-2}(x_2,\dots,y_{k-1}) \end{pmatrix} \cdot \begin{pmatrix} f_{n-k}(x_{k+1},\dots,y_n) & f_{n-k-1}(x_{k+1},\dots,y_{n-1}) \\ y_{k+1} f_{n-k-1}(x_{k+2},\dots,y_n) & y_{k+1} f_{n-k-2}(x_{k+2},\dots,y_{n-1}) \end{pmatrix}$$

Comparing the $(1,1)$ entry of this product with the corresponding entry in (20), we obtain:

$$f_n(x_1,\dots,y_n) = f_k(x_1,\dots,y_k) f_{n-k}(x_{k+1},\dots,y_n) + f_{k-1}(x_1,\dots,y_{k-1}) y_{k+1} f_{n-k-1}(x_{k+2},\dots,y_n). \tag{22}$$

Let us now mention a few consequences of the equation (22). First we remark that for $k = 1$, we have

$$f_n(x_1,\dots,x_n,y_1,\dots,y_n) = x_1 f_{n-1}(x_2,\dots,x_n,y_2,\dots,y_n) + y_2 f_{n-2}(x_3,\dots,x_n,y_3,\dots,y_n). \tag{23}$$

The formula obtained in (22) can also be considered as a generalization of an analogous classical relation for the usual Fibonacci numbers F_n; that is, if one specializes $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 1$, we get that $F_n = F_k F_{n-k} + F_{k-1} F_{n-k-1}$.

Using the recursive relation in the definition of f_n and the above equation (22), we easily obtain the following useful formula: for $1 \leqslant k < n$,

$$f_n(x_1,x_2,\dots,y_n) = f_{k+1}(x_1,\dots,x_k, f_{n-k}(x_{k+1},\dots,y_n), y_1,\dots,y_k, f_{n-k-1}(x_{k+2},\dots,y_n)) \tag{24}$$

In the free algebra $\mathbb{Z} < x_1, x_2, \dots ; y_1, y_2, \cdots >$, we can define the standard partial derivations $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial y_k}$, for $k \geqslant 1$.

Using the equation (22), we then have the following:

$$\begin{aligned} \frac{\partial f_n(x_1,\ldots,y_n)}{\partial x_k} &= f_{k-1}(x_1,\ldots,y_{k-1})f_{n-k}(x_{k+1},\ldots,y_n), \quad \text{for } 1 \leqslant k \leqslant n. \\ \frac{\partial f_n(x_1,\ldots,y_n)}{\partial y_k} &= f_{k-2}(x_1,\ldots,y_{k-2})f_{n-k}(x_{k+1},\ldots,y_n), \quad \text{for } 2 \leqslant k \leqslant n. \end{aligned} \tag{25}$$

As mentioned above, the polynomials f_n, specialized in $y_1 = \cdots = y_n = 1$, give the continuant polynomials p_n. In particular, it is easy to obtain formulas for the continuant polynomials analogous to the equations (22) and (25). On the other hand, we may as well specialize the polynomials f_n in $x_1 = \cdots = x_n = 1$; we then get a family of polynomials $r_n(y_1,\ldots,y_n)$. The first values of these polynomials are $r_1 = 1$, $r_2 = 1+y_2$, $r_3 = 1+y_2+y_3, r_4 = 1+y_2+y_3+y_4+y_2y_4$, $r_5 = 1+y_2+y_3+y_4+y_5+y_2y_4+y_2y_5+y_3y_5$. They satisfy the recurrence relation

$$r_{n+1}(y_1,\ldots,y_{n+1}) = r_n(y_1,\ldots,y_n) + r_{n-1}(y_1,\ldots,y_{n-1})y_{n+1}.$$

In particular, if we specialize further, we obtain the following sequence of natural numbers $I_n = r_n(0,1,\ldots,n-1)$ which gives the number of involutions in the symmetric group S_n. As for earlier sequences, these can be presented using a specialization of the matrices in Equation (20) (cf. also [8]).

Using the polynomials r_n and the equation (22), we easily obtain, for $1 \leqslant k \leqslant n$, that

$$\begin{aligned} &f_n(1,\ldots,1,x_{k+1},\ldots,x_n,y_1,\ldots,y_n) \\ &\quad = r_k(y_1,\ldots,y_k)x_{k+1}f_{n-k}(x_{k+1},\ldots,y_n) \\ &\qquad + r_{k-1}(y_1,\ldots,y_{k-1})y_{k+1}f_{n-k-1}(x_{k+2},\ldots,y_n). \end{aligned} \tag{26}$$

From (23), we get the formulae analogous to those given by P. M. Cohn for the continuant polynomials in [5, formulae (16), p. 148]:

$$f_n(0,x_2,\ldots,x_n,y_1,\ldots,y_n) = y_2 f_{n-2}(x_3,\ldots,x_n,y_3,\ldots,y_n)$$

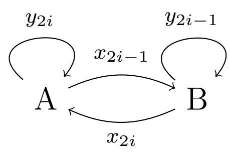
and

$$f_n(1,x_2,\ldots,x_n,y_1\ldots y_n) = f_{n-1}(x_2+y_2,x_3,\ldots,x_n,y_2,y_3,\ldots,y_n).$$

Conjugating all the matrices in the equation (20) by the invertible matrix $P(0)$, we find that

$$\begin{aligned} &\begin{pmatrix} 0 & y_1 \\ 1 & x_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & y_n \\ 1 & x_n \end{pmatrix} \\ &= \begin{pmatrix} y_1 f_{n-2}(x_2,\ldots,x_{n-1},y_2,\ldots,y_{n-1}) & y_1 f_{n-1}(x_2,\ldots,x_n,y_2,\ldots,y_n) \\ f_{n-1}(x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}) & f_n(x_1,\ldots,x_n,y_1,\ldots,y_n) \end{pmatrix}. \end{aligned} \tag{27}$$

We may also look at our noncommutative indeterminates $x_1,\ldots,x_n,y_1,\ \ldots,y_n$ in the polynomial $f_n(x_1,\ldots,x_n,y_1,\ldots,y_n)$ as arrows in a quiver E_n with two vertices A and B, where x_i is an arrow from A to B for i odd, x_i is an arrow from B to A for i even, y_i is an arrow from A to A for i odd, and y_i is an arrow from B to B for i even. The quiver E_n is the following:



The quiver E_n has $2n$ arrows. The polynomials f_i turn out to be linear combinations of paths with all the coefficients equal to one, and these paths in f_i are from A to A (hence they are cycles) when i is even, and are paths from A to B when i is odd.

Notice that we have

$$\begin{pmatrix} x_{2i-1} & 1 \\ y_{2i-1} & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} B \\ A \end{pmatrix} \text{ and } \begin{pmatrix} x_{2i} & 1 \\ y_{2i} & 0 \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}, \qquad (28)$$

so that again, in the product (20), the matrices alternatively leap from A to B.

7 Circular tilings and the polynomials c_n

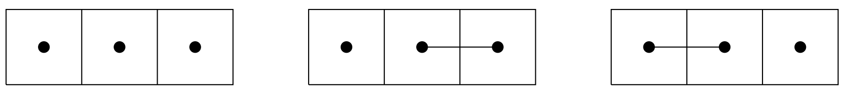
The classes of polynomials studied in this paper have a clear combinatorial interpretation. For instance:

(1) The monomials in the polynomial h_n parametrize the nonempty subsets of a set of n elements. It suffices to associate to the monomial $x_{i_1}yx_{i_2}y\dots yx_{i_t}$ of h_n the subset $\{i_1, i_2, \dots, i_t\}$ of $\{1, 2, \dots, n\}$.

(2) The monomials in the polynomial

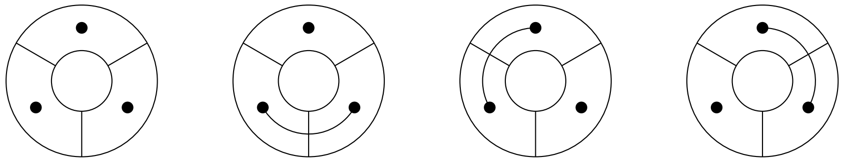
$$f_n = f_n(x_1, \dots, x_n, y_1, \dots, y_n)$$

studied in Section 6 parametrize the possible ways one can tile a strip of $1 \times n$ square cells with 1×1 squares and 1×2 dominos. Essentially, this is the standard interpretation of the Fibonacci numbers F_n via linear tilings. A *linear tiling* of a row of squares (a $1 \times n$ strip of square cells) is a covering of the strip of squares with squares and dominos (which cover two squares). For instance, the polynomial $f_3 = x_1x_2x_3 + x_1y_3 + y_2x_3$ parametrizes the set of the three linear tilings

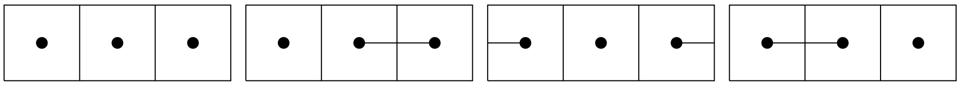


of a row of three squares. Here x_i denotes the i-th square and y_i denotes the domino that covers the $(i-1)$-th and the i-th square (y_i is the domino that "ends on the i-th square".) The Fibonacci number F_n represents the number of tilings of a strip of length n using length 1 squares and length 2 dominos.

It is also possible to consider *circular tilings*, where the deformed square are arranged in a circle [3]. For instance, the four possible circular tilings of a circle of three squares are



But for us it will be more convenient to represent the same four possible circular tilings of a circle of three squares as follows:



In this representation, we use an "out-of-phase domino" which spans the first and last cells of the tiling.

This suggests that there must also exist noncommutative polynomials c_n which parametrize the set of circular tilings of a circle of n squares. The idea is the following. Any circular tiling of a circle of n squares $s_1, \dots, s_n$ either does not contain the out-of-phase domino which spans s_n and s_1 or contains the out-of-phase domino. The circular tilings of the circle that do not contain the out-of-phase domino are in one-to-one correspondence with the linear tilings of a row of squares $s_1, \dots, s_n$, hence they are parametrized by the monomials of the polynomial $f_n(x_1, \dots, x_n, y_1, \dots, y_n)$. The circular tilings of the circle that do contain the out-of-phase domino are in one-to-one correspondence with the linear tilings of the row of squares $s_2, \dots, s_{n-1}$, hence they are parametrized by the monomials of the polynomial

$$f_{n-1}(x_2, \dots, x_{n-1}, y_2, \dots, y_{n-1}).$$

Now, as we have already said, the indeterminate x_i denotes a length 1 square in the i-th position, and the indeterminate y_i denotes a length 2 domino that ends in the i-th position. Thus we will denote the out-of-phase domino, which starts from the n-th position and ends in first one, by y_1. Hence we find that the circular tilings of a circle of n squares ($n \geqslant 1$) are parametrized by the noncommutative polynomials c_n defined by

$$\begin{aligned} c_n(x_1, \dots, x_n, y_1, \dots, y_n) &= \\ &= f_n(x_1, \dots, x_n, y_1, \dots, y_n) + y_1 f_{n-2}(x_2, \dots, x_{n-1}, y_2, \dots, y_{n-1}). \end{aligned} \tag{29}$$

Notice that:

(1) The indeterminate y_1 does not appear in the polynomials f_n, it appears in these new polynomials c_n for the first time.

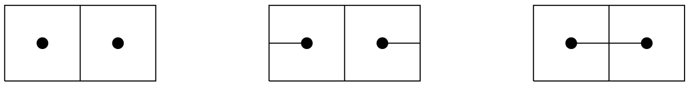
(2) In the polynomials f_n ($n \geqslant 1$), all the monomials begin with x_1 or y_2 and end with x_n or y_n. In the polynomials c_n, all the monomials begin with x_1, y_1 or y_2 and end with x_{n-1}, y_{n-1}, x_n or y_n.

(3) The first polynomials c_n are

$$\begin{aligned} &c_1(x_1, y_1) = x_1, \qquad c_2(x_1, x_2, y_1, y_2) = x_1x_2 + y_1 + y_2, \\ &c_3(x_1, x_2, x_3, y_1, y_2, y_3) = x_1x_2x_3 + x_1y_3 + y_1x_2 + y_2x_3, \\ &c_4(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = x_1x_2x_3x_4 + x_1x_2y_4 + x_1y_3x_4 \\ &\qquad\qquad + y_1x_2x_3 + y_1y_2 + y_2x_3x_4 + y_2y_4. \end{aligned}$$

(4) Once again these polynomials can be obtained by a leapfrog construction as follows: to obtain c_n you write $x_1 \dots x_n$ and replace every possible disjoint pairs $x_i x_{i+1}$ by y_{i+1} (indexing module n so that for a word terminating in x_n and starting by $x-1$ the letter x_n is erased and the letter x_1 is replaced by y_1).

(5) The polynomial $c_2(x_1, x_2, y_1, y_2) = x_1 x_2 + y_1 + y_2$ parametrizes the three circular tilings



of a circle of two squares, which we considered to be distinct. With these conventions, we have that $c_n(1, 1, \dots, 1) = L_n$ for every $n \geqslant 1$, where L_n indicates the n-th Lucas number, as desired. Here, the Lucas number L_n, defined by $L_0 = 2,\ L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ when $n \geqslant 2$, represents for $n \geqslant 1$ the number of circular tilings of a strip of length n using length 1 squares and length 2 dominos.

To relate the polynomials c_n with a suitable product of matrices we need the trace. For any ring R, the abelian group $M_n(R)$ of all $n \times n$-matrices over R can be viewed as an R-R-bimodule ${}_R M_n(R)_R$. The *trace* $\mathrm{tr}\colon {}_R M_n(R)_R \to {}_R R_R$, defined by $\mathrm{tr}(a_{ij})_{i,j} = \sum_{i=1}^n a_{ii}$, is an R-R-bimodule morphism with the further property that

$$\mathrm{tr}\left((a_{ij})_{i,j}(b_{ij})_{i,j}\right) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}.$$

From (20) and (29), we get that

Theorem 11.

$$c_n(x_1, \dots, x_n, y_1, \dots, y_n) = \mathrm{tr}\left(\begin{pmatrix} x_1 & 1 \\ y_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ y_n & 0 \end{pmatrix} \right)$$

8 The generalized Lucas polynomials ℓ_n and negative indices

Now we can define the *generalized Lucas polynomials* by the recursion formulae:

$$\begin{aligned} &\ell_0 = 2, \quad \ell_1 = x_1, \\ &\ell_n(x_1, \dots, x_n, y_1, \dots, y_n) = \ell_{n-1}(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}) x_n \\ &\qquad\qquad\qquad\qquad\qquad + \ell_{n-2}(x_1, \dots, x_{n-2}, y_1, \dots, y_{n-2}) y_n, \end{aligned} \tag{30}$$

generalizing [4, p. 142].

The first of these polynomials are

$$\begin{aligned} &\ell_0 = 2, \qquad \ell_1 = x_1, \qquad \ell_2 = x_1 x_2 + 2y_2, \\ &\ell_3 = x_1 x_2 x_3 + x_1 y_3 + 2 y_2 x_3, \\ &\ell_4 = x_1 x_2 x_3 x_4 + x_1 x_2 y_4 + x_1 y_3 x_4 + 2 y_2 x_3 x_4 + 2 y_2 y_4, \\ &\ell_5 = x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 y_5 + x_1 x_2 y_4 x_5 + x_1 y_3 x_4 x_5 \\ &\qquad + x_1 y_3 y_5 + 2 y_2 x_3 x_4 x_5 + 2 y_2 x_3 y_5 + 2 y_2 y_4 x_5, \dots \end{aligned}$$

The number of monomials in each ℓ_n is the $(n+1)$-th Fibonacci number and, for $n \geqslant 1$, one has that $\ell_n(1,1,\dots,1) = L_n$, the n-th Lucas number. The polynomial

$$\ell_n(x_1,\dots,x_n,y_1,\dots,y_n)$$

belongs to $\mathbb{Z}\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle$, though in this case also the indeterminate y_1 does not appear. But the polynomials ℓ_n are just a specialization of the polynomials f_n, as the following result shows.

Theorem 12.

$$\ell_n(x_1,\dots,x_n,y_1,\dots,y_n) = f_n(x_1,x_2,x_3\dots,x_n,y_1,2y_2,y_3,y_4,\dots,y_n)$$

for every $n \geqslant 1$.

Proof. Induction on n. The cases $n=1$ and $n=2$ are easily checked directly. For $n \geqslant 3$, assume that the theorem is true for $n-1$ and $n-2$. Then

$$\begin{aligned}
&\ell_n(x_1,\dots,x_n,y_1,\dots,y_n)\\
&= \ell_{n-1}(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1})x_n + \ell_{n-2}(x_1,\dots,x_{n-2},y_1,\dots,y_{n-2})y_n\\
&= f_{n-1}(x_1,\dots,x_{n-1},y_1,2y_2,\dots,y_{n-1})x_n + f_{n-2}(x_1,\dots,x_{n-2},y_1,2y_2\dots,y_{n-2})y_n\\
&= f_n(x_1,x_2,x_3\dots,x_n,y_1,2y_2,y_3,y_4,\dots,y_n).
\end{aligned}$$

□

Let us consider negative indices n. The sequence of Fibonacci numbers F_n can be extended to any negative index n using the recurrence formula $F_{n-2} = F_n - F_{n-1}$, and one finds that $F_{-n} = (-1)^{n+1}F_n$ for every $n \geqslant 0$. It is clear that our sequences of polynomials can be also extended to negative indices n.

Let us begin with the continuants $p_n(t_1,\dots,t_n)$, for which we have that

$$p_n(1,1,\dots,1) = F_{n+1}.$$

The difference of 1 in the indices in this formula is due to Cohn's original choice of the initial conditions $p_{-1} = 0, p_0 = 1$. Though the usual modern definition of Fibonacci numbers with $F_0 = 0, F_1 = 1$ is more appropriate, we prefer to continue using Cohn's original notation with $p_{-1} = 0$ and $p_0 = 1$.

The recursion formula must now be re-written as $p_{n-2} = p_n - p_{n-1}t_n$. Substituting $n-2$ with $-m$, we get that $p_{-m} = p_{-(m-2)} - p_{-(m-1)}t_{-(m-2)}$. We thus find that $p_0 = 1, p_{-1} = 0, p_{-2} = 1, p_{-3} = -t_{-1}, p_{-4} = t_{-1}t_{-2} + 1$, and so on. The polynomial p_{-n} for $n \geqslant 0$ turns out to be an element of the free algebra $\mathbb{Z}\langle t_{-1},t_{-2},\dots,t_{-(n-2)}\rangle$ in the noncommutative indeterminates $t_{-1},t_{-2},\dots,t_{-(n-2)}$. The general formula is given in the next proposition. Its proof is left to the reader.

Proposition 13.

$$\begin{aligned}
p_{-n}(t_{-1},t_{-2},\dots,t_{-(n-2)}) &= (-1)^n p_{n-2}(t_{-1},t_{-2},\dots,t_{-(n-2)})\\
&= p_{n-2}(-t_{-1},-t_{-2},\dots,-t_{-(n-2)})
\end{aligned}$$

for every integer $n \geqslant 0$.

Notice that relation (2) now becomes

$$\begin{pmatrix} -t_{-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -t_{-n} & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} p_{-(n+2)}(t_{-1},\dots,t_{-n}) & p_{-(n-1)}(t_{-1},\dots,t_{-(n-1)}) \\ p_{-(n-1)}(t_{-2},\dots,t_{-n}) & p_{-n}(t_{-2},\dots,t_{-(n-1)}) \end{pmatrix}. \tag{31}$$

For the generalized Fibonacci polynomials $f_n(x_1, x_2, \dots, x_n, y_1 \dots y_n)$, we have that the recursion formula

$$f_n(x_1,\dots,x_n,y_1,\dots,y_n) = f_{n-1}(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1})x_n + f_{n-2}(x_1,\dots,x_{n-2},y_1,\dots,y_{n-2})y_n$$

now becomes

$$f_{n-2}(x_1,\dots,x_{n-2},y_1,\dots,y_{n-2}) = f_n(x_1,\dots,x_n,y_1,\dots,y_n)y_n^{-1} - f_{n-1}(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1})x_ny_n^{-1}. \tag{32}$$

Thus we get that, for any integer $n \geqslant 0$, the polynomial f_{-n-2} must belong to the $\mathbb{Z}$-algebra $\mathbb{Z}\langle x_{-1}, x_{-2}, \dots, x_{-n}, y_0^{\pm1}, y_1^{\pm1}, \dots, y_{-(n-2)}^{\pm1}\rangle$, obtained from $\mathbb{Z}$ adjoining $2n$ algebraically independent elements $x_{-1}, x_{-2}, \dots, x_{-n}, y_0, y_{-1}, y_{-2}, \dots, y_{-(n-1)}$ with y_0, y_{-1}, $\dots$, $y_{-(n-1)}$ invertible. The proof of the following proposition, by induction, is left to the reader.

Proposition 14.

$$f_{-n-2}(x_{-1},x_{-2},\dots,x_{-n},y_0,y_1,\dots,y_{-(n-1)}) = f_n(-y_0^{-1}x_{-1}, -y_{-1}^{-1}x_{-2},\dots,-y_{-(n-1)}^{-1}x_{-n}, 1, y_0^{-1}, y_{-1}^{-1},\dots,y_{-(n-2)}^{-1})y_{-(n-2)}^{-1}$$

for every integer $n \geqslant 0$.

In extending the generalized Lucas polynomials ℓ_n to negative indices n, we have, like for the f_n's in (32), that $\ell_{n-2} = \ell_n y_n^{-1} - \ell_{n-1}x_ny_n^{-1}$. It is easily seen that

Proposition 15.

$$\ell_{-n}(x_1,x_0,x_{-1}\dots,x_{-(n-2)},y_0,y_{-1},y_{-2},\dots,y_{-(n-2)}) = (-1)^n\ell_n(x_1y_1^{-1}, x_0y_0^{-1}, x_{-1}y_{-1}^{-1},\dots,x_{-(n-2)}y_{-(n-2)}^{-1}; 1, y_0^{-1}, y_{-1}^{-1},\dots,y_{-(n-2)}^{-1})$$

for every integer $n \geqslant 0$.

9 The general pattern: the polynomials g_n

Now consider the following family of polynomials g_n, with $n \geqslant 0$. To define them, we need countably many noncommutative indeterminates x_{ij}, where $1 \leqslant i \leqslant j$. Set $g_0 = 1$ and

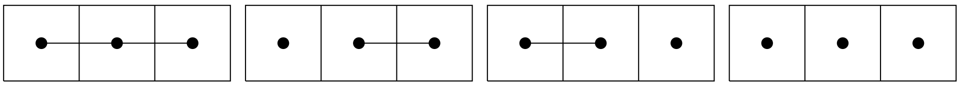
$$g_n = \sum_{i=1}^{n} g_{i-1}x_{in}, \quad \text{for} \quad n \geqslant 1. \tag{33}$$

For instance, the first polynomials g_n are

$$g_1 = x_{11}, \quad g_2 = x_{12} + x_{11}x_{22}, \quad g_3 = x_{13} + x_{11}x_{23} + x_{12}x_{33} + x_{11}x_{22}x_{33},$$
$$g_4 = x_{14} + x_{11}x_{24} + x_{12}x_{34} + x_{11}x_{22}x_{34} + x_{13}x_{44} + x_{11}x_{23}x_{44}$$
$$+ x_{12}x_{33}x_{44} + x_{11}x_{22}x_{33}x_{44}.$$

For every $n \geqslant 1$, the polynomial g_n turns out to be a polynomial with integer coefficients in the $n(n+1)/2$ indeterminates x_{ij} with $1 \leqslant i \leqslant j \leqslant n$, as is easily seen. The polynomial g_n is a sum of monic monomials that parametrize all linear tilings of a strip of n square cells, that is, all coverings of the strip of squares with rectangles of any length $1, 2, \ldots, n$. The indeterminate x_{ij} indicates the rectangle of length $j - i + 1$ that starts covering the i-th square and ends covering the j-th square.

For instance, $g_3 = x_{13} + x_{11}x_{23} + x_{12}x_{33} + x_{11}x_{22}x_{33}$ and, correspondingly, the tilings of a strip of three squares are



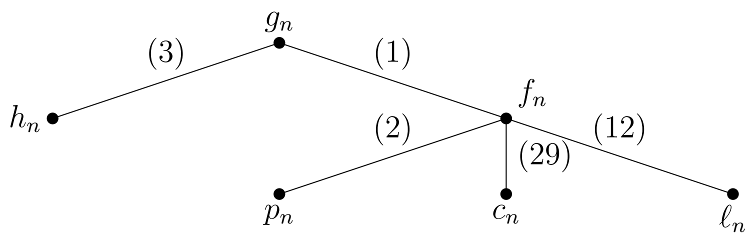
The first tiling consists of a unique rectangle of length 3. The second and the third of one rectangle of length 1 and one of length 2, in the two possible orders. The fourth tiling consists of three squares.

Remarks 16. (1) Let us show how we can recover the previous families of polynomials using the polynomials g_n. As we have seen, the family f_n parametrizes tilings of a strip of length n with tiles of length 1 (represented by the indeterminates x_i in the definition of f_n) and of length 2 (represented by the indeterminates y_i in the definition of f_n). It is clear from the equation (33) that f_n can be obtained by equating, in this expression of g_n, all the indeterminates x_{ij} to zero for $j > i + 2$. In other words, the polynomial f_n can be obtained from g_n by specializing in the polynomial g_n the indeterminates x_{ij} to zero whenever $j > i + 2$, the indeterminates x_{ii} to x_i and the indeterminates $x_{i,i+1}$ to y_{i+1}.

(2) Since the polynomials p_n are obtained from f_n by specializing the indeterminates y_i (in the definition of f_n) to 1 and the indeterminates x_i to t_i, we can also obtain the polynomials p_n by specializing the indeterminates of the polynomial g_n. To be more precise this specialization is obtained by sending x_{ij} to zero whenever $j > i + 2$, x_{ii} to t_i and $x_{i,i+1}$ to 1.

(3) The polynomial g_n is the sum of 2^{n-1} monomials, which parametrize the subsets of a set of $n - 1$ elements. Hence there is a clear immediate connection with the monomials of the polynomial h_n, which parametrize the nonempty subsets of a set of n elements. To this end, it suffices to send every monomial $x_{i_1 j_1} x_{i_2 j_2} \ldots x_{i_{t-1} j_{t-1}} x_{i_t n}$ of degree t in g_n to the subset $\{j_1, j_2, \ldots, j_{t-1}\}$ of cardinality $t - 1$ of the set $\{1, 2, \ldots, n - 1\}$. In order to get the polynomial h_n from g_{n+1}, it suffices to specialize, in the polynomial g_{n+1}, the indeterminate x_{ij} to $x_j y$ for every i and j, then multiply by $(x_{n+1}y)^{-1} = y^{-1}x_{n+1}^{-1}$, subtract 1, and finally multiply by y^{-1} on the right. That is, after the specialization, we have that $h_n = (g_{n+1}y^{-1}x_{n+1}^{-1} - 1)y^{-1}$.

(4) The "hierarchy" of the polynomials we have studied in this paper follows therefore the following pattern. Each family of polynomials is a "specialization" of the families above it, as the remarks (1), (2), (3) above, Theorem 12 and (29) show.



Notice that, in this diagram, c_n is not really obtained via some specialization from f_n, because (29) is simply the definition of the c_n's, in terms of the polynomials f_n.

Equation (33) leads to

$$(g_1, \dots, g_n) = (g_0, \dots, g_{n-1}) \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & x_{nn} \end{pmatrix}$$

Since, for $1 \leqslant l \leqslant n$, a tiling of a strip of length n is obtained by a tile of length l followed by a tiling of length $n-l$, the following formula, where we have specified explicitly the indeterminates ("the tiles") for each polynomial, is easy to get:

$$g_n(x_{ij}; 1 \leqslant i \leqslant j \leqslant n) = \sum_{l=1}^{n} x_{1l} g_{n-l}(x_{l+i,l+j}; 1 \leqslant i \leqslant j \leqslant n-l) \tag{34}$$

The row $(g_n, \dots, g_1)$ is also given by the first row of the following matrix product. This can be seen as a generalization of the equality (20).

$$\begin{pmatrix} x_{11} & 1 & 0 & \dots & 0 \\ * & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & & 0 \\ * & 0 & 0 & \dots & 1 \\ * & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_{22} & 1 & 0 & \dots & 0 \\ x_{12} & 0 & 1 & 0 & \vdots \\ * & 0 & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \dots & 1 \\ * & 0 & 0 & \dots & 0 \end{pmatrix} \cdots \begin{pmatrix} x_{nn} & 1 & 0 & \dots & 0 \\ x_{n-1,n} & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & & 0 \\ x_{2n} & 0 & 0 & \dots & 1 \\ x_{1n} & 0 & 0 & \dots & 0 \end{pmatrix}$$

10 Permanents

Let R be a nonnecessarily commutative ring, $n \geqslant 1$ be an integer, $M_n(R)$ the $n \times n$-matrix ring, and S_n be the symmetric group. Define a mapping $\text{perm}\colon M_n(R) \to R$ setting, for every matrix $A = (a_{i,j})_{i,j} \in M_n(R)$,

$$\text{perm}(A) := \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)}.$$

If $A_{i,j}$ denotes the $(n-1)\times(n-1)$-matrix that results from A removing the i-th row and the j-th column, then $\operatorname{perm}(A) := \sum_{j=1}^{n} a_{1,j}\operatorname{perm}(A_{1,j}) = \sum_{j=1}^{n}\operatorname{perm}(A_{n,j})a_{n,j}$ (it is possible to easily expand our permanent along the first row or the last row only).

Theorem 17. *For every $n \geqslant 1$,*

$$g_n(x_{ij}) = \operatorname{perm}(A_n) = \operatorname{perm}(A_n^t), \tag{35}$$

where

$$A_n = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ 1 & x_{22} & x_{23} & \dots & x_{2n} \\ 0 & 1 & x_{33} & \ddots & x_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & x_{nn} \end{pmatrix}$$

Proof. Let us show that $g_n(x_{ij}) = \operatorname{perm}(A_n)$ by induction on n. The case $n = 1$ is trivial.

Let us assume that the formula holds for n and expand $\operatorname{perm}(A_{n+1})$ along the last row: $\operatorname{perm}(A_{n+1}) = \operatorname{perm}(B_n) + g_n x_{n+1,n+1}$ where B_n is the $n \times n$ matrix given by $B_n = \begin{pmatrix} A_{n-1} & C \\ U & x_{n,n+1}\end{pmatrix}$, where C is the column $(x_{1,n+1},\dots,x_{n-1,n+1})^t$ and U is the row $(0,\dots,0,1)$. In particular, the matrix B_n is obtained from the matrix A_n changing the last column. Notice that in the expression $g_n = \sum_{i=1}^{n} g_{i-1}x_{in}$, the polynomials $g_0,\dots,g_{n-1}$ do not depend on the indeterminates x_{in}. Since the expression of $\operatorname{perm}(A_n)$ given by the inductive hypothesis is $\operatorname{perm}(A_n) = g_n$, we get that $\operatorname{perm}(B_n) = g_0x_{1,n+1} + g_1x_{2,n+1} + \dots + g_{n-1}x_{n,n+1}$. Thus we have $\operatorname{perm}(A_{n+1,n+1}) = \sum_{i=1}^{n+1} g_{i-1}x_{i,n+1} = g_{n+1}$, as desired. The fact that $\operatorname{perm}(A_n^t) = g_n$ as well is proved similarly, using the equality (34). ☐

Let us remark that, contrary to the case of permanents defined over commutative rings, we do not have in general that $\operatorname{perm}(A) = \operatorname{perm}(A^t)$.

From Theorems 17, 12 and Remarks 16((1) and (2)), we immediately get that:

Corollary 18. *The polynomials $f_n(x_1,\dots,x_n,y_1,\dots,y_n)$ and $p_n(t_1,\dots,t_n)$ are the permanents of the $n \times n$ tridiagonal matrices*

$$\begin{pmatrix} x_1 & y_2 & 0 & \dots & 0 \\ 1 & x_2 & y_3 & \dots & 0 \\ 0 & 1 & x_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 \\ 1 & t_2 & 1 & \dots & 0 \\ 0 & 1 & t_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & t_n \end{pmatrix},$$

and their transposes, respectively.

The analogue of formula (23) is the following:

Proposition 19.

$$g_n = x_{11}g_{n-1}(x_{i+1\ j+1}) + x_{12}g_{n-2}(x_{i+2\ j+2}) + x_{13}g_{n-3}(x_{i+3\ j+3}) \\ + \cdots + x_{1\ n-1}g_1(x_{i+n-1\ j+n-1}) + x_{1n}g_0$$

Proof. It suffices to apply Theorem 17 expanding the permanent along the first row. The t-th term in this expansion is

$$x_{it}\,\mathrm{perm}\begin{pmatrix} 1 & & & & & & & \\ & 1 & * & & & & & \\ & 0 & \ddots & & & * & & \\ & & & 1 & & & & \\ & & & & x_{t+1\ t+1} & x_{t+1\ t+2} & \cdots & x_{t+1\ n} \\ & & & & 1 & x_{t+2\ t+2} & & x_{t+2\ n} \\ & 0 & & & & \ddots & \ddots & \vdots \\ & & & & 0 & & 1 & x_{nn} \end{pmatrix}$$

$$= x_{it}\,\mathrm{perm}\begin{pmatrix} x_{t+1\ t+1} & x_{t+1\ t+2} & \cdots & x_{t+1\ n} \\ 1 & x_{t+2\ t+2} & & x_{t+2\ n} \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & x_{nn} \end{pmatrix} = x_{it}g_{n-t}(x_{i+t\ j+t}).$$

□

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