Some results on the structure of multipoles in the study of snarks*

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Abstract

Multipoles are the pieces we obtain by cutting some edges of a cubic graph in one or more points. As a result of the cut, a multipole M has vertices attached to a dangling edge with one free end, and isolated edges with two free ends. We refer to such free ends as semiedges, and to isolated edges as free edges. Every 3-edge-coloring of a multipole induces a coloring or state of its semiedges, which satisfies the Parity Lemma. Multipoles have been extensively used in the study of snarks, that is, cubic graphs which are not 3-edge-colorable. Some results on the states and structure of the so-called color complete and color closed multipoles are presented. In particular, we give lower and upper linear bounds on the minimum order of a color complete multipole, and compute its exact number of states. Given two multipoles M_1 and M_2 with the same number of semiedges, we say that M_1 is reducible to M_2 if the state set of M_2 is a non-empty subset of the state set of M_1 and M_2 has less vertices than M_1. The function $v(m)$ is defined as the maximum number of vertices of an irreducible multipole with m semiedges. The exact values of $v(m)$ are only known for $m \leqslant 5$. We prove that tree and cycle multipoles are irreducible and, as a byproduct, that $v(m)$ has a linear lower bound.

Keywords: cubic graph, edge-coloring, snark, multipole, Parity Lemma, states, color complete, color closed, separable, irreducible, tree, cycle, linear recurrence

1   Introduction

A cubic graph can be subdivided by an edge cut into parts known as *multipoles*. As a result of the cut, a multipole has half-edges with one free end which we call *semiedges*. An isolated edge with both free ends can be viewed as a multipole, too. We call it a *free edge*, and we refer to its free ends as *semiedges*, too. A multipole need not be connected. We call m-pole a multipole with m semiedges. According to the Parity Lemma, if a multipole can be 3-edge-colored then the number of semiedges of each color must have the same parity as m. Therefore, the possible colorings of the semiedges are not arbitrary. We refer to such colorings of the semiedges as *states*, where the ordering of the semiedges is fixed but the 3 colors are interchangeable. Multipoles and their states are used to study non-3-edge-colorable cubic graphs, called *snarks*. See, for instance, Isaacs [8], Gardner [6], Goldberg [7], and Fiol [3]. A multipole with all states permitted by the Parity Lemma is known as a *color complete* multipole, and a 3-edge-colorable multipole that has at least one state in common with any other 3-edge-colorable multipole with the same number of semiedges is known as a *color closed* multipole. Nedela and Škoviera [12] posed the question whether color closed non color complete multipoles exist for $m \geqslant 5$. Karabáš, Máčajová and Nedela [10] affirmatively answered this question, giving examples obtained by computer search. We also give examples for $m \in \{5, 6\}$ and explain a method to obtain them. Moreover, we give other results on the states and structure of multipoles, in particular of color complete multipoles. Namely, we give an exact formula for the number of states of a color complete m-pole, we prove that color complete m-poles exist for every non-trivial value of m, and we give lower and upper bounds on the minimum order of a color complete m-pole as a function of m. This gives a partial solution to the problem of finding a multipole with a given set of states or *feasibility problem* (see [10]). With this same motivation in mind, we also give the minimum numbers of states of m-poles for $m \leqslant 6$, and we study the m-poles with the minimum possible number of vertices (0 if m is even, and 1 if m is odd) which we call *minimal m-poles*. Given two multipoles M_1 and M_2 with the same number of semiedges, we say that M_1 is *reducible* to M_2 if the state set of M_2 is a non-empty subset of the state set of M_1 and M_2 has less vertices than M_1. The function $v(m)$, introduced by Fiol [3], is defined as the maximum number of vertices of an irreducible m-pole. Only a few values of $v(m)$ are known. We prove that tree and cycle multipoles are irreducible and, as a byproduct, that $v(m)$ has a linear lower bound.

2   Multipoles and Tait colorings

A *multipole* or *m-pole* $M = (V, E, \mathcal{E})$ consists of a finite set of vertices V, a set of edges E which are unordered pairs of vertices, and a set $\mathcal{E}$ whose m elements $\epsilon_1, \ldots, \epsilon_m$ are called *semiedges*. We use the notation $V(M)$, $E(M)$ and $\mathcal{E}(M)$ if we want to indicate that V, E and $\mathcal{E}$ are, respectively, the vertex, edge and semiedge sets of a certain multipole M. Each semiedge is associated either with one vertex or with another semiedge making up what is usually called an *isolated* or *free* edge. For instance, Fig. 1(a) shows a 7-pole with two free edges. Notice that a multipole can be disconnected or even be 'empty', in the

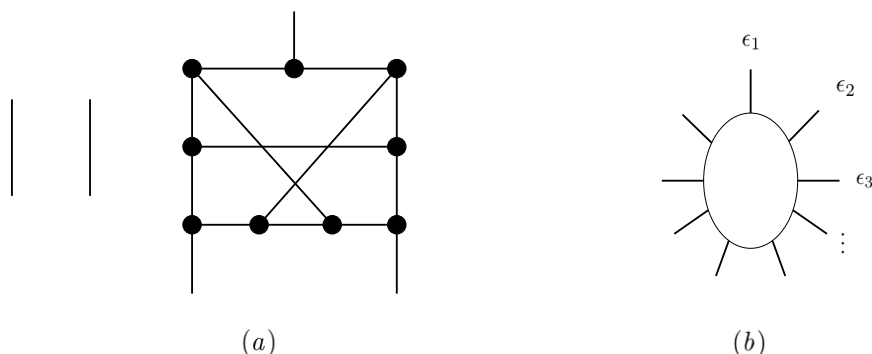sense that it can have no vertices. The diagram of a generic m-pole will be as shown in Fig. 1(b).

Figure 1: A 7-pole with two free edges (a) and a generic multipole (b).

In this paper we follow notation in [3, 4]. For instance, if the semiedge ϵ is associated with vertex u, we say that ϵ is *incident* to u and, following Goldberg's notation [7], we write $\epsilon = (u)$. Also, by joining the semiedges (u) and (v) we get the edge (u, v). The *degree* of a vertex u, denoted by $\deg(u)$, is the number of edges plus the number of semiedges incident to it. In this paper, we only consider cubic multipoles, that is, those for which $\deg(u) = 3$ for all $u \in V$. Thus, the simplest 2- and 3-poles are a free edge and a vertex with three incident semiedges. We denote them by $\boldsymbol{e}$ and $\boldsymbol{v}$, respectively.

Given a multipole M, we denote by $G[M]$ the graph obtained from M by removing all its semiedges. Using this definition, we say that a multipole N is *contained* in M, or that N is a *submultipole* of M, when $G[N]$ is a subgraph of $G[M]$. Note that if $G[N]$ is a proper subgraph of $G[M]$ then N can be obtained from M by 'cutting' (in one or more points) some of its edges. The *distance* $\partial(\epsilon, \zeta)$ between two semiedges $\epsilon = (u)$ and $\zeta = (v)$ is defined to be the distance in $G[M]$ between its incident vertices u and v.

When $G[M]$ is a tree or a forest we refer to M as a *tree multipole* or a *forest multipole* respectively. Analogously, if $G[M]$ is a cycle we say that M is a *cycle multipole*. A simple counting reasoning allows us to see that a tree m-pole has $n = m - 2$ vertices, whereas a cycle m-pole has $n = m$ vertices. In general, for every m-pole with n vertices, we have that $m \equiv n \pmod{2}$ and the following result holds:

Lemma 2.1. *Let M be an m-pole with $n = |M|$ vertices and c components. Then,*

$$n \geqslant m - 2c,$$

with equality if and only if M is a forest multipole.

Proof. For $i = 1, \ldots, c$, let $n^{(i)}$ and $m^{(i)}$ denote, respectively, the numbers of vertices and semiedges of the i-th component. Then the result follows from the fact that $n^{(i)} \geqslant m^{(i)} - 2$ with equality if and only if the i-th component is a tree multipole. $\square$

Let $C = \{1, 2, 3\}$ be a set of 'colors'. Then, a *3-edge-coloring* or *Tait coloring* of an m-pole $(V, E, \mathcal{E})$ is a mapping $\phi : E \cup \mathcal{E} \to C = \{1, 2, 3\}$ such that all the edges and/or semiedges incident with a vertex receive distinct colors, and each isolated edge has both semiedges with the same color. For example, Fig. 2 shows a Tait coloring of a 6-pole. Note that the numbers of semiedges with the same color have the same parity. The following basic lemma states that this is always the case.
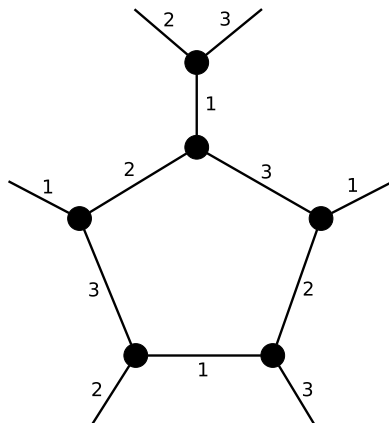

Figure 2: A Tait coloring of a 6-pole.

The Parity Lemma. *Let M be a Tait colored m-pole with m_i semiedges having color $i = 1, 2, 3$. Then,*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod{2}. \tag{1}$$

This result has been used extensively in the construction of snarks (see, for instance, Isaacs [8], Goldberg [7], Chetwynd and Wilson [2], Watkins [16], Kochol [11] and Steffen [13]).

3   Color completeness

Given a multipole M with semiedges $\epsilon_1, \ldots, \epsilon_m$, its set of semiedge colorings or *states* is

$$\mathrm{Col}(M) = \{(\phi(\epsilon_1), \ldots, \phi(\epsilon_m)) : \phi \text{ is a Tait coloring of } M\}/\sim,$$

where two states S_1 and S_2 are in the same equivalence class ($S_1 \sim S_2$) if they can be transformed into each other by a permutation of the colors.

A multipole M is *color complete* if it has all the possible semiedge colorings allowed by the Parity Lemma. We call these states *admissible*. If a given multipole M has a state S, we say that S is *realizable* in M. Two multipoles M_1 and M_2 are *color equivalent* if $\mathrm{Col}(M_1) = \mathrm{Col}(M_2)$, that is, if each state of one multipole has an equivalent in the

state set of the other. For instance, we consider the states $(1, 1, 2, 2)$ and $(3, 3, 1, 1)$ to be equivalent, and the multipoles with state sets $\{(1, 2, 3, 1, 1), (1, 2, 3, 2, 2), (1, 2, 3, 3, 3)\}$ and $\{(2, 3, 1, 2, 2), (2, 3, 1, 3, 3), (2, 3, 1, 1, 1)\}$ to be color equivalent. A state set is *realized* by the multipole M if it is color equivalent to $\mathrm{Col}(M)$. Not surprisingly, the number of states $\sigma(m) = |\mathrm{Col}(M)|$ of a color complete m-pole M coincides with the number of possible colorings of the planar regions in the m-ring of an almost triangulation [5, p. 190]. This is because of the equivalence, proved by Tait [14, 15], that the regions of a planar cubic graph G are 4-colorable if and only if G is 3-edge-colorable. Here we use the properties of 3-edge-colored multipoles to give a short derivation of the formula for such a number.

Proposition 3.1. *Let M be a color complete m-pole. Then, its number of states is*

$$\sigma(m) = \frac{1}{8}[3^{m-1} + 2 + (-1)^m 3].$$

Proof. We prove that $\sigma(m)$ satisfies the following recurrence:

$$\sigma(m) = 2\sigma(m-1) + 3\sigma(m-2) - 1, \tag{2}$$

with initial values $\sigma(2) = \sigma(3) = 1$. Note that, given the semiedges ϵ_1 and ϵ_2, the set of states can be partitioned into two subsets: Those with $\phi(\epsilon_1) = \phi(\epsilon_2)$ and those with $\phi(\epsilon_1) \neq \phi(\epsilon_2)$ for some Tait coloring ϕ. Here and henceforth, the letters a, b, c stand for the colors 1,2,3 in any order, and '$*$' denotes any color. The following facts give the recurrence:

(i) Every realizable state $(a, *, \overset{(m-2)}{\ldots}, *)$ of an $(m-1)$-multipole gives rise to two realizable states, $(b, c, *, \overset{(m-2)}{\ldots}, *)$, and $(c, b, *, \overset{(m-2)}{\ldots}, *)$, of an m-pole.

(ii) Every state $(a, *, \overset{(m-3)}{\ldots}, *)$ of an $(m-2)$-multipole gives rise to three realizable states, $(a, a, a, *, \overset{(m-3)}{\ldots}, *)$, $(b, b, a, *, \overset{(m-3)}{\ldots}, *)$, and $(c, c, a, *, \overset{(m-3)}{\ldots}, *)$, of an m-pole.

(iii) If m is odd, in (i) the realizable state $(a, \overset{(m-1)}{\ldots}, a)$ gives two equivalent states. Similarly, if m is even, in (ii) the realizable state $(a, \overset{(m-2)}{\ldots}, a)$ gives two equivalent states (in both cases, up to permutation of colors b and c).

Then, a particular solution of (2) is $\sigma(m) = 1/4$, whereas a general solution of the corresponding homogeneous equation turns out to be $\sigma(m) = \alpha 3^m + \beta(-1)^m$ where $\alpha, \beta \in \mathbb{R}$. Adding up and imposing the initial conditions we obtain the result. $\qquad \square$

Color complete multipoles can be constructed recursively by using the following result:

Proposition 3.2. *Let M_1 and M_2 be two color complete multipoles, with $m^{(1)} + r$ and $m^{(2)} + r$ semiedges respectively, where $m^{(1)}, m^{(2)} \geqslant 2$, $r \geqslant 2$. For $k = 1, \ldots, r$, choose the semiedges (u_k) from M_1 and (v_k) from M_2 and join them to obtain the edges (u_k, v_k). Then, the obtained multipole M, with $m = m^{(1)} + m^{(2)}$ semiedges, is also color complete.*

Proof. Let S be a state of M, and for $j = 1, 2$ and $i = 1, 2, 3$, let m_i, r_i, and $m_i^{(j)}$ be, respectively, the number of semiedges of M, the number of edges (u_k, v_k) of M, and the number of semiedges of M_j, having color i. Then, $m^{(j)} = m_1^{(j)} + m_2^{(j)} + m_3^{(j)}$ and $r = r_1 + r_2 + r_3$ and, by the Parity Lemma, we must have:

$$m_i^{(1)} + m_i^{(2)} \equiv m^{(1)} + m^{(2)} \pmod{2}, \tag{3}$$

$$m_i^{(1)} + r_i \equiv m^{(1)} + r \pmod{2}, \tag{4}$$

$$m_i^{(2)} + r_i \equiv m^{(2)} + r \pmod{2}. \tag{5}$$

Now, we claim that there exist states S_1 of M_1 and S_2 of M_2 inducing the state S of M. In other words, given $m_i^{(1)}$ and $m_i^{(2)}$ satisfying (3), we want to find some values of r_i, $i = 1, 2, 3$, satisfying (4) and (5). This gives

$$r_i \equiv m^{(1)} + r - m_i^{(1)} \pmod{2},$$

$$r_i \equiv m^{(2)} + r - m_i^{(2)} \pmod{2},$$

which has a solution if and only if the two right hand terms have the same parity. That is,

$$m^{(1)} - m_i^{(1)} \equiv m^{(2)} - m_i^{(2)} \pmod{2}.$$

But this is precisely the condition (3). To see why the condition on r is necessary, note first that the case $r = 0$ is discarded by the observation that a color complete multipole must be connected: otherwise, the Parity Lemma should hold for every connected component, forbidding states that would be admissible as states of the entire multipole. Hence, if r is even, then $r \geqslant 2$. Consider now the case $r = 1$. We are assuming that $m^{(1)} \geqslant 2$. If $m^{(1)} + r$ is odd, the state with $m_1^{(1)} \equiv m_2^{(1)} \equiv m_3^{(1)} \equiv \alpha$, where $\alpha \in \{0, 1\}$, imposes $r_1 \equiv r_2 \equiv r_3 \equiv 1 - \alpha$. Therefore, if $\alpha = 0$ then $r_1, r_2, r_3 \geqslant 1$, and if $\alpha = 1$ then $r_i \geqslant 2$ for some $i \in \{1, 2, 3\}$. The case with even $m^{(1)} + r$ is similar. $\square$

As a consequence, we have the following result on the minimum order $n(m)$ of a color complete m-pole:

Theorem 3.3. *For a given $m \geqslant 5$, the minimum number of vertices of a color complete m-pole satisfies the bounds:*

(a) *If m is odd, then $m + 2 \leqslant n(m) \leqslant 10m - 37$.*

(b) *If m is even, then $m + 2 \leqslant n(m) \leqslant 10m - 40$.*

Proof. First we deal with the lower bounds. For $m \geqslant 5$, it is obvious that a complete m-pole cannot have two semiedges incident to the same vertex. This would preclude a common color for both semiedges, a possibility admitted by the Parity Lemma. We have seen that a color complete multipole must be connected. Moreover, if a connected multipole with no two semiedges at distance 0 has m semiedges and only m vertices, then it must be a cycle with a semiedge on each vertex, but such an m-pole cannot be complete either

because certain states admissible by the Parity Lemma are not possible in a cycle m-pole with $m \geqslant 5$ (see Lemma 7.3). Also according to parity considerations, $m \equiv n \pmod 2$, so a multipole with m semiedges cannot have $m + 1$ vertices. Summarizing, if $m \geqslant 5$, then a complete m-pole must have at least $m + 2$ vertices.

Now we deal with the upper bounds. Color complete multipoles can be constructed recursively as seen in Proposition 3.2, and specifically with $r = 2$. Explicit color complete multipoles are known for $m = 5$, Fig. 3(a), and $m = 6$, Fig. 3(b), with 13 and 20 vertices respectively. This allows us to start the recursive construction of color complete multipoles with higher values of m. Let M_m denote a color complete m-pole of minimum order, and $|V(M_m)|$ its number of vertices. Then, for $m \geqslant 7$,

$$|V(M_m)| = |V(M_{(m-2)+6-4})| \leqslant |V(M_{m-2})| + |V(M_6)| = |V(M_{m-2})| + 20,$$

and this gives $|V(M_m)| \leqslant 10m - 37$ for m odd, and $|V(M_m)| \leqslant 10m - 40$ for m even, as claimed. $\square$

Although we think that these bounds can be improved, we already see a linear behavior. As a corollary of the recursive construction, we conclude that color complete m-poles exist for any $m \geqslant 2$.
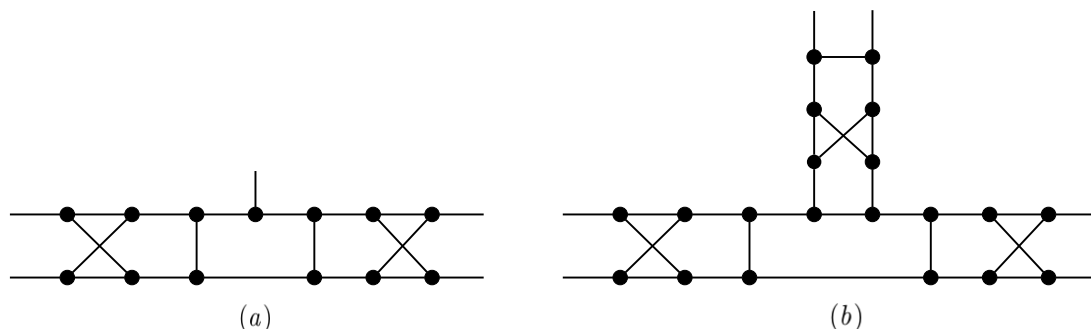

Figure 3: Color complete 5- and 6-poles.

4 Color closedness

Following the terminology of Nedela and Škoviera [12], we say that an m-pole M is *color closed* if its state set has non empty intersection with the state set of any other Tait colorable m-pole N. Also in [12] a concept of *snark irreducibility* is defined, and proved equivalent to the fact that every nontrivial edge-cut, that is, one different from three edges incident with a vertex, separates the snark into Tait colorable components. Obviously, a color complete multipole is color closed, and if $m \geqslant 4$ then a color closed m-pole M cannot be contained in an irreducible snark, because there is always at least one possible

coloring of its m semiedges that is in $\operatorname{Col}(M) \cap \operatorname{Col}(N)$ for any other Tait colorable m-pole N.

If $\mu(m)$ and $\sigma(m)$ are respectively the minimum and maximum number of states of an m-pole, then, by the pigeonhole principle, any m-pole with more than $\sigma(m) - \mu(m)$ states must be color closed. If it has less than $\sigma(m)$ states, then it is non-trivially color closed.

Proposition 4.1. *The minimum number of states of Tait colorable* 4-, 5-, *and* 6-*poles are 2, 3, and 5 respectively.*

Proof. We skip the easier cases of the 4- and 5-poles and explain in detail only the case of 6-poles.

Assume a 6-pole has state S. We may assume without loss of generality that either $S = (a, a, a, a, a, a)$, $S = (a, a, a, a, b, b)$, or $S = (a, a, b, b, c, c)$. First we prove that a 6-pole with the state (a, a, a, a, a, a) has at least 5 states. Start a Kempe interchange of *a-b* type at the first semiedge. We may assume that the resulting state is (b, b, a, a, a, a). Alternatively, start a Kempe interchange of the same type at the third semiedge. Now we know that the Kempe chain cannot finish at the first or second semiedge. We may assume that the resulting state is (a, a, b, b, a, a). Another possibility is to start the same type of Kempe interchange at the fifth semiedge. Now we know that it cannot finish at one of the first four semiedges. Hence, we assume that the resulting state is (a, a, a, a, b, b). Summarizing, we can obtain at least three new states from (a, a, a, a, a, a). Now take anyone of the three new states and start an arbitrary Kempe interchange of type *a-c*. This will give one new state, totalling five. The 6-pole consisting of 3 free edges realizes this possibility. Therefore, $\mu(6) \leqslant 5$.

We must see that if a 6-pole does not have the state (a, a, a, a, a, a) then it cannot have less than 5 states. First assume that the 6-pole has at least one state with two colors, say $S_1 = (a, a, a, a, b, b)$. Starting a Kempe interchange of type *a-b* at the last semiedge we obtain one of the states (b, a, a, a, b, a), (a, b, a, a, b, a), (a, a, b, a, b, a) or (a, a, a, b, b, a), none of them equivalent to S. Assume it is $S_2 = (a, a, a, b, b, a)$. Using arbitrary Kempe interchanges of type *a-c* we can obtain two new three-colored non-equivalent states, one from S_1 and one from S_2. This gives a total of four states $\{S_1, S_2, S_3, S_4\}$. But taking S_1 and starting a Kempe interchange of type *a-b* at the first semiedge we obtain one of the states (b, b, a, a, b, b), (b, a, b, a, b, b), (b, a, a, b, b, b), (b, a, a, a, a, b) or (b, a, a, a, b, a), the first three being equivalent to (a, a, b, b, a, a), (a, b, a, b, a, a) and (a, b, b, a, a, a). None has an equivalent state in $\{S_1, S_2, S_3, S_4\}$. Therefore, a fifth non-equivalent state S_5 exists.

Now assume that the 6-pole has only three-colored states, and that one of these states is $S_1 = (a, a, b, b, c, c)$. Using Kempe interchanges and reasoning as in the previous paragraph, we may assume that the 6-pole has the states $S_2 = (a, b, a, b, c, c)$, $S_3 = (a, c, b, b, a, c)$ and $S_4 = (a, a, b, c, b, c)$. All of them have repeated consecutive colors. But then a Kempe interchange of type *a-c* starting at the first semiedge of S_4 gives either the state (c, a, b, a, b, c) or (c, a, b, c, b, a). None of them has repeated consecutive colors, and hence none of them has an equivalent state in $\{S_1, S_2, S_3, S_4\}$. Therefore, a fifth non-equivalent state S_5 exists. Summarizing, $\mu(6) = 5$. $\qquad \square$

The construction of color complete and color closed multipoles can be easily done by aggregation. In particular, the addition of the X-shaped 4-pole of Fig. 4(a) to an arbitrary m-pole by the junction of two pairs of semiedges is useful to increase its number of possible states (note the use of the X-shaped 4-pole in Section 3, Fig. 3). In this way, we have obtained the color closed multipoles of Figs. 4(b), 5(a), and 5(b), with 4, 5, and 6 semiedges respectively, and 3, 8, and 27 states respectively. The X-shaped 4-pole is also color closed.
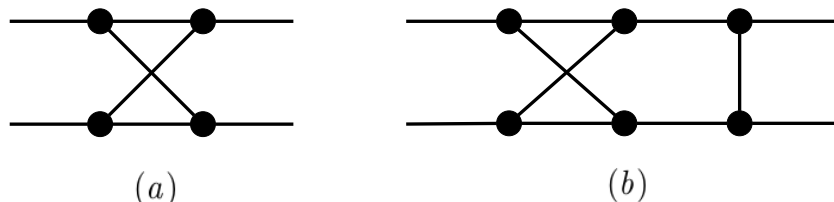
Figure 4: The X-shaped 4-pole and another color closed 4-pole.

In fact, color closed multipoles need not have a number of states close to the maximum, because many state sets are not *realizable* (that is, they are not realized by any multipole). In [10], color closed 6-poles with only 15 and 16 states are described, while the maximum number of states for 6-poles is 31.

The *minimal multipoles* are the m-poles with the minimum possible number of vertices. If m is even, the minimal m-pole is constituted by $m/2$ free edges $\boldsymbol{e}$ and it has no vertices. If m is odd, the minimal m-pole is formed by the minimal 3-pole $\boldsymbol{v}$, and $(m-3)/2$ free edges.

Figure 5: Color closed 5- and 6-poles.

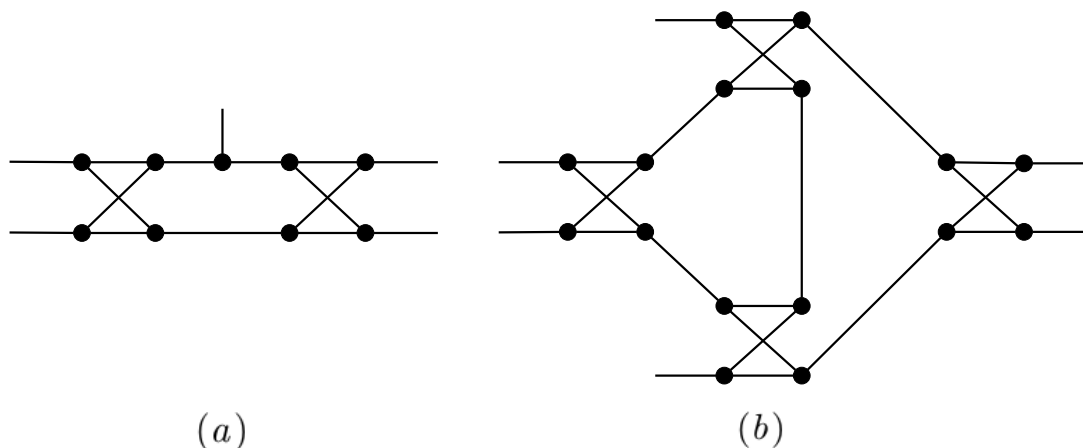**Theorem 4.2.** *Let $\rho(m)$ be the number of states of a minimal multipole.*

(a) *If m is odd, then $\rho(m) = 3^{\frac{m-3}{2}}$.*

(b) *If m is even, then $\rho(m) = \frac{1}{2}(3^{\frac{m}{2}-1} + 1)$.*

Proof. (a) The minimal 3-pole $\boldsymbol{v}$ has a unique coloring, and each one of the $(m-3)/2$ free edges has 3 possible colors.

(b) We have the following recurrence relation: $\rho(m+2) = 3\rho(m) - 1$. To show this, consider $(m+2)/2$ free edges. If the first $m/2$ free edges have the same color, say a, then two of the three possible color assignments for the last free edge, b and c, yield equivalent states because the colors are interchangeable. Otherwise, all color assignments for the last free edge yield non-equivalent states. Therefore, only two of the $3\rho(m)$ possible states are equivalent to each other, and we must discard one of them. This gives the recurrence. The initial value is $\rho(2) = 1$, corresponding to the unique state of a free edge. Using induction on m, or the same technique as in Section 3, we obtain the stated result. $\square$

From Proposition 3.1, Proposition 4.1, and Theorem 4.2, we see that the first values of $\mu(m)$, $\rho(m)$, and $\sigma(m)$ are:

$$\mu(4) = \rho(4) = 2, \qquad\qquad \sigma(4) = 4,$$
$$\mu(5) = \rho(5) = 3, \qquad\qquad \sigma(5) = 10,$$
$$\mu(6) = \rho(6) = 5, \qquad\qquad \sigma(6) = 31.$$

From the results obtained, we conjecture that the minimum numbers of states are always attained by the minimal multipoles.

5 Irreducibility

For a given 3-edge-colorable m-pole M, let $\mathrm{Col}(M)$ denote the set of states of M as defined in Section 3. Then, M is said to be *reducible* when a 3-edge-colorable m-pole N exists such that $|V(N)| < |V(M)|$ and $\mathrm{Col}(N) \subseteq \mathrm{Col}(M)$. Otherwise, we say that M is *irreducible*. Obviously, all minimal multipoles are irreducible. If a multipole is not 3-edge-colorable, it can be further decomposed into smaller multipoles until all of them are, with the aim of studying the structure of snarks in terms of 3-edge-colorable multipoles and the possible ways of joining them.

The following result was first proved in [3]:

Proposition 5.1. *Let U be a snark. Then, for any integer $m \geqslant 1$ there exists a positive integer-valued function $v(m)$ such that any m-pole M contained in U with $|M| > v(m)$ is either not Tait colorable or reducible.*

A reformulation of the same result was given later by Nedela and Škoviera [12, Th. 3.1]. In the same paper, the authors also gave credit to [3] for the introduction of the function $v(m)$, and rewrote it as $\kappa(k)$.

The known values of $v(m)$ are $v(2) = 0$, $v(3) = 1$ (both are trivial results), $v(4) = 2$ (Goldberg [7]), and $v(5) = 5$ (Cameron, Chetwynd, and Watkins [1]). For $m = 6$, Karabáš, Máčajová, and Nedela [10] proved that $v(6) \geqslant 12$.

The study of the function $v(m)$ is relevant in the study of the structure of snarks. More precisely, every value of $v(m)$ has a corresponding result for the decomposition of snarks. Moreover, according to a conjecture by Jaeger and Swart [9], every snark contains a cycle-separating edge-cut of size at most six. As commented in [10], if this were the case, then $v(6)$ would be the most interesting unknown value of $v(m)$.

The following basic lemma is simple but very useful in our study:

Lemma 5.2. *Let M be an irreducible multipole. Then, every submultipole $N \subset M$ is also irreducible.*

Proof. We proceed by contradiction. If a submultipole N of M is reducible to, say, a multipole N', then M is reducible to a multipole M' obtained replacing N by N'. This contradicts the fact that M is irreducible. $\square$

As an example of application, note that, as every m-pole with $n \geqslant 1$ vertices contains an $(m+1)$-pole with $n-1$ vertices (just delete a semiedge $\epsilon = (v)$ and its incident vertex v to create two new semiedges ϵ' and ϵ''), we have that $v(m+1) \geqslant v(m) - 1$. Hence, from the results in [10], we know that $v(7) \geqslant 11$.

Note that $v(m)$ must have the same parity as m. We can be more precise with respect to the behavior of $v(m)$ if we use the following result.

Theorem 5.3. *Let M be an irreducible multipole. Then, the disjoint union of M and one free edge $\boldsymbol{e}$ is also irreducible.*

Proof. We proceed by contradiction. Assume that the disjoint union of M and $\boldsymbol{e}$ is reducible to a submultipole M'. We denote the semiedges of $\boldsymbol{e}$ by ϵ_0 and ϵ_1, and their corresponding semiedges in M' by ϵ_0' and ϵ_1'. In the set of states of $M \cup \boldsymbol{e}$ (respectively M'), all states must have $\phi(\epsilon_0) = \phi(\epsilon_1)$ (respectively $\phi(\epsilon_0') = \phi(\epsilon_1')$). This implies that ϵ_0' and ϵ_1' are not adjacent (that is, $\partial(\epsilon_0', \epsilon_1') > 0$) unless they constitute a free edge $\boldsymbol{e}'$ also in the submultipole M', but then M would be trivially reducible to M' minus $\boldsymbol{e}'$ and we would be done. Otherwise, we may join ϵ_0 to ϵ_1 in $M \cup \boldsymbol{e}$ and ϵ_0' to ϵ_1' in M'. In the first case, we obtain M and a trivial 'closed edge' which can be ignored. In the second case, we obtain a multipole M'' whose number of vertices is $|V(M'')| = |V(M')| < |V(M + \boldsymbol{e})| = |V(M)|$ and whose set of states is a subset of the set of states of M. By definition, M would be reducible to M'', contradicting our assumption. $\square$

As a consequence, we have that $v(m) \geqslant v(m - 2)$. Therefore, the function $v(m)$ is, at least, partially monotone in the sense that its even (respectively odd) values behave monotonically.

6 Trees and forests

The number of states of a tree m-pole T_m, denoted by $t(m) = |\mathrm{Col}(T_m)|$, is easy to compute. Tree multipoles can be constructed recursively by the addition of vertices. A 3-pole with a single vertex has only one possible state. Then, $t(3) = |\mathrm{Col}(T_3)| = 1$. Every new vertex increases m one unit and doubles the number of possible states. Then, $t(m) = 2t(m-1)$. As a consequence, $t(m) = 2^{m-3}$.

An m-pole M with c components is called *separable* if there exists some m-pole N, with at least $c+1$ components, $|V(N)| = |V(M)|$, and $\mathrm{Col}(N) \subseteq \mathrm{Col}(M)$.

Now, we have the following result.

Theorem 6.1. *Every tree multipole T_m is non-separable and irreducible.*

Proof. The proof is by induction. The result is trivial for $m = 2, 3$ since $T_2 = \boldsymbol{e}$ and $T_3 = \boldsymbol{v}$. Then, assume that the results holds for some T_m with $m > 3$. For some semiedge (u), consider the $(m+1)$-pole M obtained by joining one semiedge of $\boldsymbol{v}$ to u, so 'turning' (u) into two new semiedges $\epsilon_1 = (v)_1$ and $\epsilon_2 = (v)_2$. We first prove that M is irreducible. Otherwise, there would exist an $(m+1)$-pole M' with $|V(M')| < |V(M)| = m-1$ (and hence $|V(M')| \leqslant m-3$) and $\mathrm{Col}(M') \subseteq \mathrm{Col}(M)$. Thus, M' must have at least two connected components, say, M'_1 and M'_2. Moreover, since ϵ_1 and ϵ_2 must have different colors in any Tait coloring of M, the same is true for the corresponding semiedges ϵ'_1 and ϵ'_2 in M'. Consequently, such semiedges cannot be in different components of M'. Without loss of generality, assume that $\epsilon'_1, \epsilon'_2 \in \mathcal{E}(M'_1)$. Now by joining ϵ'_1 and ϵ'_2 to two semiedges of $\boldsymbol{v}$, and doing the same operation to ϵ_1 and ϵ_2, so creating a digon, we respectively obtain the multipoles N' and N with the following properties:

- $\mathrm{Col}(N') \subseteq \mathrm{Col}(N) = \mathrm{Col}(T_m)$ (as the created digon 2-pole in N is color equivalent to $\boldsymbol{e}$).

- $|V(N')| \leqslant m-2 = |V(T_m)|$.

Thus, if $|V(N')| < m-2$, T_m would be reducible, and if $|V(N')| = m-2$, T_m would be separable. In any case, we get a contradiction.

To prove that M is also non-separable we again assume the contrary and iterate the above procedure until M is proved to be color equivalent to some small tree multipole T, whereas M' is color equivalent to a multipole N constituted by some isolated edges. But this again contradicts the fact that $\mathrm{Col}(M') \subseteq \mathrm{Col}(M)$. $\qquad \square$

As every tree m-pole T_m has $m-2$ vertices, the above theorem allows us to state that $v(m) \geqslant m-2$ for $m \geqslant 2$.

Using basically the same proof as above, we have the following:

Proposition 6.2. *Every forest multipole is non-separable and irreducible.*

The following result shows that the number of states of a forest m-pole depends only on m and its number n of vertices. Let $f(n,m)$ denote the number of states of a forest m-pole M with n vertices.

Lemma 6.3. *If $n \leqslant m - 4$, the following recurrence relation holds:*

$$f(n, m) = f(n, m - 2) + f(n + 1, m - 1),$$

with initial values $f(0, 2) = 1$ and $f(n, n + 2) = 2^{n-1}$ for $n > 0$.

Proof. First notice that, if $n = m - 2 > 0$ then a forest m-pole with n vertices is in fact a tree m-pole, with number of states 2^{m-3} or 2^{n-1}. If $n = 0$ and $m = 2$, then we have an isolated edge with only one state. This gives the starting values. If $n \leqslant m - 4$, a forest m-pole with n vertices is not a tree and has at least two components. Then, given two semiedges ϵ_1 and ϵ_2 (not in the same component), its set of states can be partitioned into two subsets: Those with $\phi(\epsilon_1) = \phi(\epsilon_2)$, and those with $\phi(\epsilon_1) \neq \phi(\epsilon_2)$, for some Tait coloring ϕ of M.

Case $\phi(\epsilon_1) = \phi(\epsilon_2)$: The two semiedges can be joined and we obtain an m'-pole M' with $m' = m - 2$ semiedges and $n' = n$ vertices. As a consequence of $n \leqslant m - 4$, the relation $n' \leqslant m' - 2$ holds. Therefore, its number of states is $f(n', m') = f(n, m - 2)$.

Case $\phi(\epsilon_1) \neq \phi(\epsilon_2)$: The two semiedges can be joined to two semiedges ϵ_1'' and ϵ_2'' of a minimal 3-pole $\boldsymbol{v}$. Let ϵ_3'' denote the remaining semiedge of $\boldsymbol{v}$. The result of the junction is a new m''-pole M'' with $m'' = m - 1$ semiedges, now including ϵ_3'', and $n'' = n + 1$ vertices. Again, the relation $n'' \leqslant m'' - 2$ holds as a consequence of $n \leqslant m - 4$. Therefore, the number of states of M'' is $f(n'', m'') = f(n + 1, m - 1)$.

Finally, adding up all states we obtain the recurrence relation. $\qquad\square$

By means of a straightforward calculation, it can be proved that, if $0 \leqslant 2k \leqslant m - 2$ and $n \leqslant m - 2(k + 1)$, then the following formula holds:

$$f(n, m) = \sum_{i=0}^{k} f(n + i, m - 2k + i)\binom{k}{i}. \tag{6}$$

As a consequence, $f(n, m)$ can be easily calculated.

Theorem 6.4.

(a) *The number of states of a forest m-pole with 0 vertices is:*

$$f(0, m) = \tfrac{1}{2}(3^{\frac{m-2}{2}} + 1).$$

(b) *The number of states of a forest m-pole with $n \geqslant 1$ vertices is:*

$$f(n, m) = 2^{n-1} 3^{\frac{m-n}{2} - 1}.$$

Proof. (*a*) Assuming $n = 0$ and $m = 2k + 2$ in (6), and using the initial values, we have:

$$f(0, m) = f(0, 2k + 2) = \sum_{i=0}^{k} f(i, i+2)\binom{k}{i} = f(0, 2) + \sum_{i=1}^{k} f(i, i+2)\binom{k}{i}$$

$$= 1 + \sum_{i=1}^{k} 2^{i-1}\binom{k}{i} = 1 + \frac{1}{2}\sum_{i=1}^{k} 2^{i}\binom{k}{i}$$

$$= 1 + \frac{1}{2}(3^k - 1) = \frac{1}{2}(3^k + 1) = \frac{1}{2}(3^{\frac{m-2}{2}} + 1).$$

(*b*) Assuming $n \geqslant 1$ and $m = n + 2k + 2$ in (6), and using the initial values, we have:

$$f(n, m) = f(n, n + 2k + 2) = \sum_{i=0}^{k} f(n+i, n+i+2)\binom{k}{i}$$

$$= \sum_{i=0}^{k} 2^{n+i-1}\binom{k}{i} = 2^{n-1}\sum_{i=0}^{k} 2^{i}\binom{k}{i} = 2^{n-1}3^k = 2^{n-1}3^{\frac{m-n}{2}-1}.$$

$\square$

Figure 6 shows the values of $f(m, n)$ for $2 \leqslant m \leqslant 14$ and $0 \leqslant n \leqslant 12$, and also the numbers of states of cycle m-poles, for which $n = m$.

The following result, consequence of Theorem 6.4, implies that the minimum number of states of a forest m-pole M is attained when M is a minimal m-pole. Note that in both cases we obtain $\rho(m)$.

Proposition 6.5. *For a fixed even value of m, the minimum of $f(n, m)$ is $\frac{1}{2}(3^{\frac{m-2}{2}} + 1)$, and for a fixed odd value of m, the minimum of $f(n, m)$ is $3^{\frac{m-3}{2}}$.*

The set of states of a tree m-pole with n vertices can be obtained algorithmically in the following way. Pick an arbitrary vertex v with at least one incident semiedge ϵ. Assume, without loss of generality, that ϵ has color a and the remaining two edges or semiedges have color b and c. The semiedge ϵ can be ignored and the rest of the tree can be viewed as a binary tree with v as the root. At each one of the $n - 1$ remaining vertices a binary coloring decision can be made, and each one of the 2^{n-1} corresponding binary sequences gives one possible state.

7 Cycles

In this section we prove the irreducibility of every cycle multipole for $m \geqslant 5$, and calculate the number of states of an arbitrary cycle multipole with $m \geqslant 1$.

Proposition 7.1. *The number $c(m)$ of states of a cycle m-pole C_m with $m \geqslant 1$ is*

$$c(m) = \tfrac{1}{3}(2^{m-1} + (-1)^m).$$

$$m$$

n	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1		2		5		14		41		122		365
1		1		3		9		27		81		243	
2	1		2		6		18		54		162		486
3		1		4		12		36		108		324	
4			3		8		24		72		216		648
5				5		16		48		144		432	
6					11		32		96		288		864
7						21		64		192		576	
8							43		128		384		1152
9								85		256		768	
10									171		512		1536
11										341		1024	
12											683		2048

Figure 6: Numbers of states of forest m-poles with n vertices. The rows $n = 0$ and $n = 1$ correspond to minimal m-poles, and the diagonal $n = m - 2$ corresponds to tree m-poles. The numbers of states of cycle m-poles, corresponding to the diagonal $n = m$, are also represented.

Proof. The coloring of the semiedges is determined by the coloring of the edges of the cycle. Then, the number $c(m)$ verifies the following recurrence:

$$c(m) = c(m - 1) + 2c(m - 2).$$

To show this, consider one arbitrary edge of the cycle, and the two edges adjacent to it. If these two edges have different color, the color of the middle edge is forced. If they have the same color, there are two possibilities for the color of the middle edge. In the first case, we delete the middle edge and make its two adjacent edges adjacent to each other: the resulting cycle has $m - 1$ edges and $c(m - 1)$ 3-edge-colorings. In the second case, we delete the middle edge and identify its two adjacent edges as if they were one: the resulting cycle has $m - 2$ edges and $c(m - 2)$ 3-edge-colorings, but in this case the deleted edge had two possible colors. This gives the recurrence for $c(m)$. Its initial values are $c(1) = 0$ and $c(2) = 1$. Using induction on m, or the same technique as in Section 3, we can prove the result. $\square$

The set of states of a cycle m-pole can be obtained algorithmically in the following way. Cut an arbitrary edge (u, v) of the cycle to obtain a tree $(m + 2)$-pole, retaining the condition that the resulting semiedges $(u)_1$ and $(v)_1$ must have the same color. Assume, without loss of generality, that they have color a. As a result of the cut, u and v are at opposite ends of the tree. Ignore for the moment v, its nearest vertex and their incident semiedges. The result is a tree m-pole whose set of states can be determined as in the previous section, picking u as the root and ignoring $(u)_1$ to obtain a binary tree. Now it is straightforward to extend the states of the partial tree m-pole to the states of the whole tree $(m + 2)$-pole, with the condition that $(v)_1$ has color a to match the color of $(u)_1$. This is an alternative way to obtain the enumeration result on the states of cycle multipoles.

Now we show that, as the tree multipoles, cycle multipoles are also irreducible. To this end, we need some previous lemmas.

Lemma 7.2. *Let $\epsilon_1, \epsilon_2, \epsilon_3$ be three (ordered) semiedges of a m-tree multipole T_m with $m \geqslant 3$. Then, the state $(\ldots, \epsilon_1, \epsilon_2, \epsilon_3, \ldots) = (\ldots, a, b, a, \ldots)$ is realizable unless one of the following conditions holds:*

(i) $\partial(\epsilon_1, \epsilon_3) = 0$.

(ii) $\partial(\epsilon_1, \epsilon_2) = \partial(\epsilon_2, \epsilon_3) = 1$.

Proof. Let $\epsilon_i = (v_i)$, $i = 1, 2, 3$. First note that, as ϵ_1 and ϵ_2 are distinct, a coloring of the form $(\ldots, \epsilon_1, \epsilon_2, \epsilon_3, \ldots) = (\ldots, a, b, *, \ldots)$ is always possible. Then, the only cases where color $*$ cannot be a are (i) (trivially since then $v_1 = v_3$) and (ii) since, then, edge $v_1 v_2$ must have color c and, hence, edge $v_2 v_3$ must have color a which is forbidden. $\square$

Lemma 7.3. *Let $\epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_{i+r-1}$, be r successive semiedges of a m-cycle multipole C_m with $r \geqslant m$ (subindexes understood modulo m). Then, the following states are not realizable:*

(i) *For $r = 3$, $(\ldots, \epsilon_i, \epsilon_{i+1}, \epsilon_{i+2}, \ldots) = (\ldots, a, b, a, \ldots)$.*

(ii) *For $r = 4$, $(\ldots, \epsilon_i, \epsilon_{i+1}, \epsilon_{i+2}, \epsilon_{i+3}, \ldots) = (\ldots, a, b, b, c, \ldots)$.*

Proof. Let $\epsilon_{i+j} = (v_{i+j})$, $j = 0, \ldots, 3$. Case (i) is proved as in Lemma 7.2. Case (ii) follows also easily by considering the two different possible colors, a and c, of edge $v_{i+1} v_{i+2}$ and concluding that both lead to a contradiction. $\square$

Theorem 7.4. *Every cycle multipole C_m with $m \geqslant 5$ is irreducible.*

Proof. The proof is by contradiction. Assume that C_m is reducible to an m-pole T. As $|T| \leqslant m - 2$, T is either a tree multipole or it has at least two components. Since $\mathrm{Col}(T) \subseteq \mathrm{Col}(C_m)$, there must be a cyclic ordering $(\epsilon_0, \ldots, \epsilon_{m-1}, \epsilon_0, \ldots)$ of the semiedges of T, corresponding to the successive semiedges of C_m, such that every state of T is also realizable in C_m. However, we will prove that, no matter how we choose such an ordering, this is not the case. We consider three possibilities:

(*a*) T is a tree.

    (*a*1) Assume all the semiedges of T come into pairs (that is, every pair of them is incident with a given vertex). Without loss of generality, suppose that ϵ_i and ϵ_{i+1} are incident to u_i for $i = 0, 2, \ldots, (m-2)/2$. Then, by Lemma 7.2, in the cyclic ordering and in order to avoid the states $(\ldots, a, b, a, \ldots)$ we are forced to take $(\ldots, \epsilon_1, \epsilon_j, \epsilon_2, \epsilon_{j+1}, \ldots)$ with $j \geqslant 3$. But every choice for the semiedge ϵ_k next to ϵ_{j+1} allows the state $(\ldots, a, b, a, \ldots)$ for the semiedges $(\ldots, \epsilon_2, \epsilon_{j+1}, \epsilon_k, \ldots)$, which is not realizable in C_m by Lemma 7.3.

    (*a*2) Otherwise, T must have some semiedge ζ at nonzero distance from every other semiedge $(\epsilon_i, \epsilon_{i+1})$ of incident semiedges. Thus, at some point of the cyclic ordering, we must have $(\ldots, \zeta, \epsilon_j, \epsilon_k, \ldots)$ where $j \in \{i, i+1\}$. Then, again, all possible choices of ϵ_k allow the state $(\ldots, a, b, a, \ldots)$.

(*b*) T has two components X, Y: Let X and Y have semiedges ϵ_i and ζ_j, respectively. Notice that either X or Y has at least three semiedges. Then, at some point of the cyclic ordering, we must go from, say, $\epsilon_1 \in \mathcal{E}(X)$ to $\zeta_2 \in \mathcal{E}(Y)$. Up to symmetries, there are three possibilities for the next semiedges to ϵ_1 and ζ_2 (see Fig. 7 where 'cyclic adjacency' between semiedges is represented by dotted lines). With no loss on clarity, we may omit the trailing periods in the notation of states:
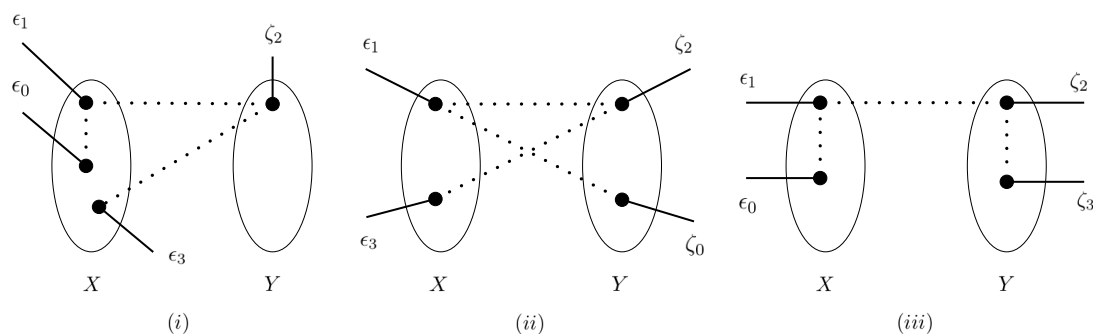

Figure 7: Possible successions of semiedges.

    (*b*1) Case $(\ldots, \epsilon_0, \epsilon_1, \zeta_2, \epsilon_3, \ldots)$: Here, for any Tait coloring ϕ of T, and in order to avoid the state (a, b, a) for the semiedge triples $(\epsilon_0, \epsilon_1, \zeta_2)$ and $(\epsilon_1, \zeta_2, \epsilon_3)$, it must be $\phi(\epsilon_0) = \phi(\epsilon_1)$ and $\phi(\epsilon_1) \neq \phi(\epsilon_3)$. Thus, the state of the successive semiedges must be $(a, a, *, b)$. Now consider the a-b Kempe chain in X, starting from ϵ_3 and interchange its colors to get the new Tait coloring ϕ'. If it ends to a semiedge different from ϵ_0, ϵ_1 we get the new state $(a, a, *, a)$. Otherwise, if it ends to ϵ_0 or ϵ_1 we get $(b, a, *, a)$ or $(a, b, *, a)$ respectively. In all cases we can adequately chose the color $*$ to get the state (a, b, a) for three successive semiedges.

(b2) Case $(\ldots, \zeta_0, \epsilon_1, \zeta_2, \epsilon_3, \ldots)$: To avoid the state (a, b, a) for the two semiedge triples it must be $\phi(\zeta_0) \neq \phi(\zeta_2)$ and $\phi(\epsilon_1) \neq \phi(\epsilon_3)$. Thus, the state of the successive semiedges can be chosen to be (a, a, b, b). Let us consider the different possibilities for the semiedge following ϵ_3. If this is $\epsilon_4 \in \mathcal{E}(X)$, the semiedges $(\epsilon_1, \zeta_2, \epsilon_3, \epsilon_4)$ follow the pattern as in (i), which has already been dealt with. Alternatively, if such a semiedge is $\zeta_4 \in \mathcal{E}(Y)$, it cannot have color c because of Lemma 7.3(ii). Thus, suppose that ζ_4 has a color different from c, say a, and consider the a-b Kempe chain in Y starting from ζ_4 and interchange its colors. Then, reasoning as above, we obtain the realizable state (a, b, a) for three successive semiedges.

(b3) Case $(\ldots, \epsilon_0, \epsilon_1, \zeta_2, \zeta_3, \ldots)$: Now, to avoid the state (a, b, a) for the two semiedge triples it must be $\phi(\epsilon_0) = \phi(\epsilon_1)$ and $\phi(\zeta_2) = \phi(\zeta_3)$, so that we can chose the state of the successive semiedges to be (a, a, b, b). Thus, the semiedge following ζ_3 cannot have color c because of Lemma 7.3(ii). For the situation of such a semiedge, we consider again two possibilities: If $\zeta_4 \in Y$ and it has color a, the interchanging of color in the a-c Kempe chain leads again to the state (a, a, b, b, c). Otherwise, if ζ_4 has color b, the b-c Kempe chain leads again to either such a state or to the color c for some of the semiedges ζ_2 or ζ_3. But, again, both alternatives led to a realizable state (c, b, c) (changing the color of ϵ_1 from a to c, if necessary). Finally, if $\epsilon_4 \in X$ and it has colors a or b, we reason as above with the respective a-c or b-c Kempe chains to reach the states of type (a, a, b, b, c) or (a, b, a).

(c) T has at least three components X, Y, Z: Denote the semiedges of X and Y as above and let Z have semiedges η_k. Note that in the cyclic ordering, and to avoid the state (a, b, a), we cannot have 3 successive semiedges in different components X, Y, Z. Then, the only possibility to be considered, being different from those in (b), is $(\epsilon_0, \zeta_1, \zeta_2, \eta_3)$. Then it must be $\phi(\zeta_1) = \phi(\zeta_2)$ and the corresponding state can be chosen to be (a, b, b, c). But this is one of the non-desired states of Lemma 7.3.

Summarizing, T has always a state not realizable in C_m. But this is in contradiction with our assumption. Thus, we conclude that C_m is irreducible. $\qquad \square$

As every cycle m-pole C_m has m vertices, the above proposition allows us to state that if $m \geqslant 5$ then $v(m) \geqslant m$.

8 Conclusions

We calculated the exact number of states of tree, cycle, minimal and color complete multipoles with m semiedges, and proved that color complete multipoles exist for every non trivial value of m. The minimum order of a color complete multipole has lower and upper linear bounds in terms of m. Moreover, we proved that results on the behavior of the function $v(m)$, which is only partially known but relevant in the study of the structure

of snarks, can be obtained by the analysis of reducibility of multipoles, in particular tree and cycle multipoles.

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