Infinite gammoids

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Abstract

Finite strict gammoids, introduced in the early 1970's, are matroids defined via finite digraphs equipped with some set of sinks: a set of vertices is independent if it admits a linkage to these sinks. In particular, an independent set is maximal (i.e. a base) precisely if it is linkable onto the sinks.

In the infinite setting, this characterization of the maximal independent sets need not hold. We identify a type of substructure as the unique obstruction. This allows us to prove that the sets linkable onto the sinks form the bases of a (possibly non-finitary) matroid if and only if this substructure does not occur.

Keywords: Infinite matroids; strict gammoids; transversal matroid; linkage; infinite digraphs; rays

0 Introduction

Infinite matroid theory has seen vigorous development (e.g. [1], [5] and [7]) since Bruhn et al. [6] in 2010 gave five equivalent sets of axioms for infinite matroids in response to a problem proposed by Rado [19] (see also Higgs [13] and Oxley [15]). Our contribution to the development focusses on the class of gammoids. In this first paper, the main object of investigation is the bases of infinite strict gammoids. (The second one considers other aspects including duality and minors [2].)

The concept of gammoids originated from the transversal matroids of Edmonds and Fulkerson [11]. A transversal matroid can be defined by taking as its independent sets the subsets of a fixed vertex class of a bipartite graph matchable to the other vertex class. Perfect [17] introduced the class of gammoids by replacing matchings in bipartite graphs with disjoint directed paths in digraphs. Later, Mason [14] started the study of a subclass of gammoids known as strict gammoids.

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To be precise, let a *dimaze* (short for *di*rected *maze*) be a digraph with a fixed subset of the vertices of out-degree 0, called *exits*. A dimaze *contains* another dimaze, if, in addition to digraph containment, the exits of the former include those of the latter. In the context of digraphs, any path or ray (i.e. infinite path) is forward oriented. A set of vertices of (the digraph of) the dimaze is independent if it is linkable to the exits by a collection of disjoint directed paths. The set of all linkable sets is the *linkability system* of the dimaze. When the linkability system is the set of independent sets of a matroid on the vertex set of the dimaze, we say that the dimaze defines a matroid. Any matroid arising in this way is called a *strict gammoid*.

Mason [14] proved that every finite dimaze defines a strict gammoid. When a dimaze is infinite, Perfect [17] gave sufficient conditions for when some subset of the linkability system gives rise to a matroid. Any such matroid is *finitary*, in the sense that a set is independent as soon as all its finite subsets are. Since finitary matroids were the only ones known at that time, infinite dimazes whose linkability systems are non-finitary[1] were not considered to define matroids.

With infinite matroids canonically axiomatized in a way that allows for non-finitary ones, a natural question is whether every infinite dimaze now defines a matroid. In general, the answer is still negative, as the linkability system may fail to satisfy one of the infinite matroid axioms (IM), which asks for the existence of certain maximal independent sets. Observe that in any finite dimaze, a set is linkable onto the exits if and only if it is maximally independent. However, in an infinite dimaze, sets which are linkable onto the exits need not be maximally independent. It turns out that if they all are, we have a matroid.

So the question arises: In which dimazes is every set that is linkable onto the exits maximally independent? Investigation of this question leads us to the following example of a dimaze. An *alternating ray* is a digraph obtained from an undirected ray, i.e. a 1-way infinite path, by directing the edges in such a way that the first vertex and infinitely many others have out-degree 0. An *alternating comb* is a dimaze constructed by linking all the vertices of out-degree 0 of an alternating ray onto a set of exits by (possibly trivial) disjoint directed paths which meet the ray exactly at their initial vertices (see for example Figure 3a). It will be easy to see that, in an alternating comb, the set of vertices of out-degree 2 is linkable either onto the exits or to a proper subset thereof; hence, the set is not maximally independent and yet linkable onto the exits.

By proving that alternating combs form the unique obstruction to the characterization of maximal independent sets as sets linkable onto the exits, we are able to establish the following.

Theorem. *Given a dimaze, the vertex sets linkable onto the exits form the bases of a matroid if and only if the dimaze contains no alternating comb. The independent sets of this matroid are precisely the linkable sets of vertices.*

The non-trivial direction implies that a dimaze that fails to define a matroid contains

[1] For example, in the dimaze on an infinite star directed from the centre towards the exits at the leaves, every finite set of vertices is independent, but the whole vertex set is not.

an alternating comb. While a dimaze containing an alternating comb may still define a matroid, the set of bases is a proper subset of the sets linkable onto the exits and can be difficult to describe.

We collect definitions and give examples of infinite dimazes which do not define a matroid in Section 1. In Section 2, we prove that the independence augmentation axiom as well as a variant (Lemma 2.8) hold in general. (These are applied in [8] to characterize nearly finitary linkability systems.) After rephrasing a proof of the linkage theorem of Pym [18], we prove the main result (Theorem 2.5). In Section 3, we consider the question whether every strict gammoid can be defined by a dimaze without any alternating comb. We answer this negatively via the intermediate step of showing that any tree, when viewed as a bipartite graph, defines a transversal matroid, a statement which is of independent interest.

1 Preliminaries

In this section, we present relevant definitions. For notions not found here, we refer to [6] and [16] for matroid theory, and [9] for graph theory.

Given a set E and a family of subsets $\mathcal{I} \subseteq 2^E$, let $\mathcal{I}^{\max}$ denote the maximal elements of $\mathcal{I}$ with respect to set inclusion. For a set $I \subseteq E$ and $x \in E$, we also write $I + x$ and $I - x$ for $I \cup \{x\}$ and $I \setminus \{x\}$ respectively.

Definition 1.1. [6] A *matroid* M is a pair $(E, \mathcal{I})$ where E is a set and $\mathcal{I} \subseteq 2^E$ which satisfies the following:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \subseteq I'$ and $I' \in \mathcal{I}$, then $I \in \mathcal{I}$.

(I3) For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \in \mathcal{I}$ and $I \subseteq X \subseteq E$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

The set E is the *ground set* and the elements in $\mathcal{I}$ are the *independent sets* of M. Equivalently, matroids can be defined with base axioms. A collection of subsets $\mathcal{B}$ of E is the set of bases of a matroid if and only if the following three axioms hold:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.

(BM) The set $\mathcal{I}$ of all subsets of elements in $\mathcal{B}$ satisfies (IM).

The ground sets we consider will be sets of vertices of digraphs or bipartite graphs. We usually identify a matroid with its set of independent sets.

Connectivity in finite matroids stems from graph connectivity and is a well established part of the theory. In the infinite setting, Bruhn and Wollan [7] gave the following rank-free definition of connectivity that is compatible with the finite one. For an integer $k \geqslant 0$, a *k-separation* of a matroid is a partition of E into X and Y such that both $|X|, |Y| \geqslant k$ and for any bases B_X, B_Y of $M \setminus Y$ and $M \setminus X$ respectively, the number of elements to be deleted from $B_X \cup B_Y$ to get a base of M is less than k. A matroid is *k-connected* if there are no l-separations for any $l < k$. We will show that there are highly connected strict gammoids that are in a sense far from being finitary.

Next, suppose we are given a digraph D, and a subset B_0 of the vertices of out-degree 0.[2] The pair (D, B_0) is called a *dimaze* and B_0 the set of *exits*. A dimaze (D, B_0) *contains* a dimaze (D', B_0'), if D' is a subdigraph of D and $B_0' \subseteq B_0$. A *linkage* $\mathcal{P}$ is a set of disjoint directed paths such that the terminal vertex of each path is in B_0. Let $\mathrm{Ini}(\mathcal{P})$ and $\mathrm{Ter}(\mathcal{P})$ be respectively the set of initial vertices and that of terminal vertices of paths in $\mathcal{P}$. A set $I \subseteq V(D)$ is *linkable* or *independent* if there is a linkage $\mathcal{P}$ *from* I, i.e. $\mathrm{Ini}(\mathcal{P}) = I$. Suppose further that $\mathrm{Ter}(\mathcal{P}) = B_0$, then I is linkable *onto* B_0. The collection of linkable sets is called the *linkability system*. Note that, by adding trivial paths if required:

$$\text{Any linkable set in } (D, B_0) \text{ can be extended to one linkable onto } B_0. \tag{1}$$

Definition 1.2. Let (D, B_0) be a dimaze. The pair of $V(D)$ and the set of linkable subsets is denoted by $M_L(D, B_0)$. A *strict gammoid* is a matroid isomorphic to $M_L(D, B_0)$ for some (D, B_0). A *gammoid* is a matroid restriction of a strict gammoid. Given a gammoid M, (D, B_0) is called a *presentation* of M if $M = M_L(D, B_0) \restriction X$ for some $X \subseteq V(D)$. We say that the dimaze (D, B_0) *defines* a matroid if $M_L(D, B_0)$ is a matroid.[3]

Given vertex subsets A and B of a digraph D, an *A–B separator* S is a set of vertices such that there are no paths from A to B avoiding S. A separator is *on* a linkage $\mathcal{P}$ if it consists of exactly one vertex on each path in $\mathcal{P}$. The celebrated Aharoni-Berger-Menger theorem [3] states that there exist a linkage from a subset of A to B and an A–B separator on this linkage.

Mason [14] (see also [17]) showed that $M_L(D, B_0)$ is a matroid for any finite dimaze (D, B_0). However, this is not the case for infinite dimazes. For example, let D be a complete bipartite graph between an uncountable set X and a countably infinite set B_0 with all the edges directed towards B_0. Then $I \subseteq X$ is independent if and only if I is countable. So there is no maximal independent set in X, hence $M_L(D, B_0)$ does not satisfy the axiom (IM).

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[2]The assumption on B_0 incurs no loss of generality, as we may delete the out-going edges from B_0 without changing the linkability system. Moreover, this assumption excludes unwanted trivial cases in subsequent constructions by forcing vertices having out-going edges to lie outside B_0.

[3] In particular, B_0 is always a base when $M_L(D, B_0)$ is a matroid.

Example 1.3. Here is a counterexample whose digraph is locally finite. Let D be the digraph obtained by directing upwards or leftwards the edges of the subgraph of the grid $\mathbb{Z} \times \mathbb{Z}$ induced by $\{(x, y) : y > 0 \text{ and } y \geqslant x \geqslant 0\}$ and let $B_0 := \{(0, y) : y > 0\}$, see Figure 1. Then $I := \{(x, x) : x > 0\}$ is linkable onto a set $J \subseteq B_0$ if and only if J is infinite. Therefore, $I \cup (B_0 \setminus J)$ is independent if and only if J is infinite. Hence, I does not extend to a maximal independent set in $X := I \cup B_0$.

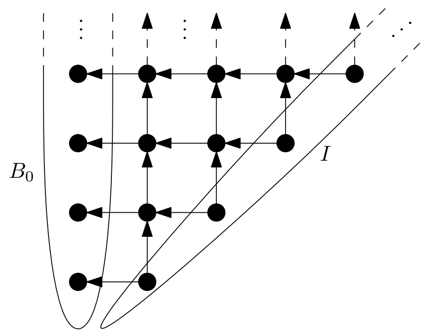

Figure 1: A locally finite dimaze which does not define a matroid

If D' is a subdigraph of D and $B_0' \subseteq B_0$, then (D, B_0) contains (D', B_0') as a *subdimaze*. A dimaze (D', B_0') is a *subdivision* of (D, B_0) if it can be obtained from (D, B_0) as follows. We first add an extra vertex b_0 and the edges $\{(b, b_0) : b \in B_0\}$ to D. Then the edges of this resulting digraph are subdivided to define a digraph D''. Set B_0' as the in-neighbourhood of b_0 in D'' and D' as $D'' - b_0$. Note that this defaults to the usual notion of subdivision if $B_0 = \emptyset$.

The following dimazes play an important role in our investigation. An undirected ray is a graph with an infinite vertex set $\{x_i : i \geqslant 1\}$ and the edge set $\{x_i x_{i+1} : i \geqslant 1\}$. We orient the edges of an undirected ray in different ways to construct three dimazes:

1. R^A by orienting (x_{i+1}, x_i) and (x_{i+1}, x_{i+2}) for each odd $i \geqslant 1$ and the set of exits is empty;

2. R^I by orienting (x_{i+1}, x_i) for each $i \geqslant 1$ and x_1 is the only exit;

3. R^O by orienting (x_i, x_{i+1}) for each $i \geqslant 1$ and the set of exits is empty.

A subdivision of R^A, R^I and R^O is called *alternating ray*, *incoming ray* and *(outgoing) ray*, respectively.

Let $Y = \{y_i : i \geqslant 1\}$ be a set disjoint from X. We extend the above types of rays to combs by adding edges (and their terminal vertices) to the corresponding digraphs and declaring the resulting sinks to be the exits:

1. C^A by adding no edges to R^A;

2. C^I by adding the edges (x_i, y_i) to R^I for each $i \geqslant 2$;

3. C^O by adding the edges (x_i, y_i) to R^O for each $i \geqslant 2$.

Any subdivision of C^A, C^I and C^O is called *alternating comb*, *incoming comb* and *outgoing comb*, respectively. The subdivided ray in any comb is called the *spine* and the paths to the exits are the *spikes*.

A dimaze (D, B_0) is called $\mathcal{H}$*-free* for a set $\mathcal{H}$ of dimazes if it does not have a subdimaze isomorphic to a subdivision of an element in $\mathcal{H}$. A (strict) gammoid is called $\mathcal{H}$*-free* if it admits an $\mathcal{H}$-free presentation. In general, an $\mathcal{H}$-free gammoid may admit a presentation that is not $\mathcal{H}$-free (see Figure 3 for $\mathcal{H} = \{C^A\}$).

For notations about paths, we follow [9, p. 7]; and for a path P, we also write P for $V(P)$.

2 Dimazes and matroid axioms

The aim of this section is to give a sufficient condition for a dimaze (D, B_0) to define a matroid. As (I1) and (I2) hold for $M_L(D, B_0)$, we need only consider (I3) and (IM).

2.1 Linkability system and proof of (I3)

We prove that (I3) holds in any $M_L(D, B_0)$ using a classical result due to Grünwald [12], which can be formulated as follows (see also [9, Lemmas 3.3.2 and 3.3.3]). Let (D, B_0) be a dimaze and $\mathcal{P}$ a linkage. A (finite) $\mathcal{P}$*-alternating walk* is a sequence $W = w_0 e_0 w_1 e_1 \ldots e_{n-1} v_n$ of vertices w_i and distinct edges e_i of D, such that every $e_i \in W$ is incident with w_i and w_{i+1}, and the following properties hold for each $0 \leqslant i < n$:

(W1) $e_i = (w_{i+1}, w_i)$ if and only if $e_i \in E(\mathcal{P})$;

(W2) if $w_i = w_j$ for any $j \neq i$, then $w_i \in V(\mathcal{P})$;

(W3) if $w_i \in V(\mathcal{P})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{P}) \neq \emptyset$ (with $e_{-1} := e_0$).

Lemma 2.1. *Let (D, B_0) be a dimaze, $\mathcal{P}$ a linkage, and Ini$(\mathcal{P}) \subseteq X \subseteq V$.*

(i) If there is a $\mathcal{P}$-alternating walk from $X \setminus$ Ini$(\mathcal{P})$ to $B_0 \setminus$ Ter$(\mathcal{P})$, then there is a linkage $\mathcal{P}'$ with Ini$(\mathcal{P}) \subsetneq$ Ini$(\mathcal{P}') \subseteq X$ and Ter$(\mathcal{P}) \subsetneq$ Ter$(\mathcal{P}') \subseteq B_0$.

(ii) If there is not any $\mathcal{P}$-alternating walk from $X \setminus$ Ini$(\mathcal{P})$ to $B_0 \setminus$ Ter$(\mathcal{P})$, then there is an X–B_0 separator on $\mathcal{P}$.

Proposition 2.2. *Let (D, B_0) be a dimaze. Then $M_L(D, B_0)$ satisfies (I3).*

Proof. Let $I, B \in M_L(D, B_0)$ such that B is maximal but I is not. Then we have a linkage $\mathcal{Q}$ from B and another $\mathcal{P}$ from I. We may assume $\mathcal{P}$ misses some $v_0 \in B_0$.

If there is a $\mathcal{P}$-alternating walk from $(B \cup I) \setminus \mathrm{Ini}(\mathcal{P})$ to $B_0 \setminus \mathrm{Ter}(\mathcal{P})$, then we can extend I in $B \setminus I$ by Lemma 2.1(i).

On the other hand, if no such walk exists, we draw a contradiction to the maximality of B. In this case, by Lemma 2.1(ii), there is a $(B \cup I)$–B_0 separator S on $\mathcal{P}$. For every $v \in B$, let Q_v be the path in $\mathcal{Q}$ starting from v. Let s_v be the first vertex of S that Q_v meets and P_v the path in $\mathcal{P}$ containing s_v. Let us prove that $\mathcal{Q}' := \{Q_v s_v P_v : v \in B\}$ is a linkage.

Suppose v and v' are distinct vertices in B such that $Q_v s_v P_v$ and $Q_{v'} s_{v'} P_{v'}$ meet each other. As $\mathcal{P}$ and $\mathcal{Q}$ are linkages, without loss of generality, we may assume $Q_v s_v$ meets $s_{v'} P_{v'}$ at some $s \notin S$. Then $Q_v s P_{v'}$ is a path from B to B_0 avoiding the separator. This contradiction shows that $\mathcal{Q}'$ is indeed a linkage from B to B_0. As $\mathcal{Q}'$ does not cover v_0, $B + v_0$ is independent which contradicts the maximality of B. □

2.2 Linkage theorem and (IM)

For any dimaze (D, B_0), now that we know $M_L(D, B_0)$ satisfies (I3), it remains to investigate (IM). If D is finite, then the following holds:

A set is maximally independent if and only if it is linkable onto the exits. (†)

When D is infinite, (†) need not hold; for instance, the dimaze in Example 1.3, which does not even define a matroid. Using the Aharoni-Berger-Menger theorem [3] and the linkage theorem [18] (see also [10]), we prove that when (†) holds, $M_L(D, B_0)$ is a matroid.

Now the natural question is: in which dimazes is every set, that is linkable onto the exits, a maximal independent set? Consider the alternating comb given in Figure 3a. Using the notation there, the set $X := \{x_i : i \geqslant 1\}$ can be linked onto B_0 by the linkage $\{(x_i, y_{i-1}) : i \geqslant 1\}$ or to $B_0 - x_0$ by the linkage $\{(x_i, y_i) : i \geqslant 1\}$. Hence, X is a non-maximal independent set that is linkable onto B_0. More generally, if a dimaze (D, B_0) contains an alternating comb C, then the vertices of out-degree 2 on C together with $B_0 - C$ is a non-maximal set linkable onto B_0. So an answer to the above question must exclude dimazes containing an alternating comb. We will prove that dimazes without any alternating comb are precisely the answer.

One might think that the following proof strategy should work: If the characterization of maximal independent sets does not hold, then there are two linkages, a blue one from a set and a red one from a proper superset, both covering the exits. To construct an alternating comb, one starts with finding an alternating ray. For that, a first attempt is to "alternate" between the red and blue linkages, i.e. to repeat the following: go forward along the red linkage, change to the blue one at some common vertex, and then go backwards on the blue linkage, and change again to the red one. It is not the case that this construction always gives rise to an alternating ray (because vertices might be visited twice). But supposing that we do get an alternating ray, a natural way to extend it to an alternating comb is to use the terminal segments of one fixed linkage. However, this alternating ray can have two distinct vertices of in-degree 2 which lie on the same path of the fixed linkage.

Appropriate choices to alternate between the linkages will be provided by the *proof* of the linkage theorem of Pym [18]. So we outline the proof, rephrased for our purpose.

Linkage Theorem. *Let D be a digraph and two linkages be given: the "red" one, $\mathcal{P} = \{P_x : x \in X_{\mathcal{P}}\}$, from $X_{\mathcal{P}}$ onto $Y_{\mathcal{P}}$ and the "blue" one, $\mathcal{Q} = \{Q_y : y \in Y_{\mathcal{Q}}\}$, from $X_{\mathcal{Q}}$ onto $Y_{\mathcal{Q}}$. Then there is a set X^∞ satisfying $X_{\mathcal{P}} \subseteq X^\infty \subseteq X_{\mathcal{P}} \cup X_{\mathcal{Q}}$ which is linkable onto a set Y^∞ satisfying $Y_{\mathcal{Q}} \subseteq Y^\infty \subseteq Y_{\mathcal{Q}} \cup Y_{\mathcal{P}}$.*

Proof outline. We construct a sequence of linkages converging to a linkage with the desired properties. For each integer $i \geqslant 0$, we will specify a vertex on each path in $\mathcal{P}$. For each $x \in X_{\mathcal{P}}$, let $f_x^0 := x$. Let $\mathcal{Q}^0 := \mathcal{Q}$. For each $i > 0$ and each $x \in X_{\mathcal{P}}$, let f_x^i be the last vertex v on $f_x^{i-1}P_x$ such that $(f_x^{i-1}P_x\mathring{v}) \cap V(\mathcal{Q}^{i-1}) = \emptyset$. For $y \in Y_{\mathcal{Q}}$, let t_y^i be the first vertex $v \in Q_y$ such that the terminal segment $\mathring{v}Q_y$ does not contain any f_x^i. Let

$$\begin{aligned}
\mathcal{A}^i &:= \{Q_y \in \mathcal{Q} : t_y^i \neq f_x^i \ \forall x \in X_{\mathcal{P}}\},\\
\mathcal{B}^i &:= \{P_x f_x^i Q_y : x \in X_{\mathcal{P}}, y \in Y_{\mathcal{Q}} \text{ and } f_x^i = t_y^i\},\\
\mathcal{C}^i &:= \{P_x \in \mathcal{P} : f_x^i \in Y_{\mathcal{P}} \text{ and } f_x^i \neq t_y^i \ \forall y \in Y_{\mathcal{Q}}\},
\end{aligned}$$

and $\mathcal{Q}^i := \mathcal{A}^i \cup \mathcal{B}^i \cup \mathcal{C}^i$. It can be shown that $\mathcal{Q}^i$ is a linkage. Moreover, for any $x \in X_{\mathcal{P}}$, $\{f_x^i\}_{i\geqslant 0}$ eventually settles at a vertex f_x^∞ as $i \to \infty$; similarly for any $y \in Y_{\mathcal{Q}}$, $\{t_y^i\}_{i\geqslant 1}$ settles at some t_y^∞. Then $\mathcal{Q}^\infty$, defined as the union of the following three sets,

$$\begin{aligned}
\mathcal{A}^\infty &:= \{Q_y \in \mathcal{Q} : t_y^\infty \neq f_x^\infty \ \forall x \in X_{\mathcal{P}}\},\\
\mathcal{B}^\infty &:= \{P_x f_x^\infty Q_y : x \in X_{\mathcal{P}}, y \in Y_{\mathcal{Q}} \text{ and } f_x^\infty = t_y^\infty\},\\
\mathcal{C}^\infty &:= \{P_x \in \mathcal{P} : f_x^\infty \in Y_{\mathcal{P}} \text{ and } f_x^\infty \neq t_y^\infty \ \forall y \in Y_{\mathcal{Q}}\},
\end{aligned}$$

is a linkage satisfying the requirements. □

We can now prove the following.

Lemma 2.3. *Given a dimaze (D, B_0), suppose that every independent set linkable onto the exits is maximal, then the dimaze defines a matroid.*

Proof. Since (I1) and (I2) are obviously true for $M_L(D, B_0)$, and that (I3) holds by Proposition 2.2, to prove the theorem, it remains to check that (IM) holds.

Let I be independent and a set $X \subseteq V$ such that $I \subseteq X$ be given. Suppose there is a "red" linkage from I to B_0. Apply the Aharoni-Berger-Menger theorem on X and B_0 to get a "blue" linkage $\mathcal{Q}$ from $B \subseteq X$ to B_0 and an X–B_0 separator S on the blue linkage. Consider the subdigraph of D induced by those vertices separated from B_0 by S. Let H be obtained from this digraph by deleting the edges with initial vertex in S. Since every linkage from H to B_0 goes through S, a subset of $V(H)$ is linkable in (D, B_0) if and only if it is linkable in (H, S). Use the linkage theorem to find a linkage $\mathcal{Q}^\infty$ from X^∞ with $I \subseteq X^\infty \subseteq I \cup B \subseteq X$ onto S.

Let $Y \supseteq X^\infty$ be any independent set in $M_L(H, S)$. By applying the linkage theorem on a linkage from Y to S and $\mathcal{Q}^\infty$ in (H, S), we may assume that Y is linkable onto S by a linkage $\mathcal{Q}'$. Concatenating $\mathcal{Q}'$ with segments of paths in $\mathcal{Q}$ starting from S and adding trivial paths from $B_0 \setminus V(\mathcal{Q})$ gives us a linkage from $Y \cup (B_0 \setminus V(\mathcal{Q}))$ onto B_0. By the hypothesis, $Y \cup (B_0 \setminus V(\mathcal{Q}))$ is a maximal independent set in $M_L(D, B_0)$.

Applying the above statement on X^∞ shows that $X^\infty \cup (B_0 \setminus V(\mathcal{Q}))$ is also maximal in $M_L(D, B_0)$. It follows that Y cannot be a proper superset of X^∞. Hence, X^∞ is maximal in $M_L(H, S)$, and so also in $M_L(D, B_0) \cap 2^X$. This completes the proof that $M_L(D, B_0)$ is a matroid. $\square$

Next we show that containing an alternating comb is the only reason that the characterization (†) fails.

Lemma 2.4. *Let (D, B_0) be a C^A-free dimaze. Then a set $B \subseteq V$ is maximal in $M_L(D, B_0)$ if and only if it is linkable onto B_0.*

Proof. The forward direction follows trivially from (1).

For the backward direction, let I be a non-maximal subset that is linkable onto B_0, by a "blue" linkage $\mathcal{Q}$. Since I is not maximal, there is $x_0 \notin I$ such that $I + x_0$ is linkable to B_0 as well, by a "red" linkage $\mathcal{P}$. Construct an alternating comb inductively as follows:

Running the proof of the linkage theorem on $\mathcal{P}$ and $\mathcal{Q}$, we get a linkage $\mathcal{Q}^\infty$ from $I + x_0$ onto B_0 consisting of only $\mathcal{B}^\infty$, as $Y_\mathcal{P} \subseteq Y_\mathcal{Q}$ and $X_\mathcal{Q} \subseteq X_\mathcal{P}$. So each path in $\mathcal{Q}^\infty$ consists of a red initial and a blue terminal segment.

Start the construction with x_0. For $k \geqslant 1$, if x_{k-1} is defined, let Q_k be the blue path containing $p_{k-1} := f^\infty_{x_{k-1}}$. We will prove that $p_{k-1} \notin I$ so that we can define q_k to be the last vertex on $Q_k \mathring{p}_{k-1}$ that is on a path in $\mathcal{Q}^\infty$. Since the blue segments of $\mathcal{Q}^\infty$ are disjoint, q_k lies on a red path P_{x_k}. We continue the construction with x_k.

Claim 1. *$p_{k-1} \notin I$ and hence, the blue segment $q_k Q_k p_{k-1}$ is non-trivial. The red segment $q_k P_{x_k} p_k$ is also non-trivial.*

Proof. We prove the claim by induction. Clearly, $p_0 \notin I$, so the claim holds for $k = 1$. For $k \geqslant 2$, assume that $p_{k-2} \notin I$. We argue that $q_{k-1} \neq p_{k-1}$. Suppose they are equal for a contradiction. Then the path $P_{x_{k-1}} q_{k-1} Q_{k-1}$ is in $\mathcal{B}^\infty$. Since $q_{k-1} Q_{k-1} p_{k-2}$ is non-trivial, p_{k-2} and p_{k-1} are distinct vertices of the form f^∞_x on $P_{x_{k-1}} q_{k-1} Q_{k-1}$. This contradicts that $P_{x_{k-1}} q_{k-1} Q_{k-1}$ is in $\mathcal{B}^\infty$. Hence, we have $p_{k-1} \neq q_{k-1}$. This shows that the red segment $q_{k-1} P_{x_{k-1}} p_{k-1}$ is non-trivial, and so $p_{k-1} \notin I$. $\square$

We now show that $p_1 Q_1 \cup \bigcup_{k=2}^\infty q_j Q_j p_{j-1} \cup q_k P_{x_k} p_k Q_{k+1}$ is an alternating comb.

Claim 2. *$x_j \neq x_k$ for any distinct j and k.*

Proof. For $l \geqslant 0$, let i_l be the least integer such that $f^{i_l}_{x_l} = f^\infty_{x_l}$. We show that $i_{k-1} < i_k$. By the definition of q_k and i_{k-1}, $q_k Q_k$ is a segment of a path in $\mathcal{Q}^i$ for any $i < i_{k-1}$, so $f^i_{x_k}$ is on the segment $P_{x_k} q_k$, and $P_{x_k} f^{i_{k-1}}_{x_k} \subseteq P_{x_k} q_k$. Since $q_k P_{x_k} p_k$ is non-trivial, $P_{x_k} q_k \subsetneq P_{x_k} f^\infty_{x_k}$. We conclude that $f^{i_{k-1}}_{x_k} \neq f^\infty_{x_k}$. By the definition of i_k, we have $i_k > i_{k-1}$. Hence, $x_j \neq x_k$ for any $j \neq k$. $\square$

Since $q_k P_{x_k} p_k Q_{k+1}$ is a segment of the path on x_k in the linkage $\mathcal{Q}^\infty$, it is disjoint from $\mathring{q}_j Q_j \mathring{p}_{j-1}$ by the definition of q_j. Moreover, by Claim 2, all the segments of the form $q_k P_{x_k} p_k Q_{k+1}$ are disjoint, and so are those of the form $q_j Q_j p_{j-1}$. Hence, we have an alternating comb. This contradiction completes the proof. $\square$

We have all the ingredients to prove the main result.

Theorem 2.5. *Given a dimaze, the vertex sets linkable onto the exits form the bases of a matroid if and only if the dimaze contains no alternating comb. The independent sets of this matroid are precisely the linkable sets of vertices.*

Proof. For the first statement, the backward direction follows from Lemma 2.3 and Lemma 2.4. To see the forward direction, suppose there is an alternating comb C. Let B_1 be the union of the vertices of out-degree 2 on C with $B_0 - C$. Then B_1 is linkable onto B_0, and so is $B_1 + v$ for any $v \in B_0 \cap C$. But B_1 and $B_1 + v$ violate the base axiom (B2). The second statement follows from the first and (1). □

Corollary 2.6. *Any dimaze which does not define a matroid contains an alternating comb.*

We remark that even after forbidding alternating combs (or any ray at all), there are dimazes defining interesting strict gammoids. The existence of *wild* matroids, i.e. matroids containing a circuit which has an infinite intersection with a cocircuit, was first demonstrated in [5]. It turns out that strict gammoids are a rich source of wild matroids.

Lemma 2.7. *Suppose that $M_L(D, B_0)$ is a strict gammoid such that there is a circuit containing infinitely many vertices linkable to a fixed exit b in B_0. Then $M_L(D, B_0)$ is a wild matroid.*

Proof. The fundamental cocircuit of b with respect to B_0, consisting of all the vertices linkable to b, intersects the given circuit at infinitely many vertices. □

A concrete example is that $V(D) = \{v_i, b_i : i \geqslant 1\}$ with $B_0 = \{b_i : i \geqslant 1\}$ and $E(D) = \{(v_i, b_i), (v_1, b_i), (v_i, b_1) : i \geqslant 1\}$. Then (D, B_0) is a C^A-free dimaze, and $\{v_i, b_1 : i \geqslant 1\}$ is an infinite circuit satisfying the lemma.

2.3 Nearly finitary linkability system

Although forbidding alternating combs ensures that we get a strict gammoid, not every strict gammoid arises this way. It turns out that when a dimaze gives rise to a nearly finitary ([4]) linkability system, the dimaze defines a matroid regardless of whether it contains an alternating comb or not. We will show this using the proof of the linkage theorem.

Lemma 2.8. *Let (D, B_0) be a dimaze. Then $M_L(D, B_0)$ satisfies the following:*

($*$) *For all independent sets I and J with $J \setminus I \neq \emptyset$, for every $v \in I \setminus J$ there exists $u \in J \setminus I$ such that $J + v - u$ is independent.*

Proof. We may assume that $I \setminus J = \{v\}$. Let $\mathcal{Q} = (Q_y)_{y \in Y_\mathcal{Q}}$ be a "blue" linkage from J onto some $Y_\mathcal{Q} \subseteq B_0$ and $\mathcal{P}$ a "red" one from I. The linkage theorem yields a linkage $\mathcal{Q}^\infty$, which we will show to witness the independence of a desired set. We use the notations introduced in its proof. For each $y \in Y_\mathcal{Q}$, let t_y^0 be the initial vertex of Q_y.

For $i > 0$ it is not hard to derive the following facts from the definitions of $\mathcal{Q}^i, f_x^i$ and t_y^i:

$$x \in I \cap \mathrm{Ini}(\mathcal{Q}^{i-1}) \quad \Longrightarrow \quad f_x^i = f_x^{i-1}; \tag{2}$$

$$t_y^0 \in \mathrm{Ini}(\mathcal{A}^i) \quad \Longleftrightarrow \quad \forall x \in I, f_x^i \notin Q_y; \tag{3}$$

$$x \in I \setminus \mathrm{Ini}(\mathcal{Q}^i) \quad \Longleftrightarrow \quad \exists y \in Y_{\mathcal{Q}}, x' \in I \text{ s.t. } f_x^i \in Q_y \mathring{f}_{x'}^i. \tag{4}$$

Claim. *For $i \geqslant 0$, either $\mathcal{Q}^i = \mathcal{Q}^\infty$ or there is some x^i such that:*

U_i: $(J+v) \setminus Ini(\mathcal{Q}^i) = \{x^i\}$ and $I - x^i \subseteq Ini(\mathcal{B}^i)$;

D_i: $\forall y \in Y_{\mathcal{Q}}$, if $t_y^0 \in I$ then $\exists! x \in I - x^i$ s.t. $f_x^i \in Q_y$; no such x otherwise.

Proof. With $x^0 := v$, the claim clearly holds for $i = 0$. Given $i > 0$, to prove the claim, we may assume that $\mathcal{Q}^{i-1} \neq \mathcal{Q}^\infty$ and U_{i-1} and D_{i-1} hold.

By definition of $f_{x^{i-1}}^i$, either

$$f_{x^{i-1}}^i \in t_{y^i}^{i-1} Q_{y^i} \text{ for some unique } y^i \in Y_{\mathcal{Q}} \text{ or } f_{x^{i-1}}^i \in Y_{\mathcal{P}} \setminus Y_{\mathcal{Q}}.$$

Note that by (2) only x^{i-1} can be a vertex such that $f_{x^{i-1}}^i \neq f_{x^{i-1}}^{i-1}$. Hence, $t_y^i = t_y^{i-1}$ for all $y \in Y_{\mathcal{Q}}$ except possibly y^i which satisfies $t_{y^i}^i = f_{x^{i-1}}^i$. So by (4), we have $x^{i-1} \in \mathrm{Ini}(\mathcal{Q}^i)$.

Case (i): Suppose that there exists $x \in I - x^{i-1}$ such that $f_{x^{i-1}}^i$ and f_x^i are on the same path Q_{y^i}. By D_{i-1}, $t_{y^i}^0 \in I$ and x is unique. Then D_i holds for $x^i := x$. In particular, by (3), $J \setminus I \subseteq \mathrm{Ini}(\mathcal{A}^i)$.

We now prove U_i. As $x^i \in \mathrm{Ini}(\mathcal{B}^{i-1})$, $t_{y^i}^{i-1} = f_{x^i}^{i-1}$, so $f_{x^{i-1}}^i \in \mathring{f}_{x^i}^i Q_{y^i}$, which implies that $x^i \notin \mathrm{Ini}(\mathcal{Q}^i)$ by (4). Given $x \in I - \{x^{i-1}, x^i\}$, then $x \in \mathrm{Ini}(\mathcal{B}^{i-1})$ by U_{i-1}. So there exists $y \neq y^i$, such that $f_x^i = f_x^{i-1} = t_y^{i-1} = t_y^i$. It follows that $x \in \mathrm{Ini}(\mathcal{B}^i)$, and $I - x^i \subseteq \mathrm{Ini}(\mathcal{B}^i)$. Therefore, $(J+v) \setminus \mathrm{Ini}(\mathcal{Q}^i) = \{x^i\}$.

Case (ii): Suppose that there does not exist any $x \in I - x^{i-1}$ such that f_x^i is on the path Q_{y^i} containing $f_{x^{i-1}}^i$, if such a path exists. In this case, $t_{y^i}^0 \in J \setminus I$. By D_{i-1}, (2) and (4), we have $I - x^{i-1} \subseteq \mathrm{Ini}(\mathcal{B}^i)$. Hence, $I \subseteq \mathrm{Ini}(\mathcal{Q}^i)$, and $\mathcal{Q}^\infty = \mathcal{Q}^i$. □

If for some integer $i > 0$, case (ii) holds, then by (3), only $u := t_{y^i}^0 \in J \setminus I$ can fail to be in $\mathrm{Ini}(\mathcal{A}^\infty)$. Otherwise, case (i) holds for each integer $i \geqslant 0$, so that $J \setminus I$ is a subset of $\mathrm{Ini}(\mathcal{A}^i)$ and hence a subset of $\mathrm{Ini}(\mathcal{Q}^\infty)$. In either situation, since $I = X_{\mathcal{P}} \subseteq \mathrm{Ini}(\mathcal{Q}^\infty)$, we conclude that there is some $u \in J \setminus I$ such that $J + v - u$ is independent. □

Let $(E, \mathcal{I})$ be a set system. The *finitarisation* $\mathcal{I}^{\mathrm{fin}}$ of $\mathcal{I}$ consists of sets which have all their finite subsets in $\mathcal{I}$. $(E, \mathcal{I})$ is called *nearly finitary* if for any maximal element $B \in \mathcal{I}^{\mathrm{fin}}$ there is an $I \in \mathcal{I}$ such that $|B \setminus I| < \infty$.

Theorem 2.9. *Let (D, B_0) be a dimaze. If $M_L(D, B_0)$ is nearly finitary, then it is a matroid.*

Proof. Since $M_L(D, B_0)$ satisfies (I1), (I2) and $(*)$, by [4, Lemma 4.15], it also satisfies (IM). Hence, by Proposition 2.2, it is a matroid. □

The theorem shows that dimazes which contain an alternating comb may also define matroids.

Example 2.10. We construct a dimaze (D, B_0) which defines a nearly finitary linkability system, by identifying the corresponding exits of $n > 1$ copies of C^O (see Figure 2). Note that (D, B_0) contains an alternating comb; and $M_L(D, B_0)$ is not finitary (a vertex not in B_0 together with all reachable vertices in B_0 form an infinite circuit[4]).

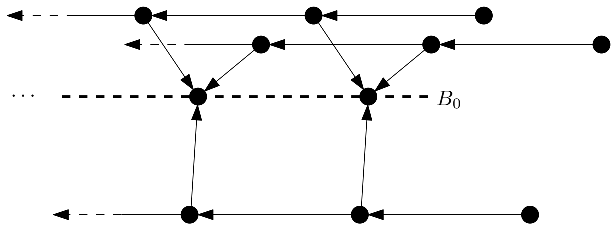

Figure 2: A dimaze that defines a nearly finitary linkability system

We check that $M_L(D, B_0)$ is nearly finitary. Suppose B is a maximal element in $M_L(D, B_0)^{\text{fin}}$. Let I be the set obtained from B by deleting the last vertex, if exists, of B on each ray in $D - B_0$; and $T := B \setminus I$. Fix an enumeration $i_1, i_2, \ldots$ for I such that D contains a ray starting in i_{k+1} that avoids $I_k := \{i_1, \ldots, i_k\}$ for each $k \geqslant 0$ with $i_{k+1} \notin B_0$. For any integer $k \geqslant 1$, let T_k consist of exactly one vertex on each ray in $D - B_0$ (that hits I_k): the first one in B after the last vertex of I_k. Note that there are only finitely many linkages from I_k to B_0 avoiding T_k. In fact, there is at least one: the restriction to I_k of a linkage of the finite subset $I_k \cup T_k$ of B. Applying the infinity lemma ([9, Proposition 8.2.1]), with the kth set consisting of the finite non-empty collection of linkages from I_k to B_0 avoiding T_k, we obtain a linkage from I to B_0. Hence, $I \in M_L(D, B_0)$. As B is arbitrary and $|T| \leqslant n$, we conclude that $M_L(D, B_0)$ is nearly finitary.

On the other hand, Theorem 2.9 does not imply Theorem 2.5: there are non-nearly finitary C^A-free strict gammoids. In fact they can be highly connected and have non-nearly finitary duals.

Lemma 2.11. *Given a matroid M and a base B, if $E \setminus B$ contains infinitely many elements which are not in any finite circuit, then M is not nearly finitary.*

Proof. Extend B to a base B^{fin} of M^{fin}. As $|B^{\text{fin}} \setminus B| = \infty$, so is $|B^{\text{fin}} \setminus B'|$ for any other base B' of M inside B^{fin}. So M is not nearly finitary. $\square$

Example 2.12. For any integer $k \geqslant 2$, there is a k-connected strict gammoid $M = M_L(D, B_0)$ such that the underlying graph of D is rayless, and neither M nor its dual is nearly finitary.

Consider a rooted tree T of depth 3 where each internal vertex has infinitely many children, and each edge is directed towards $L_0 \cup L_2$ where L_i is the set of vertices at

[4]Any circuit in a finitary matroid is finite.

distance i from the root for $0 \leqslant i \leqslant 3$. Let D be a digraph with $V = V(T) \cup X \cup Y$, where each of X and Y is an extra set of k vertices; and $E(D) = E(T) \cup \{(x, b), (v, y) : x \in X, b \in B_0, v \in V \setminus B_0, y \in Y\}$, where $B_0 = L_0 \cup L_2 \cup Y$. Since (D, B_0) is C^A-free, by Theorem 2.5, $M = M_L(D, B_0)$ is a matroid.

As no vertex in L_1 lies in a finite circuit, applying Lemma 2.11 with the base B_0 shows that M is not nearly finitary. Similarly, as no vertex in L_2 lies in a finite cocircuit, the same lemma with $V \setminus B_0$ shows that M^* is not nearly finitary.

For any $l < k$, it is not difficult to see that in any bipartition of V into sets P, Q of size at least l, there is a linkage from $P_1 \subseteq P \setminus B_0$ to $Q \cap B_0$ and from $Q_1 \subseteq Q \setminus B_0$ to $P \cap B_0$ of size at least l. It follows that $P_1 \cup (P \cap B_0) \cup Q_1 \cup (Q \cap B_0)$ contains at least l vertices more than B_0. Hence, (P, Q) is not an l-separation. So M is k-connected.

So far we have seen that if a dimaze (D, B_0) is C^A-free or that $M_L(D, B_0)$ is nearly finitary, then $M_L(D, B_0)$ is a matroid. However, there are examples of strict gammoids that lie in neither of the two classes. All our examples of dimazes that do not define a matroid share another feature other than possessing an alternating comb: there is an independent set I that cannot be extended to a maximal in $I \cup B_0$. In view of this, we propose the following.

Conjecture 2.13. *Suppose that for all $I \in M_L(D, B_0)$ and $B \subseteq B_0$, there is a maximal independent set in $I \cup B$ extending I. Then (IM) holds for $M_L(D, B_0)$.*

3 Dimazes with alternating combs

We have seen in Section 2 that forbidding alternating combs in a dimaze guarantees that it defines a strict gammoid. However, the alternating comb in Figure 3 defines a matroid. On the other hand, this strict gammoid is isomorphic to the one defined by the incoming comb via the isomorphism given in the figure. So one might hope that *every* strict gammoid is C^A-free. In general, this is not the case and the aim of this section is to construct a counterexample.

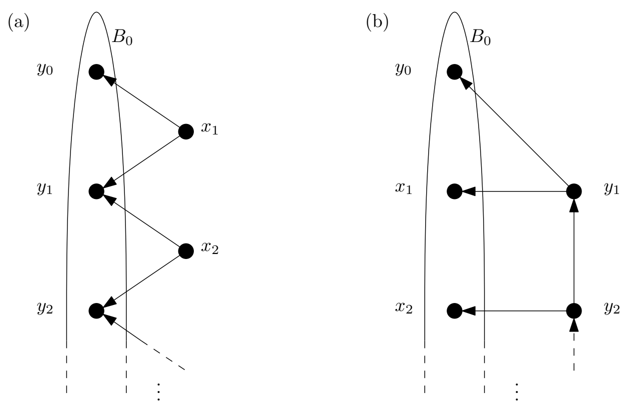

Figure 3: An alternating comb and an incoming comb defining isomorphic strict gammoids

3.1 Finite circuits, cocircuits and alternating combs

We first give a necessary condition of any strict gammoid defined by a C^A-free dimaze.

Lemma 3.1. *If a dimaze (D, B_0) is C^A-free, then $M_L(D, B_0)$ contains a finite circuit or a finite cocircuit.*

Proof. Suppose the lemma does not hold. Then every finite subset of V is independent and coindependent, and B_0 is infinite. We construct a sequence $(R_k : k \geqslant 1)$ of finite subdigraphs of D that gives rise to an alternating comb for a contradiction.

Let $v_1 \notin B_0$ and R_1 a path from v_1 to B_0. For $k \geqslant 1$, we claim that there is a path P_k from v_k to B_0 such that $P_k \cap V(R_k) = \{v_k\}$, a vertex w_k on $\mathring{v}_k P_k$, and a vertex $v_{k+1} \notin V(R_k) \cup P_k$ with $(v_{k+1}, w_k) \in E(D)$. Let $R_{k+1} := R_k \cup P_k \cup (v_{k+1}, w_k)$.

Indeed, since any finite set containing v_k is independent, there is a path from v_k avoiding any given finite set disjoint from v_k. Hence, there is a set $\mathcal{F}$ of $|V(R_k)| + 1$ disjoint paths (except at v_k) from v_k to B_0 avoiding the finite set $V(R_k) - v_k$. Since $V(\mathcal{F}) \cup R_k$ is coindependent, its complement contains a base B, witnessed by a linkage $\mathcal{P}$. Since $|V(\mathcal{F}) \cap B_0| > |V(R_k)|$ and $\mathrm{Ter}(\mathcal{P}) = B_0$, there is a path $P \in \mathcal{P}$ that is disjoint from R_k and ends in $V(\mathcal{F}) \cap B_0$. As the terminal vertex of P is in $V(\mathcal{F})$, but its initial vertex is not, there is an edge (v_{k+1}, w_k) of P such that w_k is on some path $P_k \in \mathcal{F}$, but v_{k+1} is not (on P_k). Then the vertices v_{k+1}, w_k and the path P_k satisfy the requirements of the claim. By induction, the claim holds for all $k \geqslant 1$.

Let $R := \bigcup_{k \geqslant 1} R_k$. Then $(R, V(R) \cap B_0)$ is an alternating comb in (D, B_0). This contradiction completes the proof. □

A matroid is *infinitely connected* if it does not have any k-separation for any integer k. The only infinitely connected finite matroids are uniform matroids of rank about half of the size of the ground set (see [16, Chapter 8]) and they are strict gammoids. It seems natural to look for an infinitely connected infinite matroid among strict gammoids, but the previous lemma gives us a partial negative result because the bipartition of any finite circuit of size k against the rest is a k-separation. It remains open whether there is an infinitely connected infinite gammoids.

Corollary 3.2. *If an infinite dimaze (D, B_0) is C^A-free, then $M_L(D, B_0)$ is not infinitely connected.*

3.2 Trees and transversal matroids

To give a strict gammoid not definable by a dimaze without alternating comb, we need only construct a strict gammoid without any finite circuit or cocircuit. A particular example is furnished by turning a transversal matroid defined on a tree to a strict gammoid. We prove a more general result that any tree gives rise to a transversal matroid. The definitions are recalled here.

Given a bipartite graph G, fix an ordered bipartition (V, W) of $V(G)$; this induces an ordered bipartition of any subgraph of G. A subset of V is *independent* if it is matchable to W. Let $M_T(G)$ be the pair of V and the collection of independent sets. It is clear that (I1), (I2) hold for $M_T(G)$. When G is finite, (I3) also holds [11]. The proof of this fact which uses alternating paths can be extended to show that (I3) also holds when G is infinite.

Let m be a matching. An edge in m is called an m–edge. An *m–alternating path* is a path or a ray that starts from a vertex in V such that the edges alternate between the m–edges and the non-m–edges, as long as possible. An *m–m' alternating path* is defined analogously with m', also a matching, replacing the role of the non-m–edges.

Lemma 3.3. *For any bipartite graph G, $M_T(G)$ satisfies (I3).*

Proof. Let $I, B \in M_T(G)$ such that B is maximal but I is not. As I is not maximal, there is a matching m of $I + x$ for some $x \in V \setminus I$. Let m' be a matching of B to W. Start an m–m' alternating path P from x. By maximality of B, the alternating path is not infinite and cannot end in $W \setminus V(m')$. So we can always extend it until it ends at some $y \in B \setminus I$. Then $m\Delta E(P)$ is a matching of $I + y$, which completes the proof. □

If $M_T(G)$ is a matroid, it is called a *transversal matroid*. For $X \subseteq V$, the restriction of $M_T(G)$ to X is also a transversal matroid, and can be defined by the independent sets of the subgraph of G induced by $X \cup N(X)$.

Suppose now G is a tree rooted at a vertex in W. By upwards (downwards), we mean towards (away from) the root. For any vertex set Y, let $N^{\uparrow}(Y)$ be the upward neighbourhood of Y, and $N^{\downarrow}(Y)$ the set of downward neighbours. An edge is called *upward* if it has the form $\{v, N^{\uparrow}(v)\}$ where $v \in V$, otherwise it is *downward*.

We will prove that $M_T(G)$ is a matroid. To build a maximal independent set whose existence is required by (IM), we inductively construct a sequence of matchings $(m^\alpha : \alpha \geqslant 0)$, indexed by ordinals.

Given $m^{\beta-1}$, to define m^β, we consider the vertices not matched by $m^{\beta-1}$ that do not have unmatched children for the first time at step $\beta - 1$. We ensure that any such vertex v that is also in I is matched in step β, by exchanging v with a currently matched vertex r_v that is not in I.

When every vertex that has not been considered has an unmatched child, we stop the algorithm. We then prove that the union of the matched vertices and those unconsidered vertices is a maximal independent superset of I.

Theorem 3.4. *For any tree G with an ordered bipartition (V, W), $M_T(G)$ is a transversal matroid.*

Proof. To prove that $M_T(G)$ is a matroid, it suffices to prove that (IM) holds. Let an independent set $I \subseteq X \subseteq V$ be given. Without loss of generality, we may assume that $X = V$.

We start by introducing some notations. Root G at some vertex in W. Given an ordinal α and a matching m^α, let $I^\alpha := V(m^\alpha) \cap V$ and $W^\alpha := V(m^\alpha) \cap W$. Given a sequence of matchings $(m^{\alpha'} : \alpha' \leqslant \alpha)$, let

$$C^\alpha := \{v \in V \setminus I^\alpha : N^{\downarrow}(v) \subseteq W^\alpha \text{ but } N^{\downarrow}(v) \not\subseteq W^{\alpha'} \ \forall \alpha' < \alpha\}.$$

Note that $C^\alpha \cap C^{\alpha'} = \emptyset$ for $\alpha' \neq \alpha$. For each $w \in W \setminus W^\alpha$, choose one vertex v_w in $N^\downarrow(w) \cap C^\alpha$ if it is not empty. Let

$$S^\alpha := \{v_w : w \in W \setminus W^\alpha \text{ and } N^\downarrow(w) \cap C^\alpha \neq \emptyset\}.$$

Denote the following statement by $A(\alpha)$:

> There is a pairwise disjoint collection $\mathcal{P}^\alpha := \{P_v : v \in I \cap C^\alpha \setminus S^\alpha\}$ of m^α–alternating paths such that each P_v starts from $v \in I \cap C^\alpha \setminus S^\alpha$ with a downward edge and ends at the first vertex r_v in $I^\alpha \setminus I$.

Start the inductive construction with m^0, which is the set of upward edges that is contained in every matching of I. It is not hard to see that $C^0 \cap I = \emptyset$, so that $A(0)$ holds trivially.

Let $\beta > 0$. Given the constructed sequence of matchings $(m^\alpha : \alpha < \beta)$, suppose that $A(\alpha)$ holds for each $\alpha < \beta$. Construct a matching m^β as follows.

If β is a successor ordinal, let

$$m^\beta := E(S^{\beta-1}, N^\uparrow(S^{\beta-1})) \cup (m^{\beta-1} \Delta E(\mathcal{P}^{\beta-1})).$$

By $A(\beta - 1)$, the paths in $\mathcal{P}^{\beta-1}$ are disjoint. So $m^{\beta-1}\Delta E(\mathcal{P}^{\beta-1})$ is a matching. Using the definition of $S^{\beta-1}$, we see that m^β is indeed a matching. Observe also that

$$I^{\beta-1} \cap I \subseteq I^\beta \cap I; \tag{5}$$
$$W^{\beta-1} \subseteq W^{\beta-1} \cup N^\uparrow(S^{\beta-1}) = W^\beta. \tag{6}$$

If β is a limit ordinal, define m^β by

$$e \in m^\beta \iff \exists \beta' < \beta \text{ such that } e \in m^\alpha \ \forall \alpha \text{ with } \beta' \leqslant \alpha < \beta. \tag{7}$$

As m^α is a matching for every ordinal $\alpha < \beta$, we see that m^β is a matching in this case, too.

Suppose that a vertex $u \in (V \cap I) \cup W$ is matched to different vertices by m^α and $m^{\alpha'}$ for some $\alpha, \alpha' \leqslant \beta$. Then there exists some ordinal $\alpha'' + 1$ between α and α' such that u is matched by an upward $m^{\alpha''}$–edge and by a downward $m^{\alpha''+1}$–edge. Hence, the change of the matching edges is unique. This implies that for any α, α' with $\alpha \leqslant \alpha' \leqslant \beta$, by (5) and (6), we have

$$I^\alpha \cap I \subseteq I^{\alpha'} \cap I; \tag{8}$$
$$W^\alpha \subseteq W^{\alpha'}. \tag{9}$$

Moreover, for an upward m^β–edge vw with $v \in V$, we have

$$v \in I^0 \text{ or } \exists \alpha < \beta \text{ such that } v \in C^\alpha \text{ and } w \notin W^\alpha. \tag{10}$$

We now prove that $A(\beta)$ holds. Given $v_0 = v \in I \cap C^\beta \setminus S^\beta$, we construct a decreasing sequence of ordinals starting from $\beta_0 := \beta$. For an integer $k \geqslant 0$, suppose that $v_k \in$

$I \cap C^{\beta_k}$ with $\beta_k \leqslant \beta$ is given. By (8), $I^0 \subseteq I^{\beta_k}$, so $v_k \notin I^0$ and hence there exists $w_k \in N^{\downarrow}(v_k) \setminus W^0$.[5] Since $N^{\downarrow}(v_k) \subseteq W^{\beta_k} \subseteq W^{\beta}$, w_k is matched by m^{β} to some vertex v_{k+1}. In fact, as $w_k \notin W^0$, $v_{k+1} \notin I^0$. Let β_{k+1} be the ordinal with $v_{k+1} \in C^{\beta_{k+1}}$. Since $v_{k+1}w_k$ is an upward edge and $N^{\downarrow}(v_k) \subseteq W^{\beta_k}$, we have by (10) that $w_k \in W^{\beta_k} \setminus W^{\beta_{k+1}}$. By (9), $\beta_k > \beta_{k+1}$.

As there is no infinite decreasing sequence of ordinals, we have an m^{β}–alternating path $P_v = v_0w_0v_1w_1 \cdots$ that stops at the first vertex $r_v \in V \setminus I$.

The disjointness of the P_v's follows from that every vertex has a unique upward neighbour and, as we just saw, that $\mathring{v}P_v$ cannot contain any vertex $v' \in C^{\beta}$. So $A(\beta)$ holds.

We can now go on with the construction.

Let $\gamma \leqslant |V|$ be the least ordinal such that $C^{\gamma} = \emptyset$.[6] Let $C := \bigcup_{\alpha<\gamma} C^{\beta}$ and $U := V \setminus (I^0 \cup C)$; so V is partitioned into I^0, C and U. As $C^{\gamma} = \emptyset$, every vertex in U can be matched downwards to a vertex that is not in W^{γ}. These edges together with m^{γ} form a matching m^B of $B := U \cup I^{\gamma}$, which we claim to be a witness for (IM). By (8), $I^0 \cup (C \cap I) \subseteq I^{\gamma}$, hence, $I \subseteq B$.

Suppose B is not maximally independent for a contradiction. Then there is an m^B–alternating path $P = v_0w_0v_1w_1 \cdots$ such that $v_0 \in V \setminus B$ that is either infinite or ends with some $w_n \in W \setminus V(m^B)$. We show that neither occurs.

Claim 1. *P is finite.*

Proof. Suppose P is infinite. Since $v_0 \notin B$, P has a subray $R = w_iP$ such that w_iv_{i+1} is an upward m^B–edge. So $w_jv_{j+1} \in m^B$ for any $j \geqslant i$. As vertices in U are matched downwards, $R \cap U = \emptyset$. As $m^B \Delta E(R)$ is a matching of $B \supseteq I$ in which every vertex in $R \cap V$ is matched downwards, $R \cap I^0 = \emptyset$ too. So for any $j \geqslant i$, there exists a unique β_j such that $v_j \in C^{\beta_j}$.

Choose $k \geqslant i$ such that β_k is minimal. But with a similar argument used to prove $A(\beta)$, we have $\beta_k > \beta_{k+1}$. Hence P cannot be infinite. □

Claim 2. *P does not end in $W \setminus V(m^B)$.*

Proof. Suppose that P ends with $w_n \in W \setminus V(m^B)$. Certainly, v_n can be matched downwards (either to w_{n-1} or w_n) in a matching of $B \supseteq I$. Hence, $v_n \notin I^0$. It is easy to check that for $v \in C^{\alpha}$, $N(v) \subseteq W^{\alpha+1}$. Hence, as $w_n \in W \setminus W^{\gamma}$, $v_n \notin C$. Hence, $v_n \in U$. It follows that for each $0 < i \leqslant n$, v_i is matched downwards and so does not lie in I^0. As $v_0 \notin B$, $v_0 \in C$. It follows that $w_0 \in W^{\gamma}$ and $v_1 \in C$. Repeating the argument, we see that $v_n \in C$, which is a contradiction. □

We conclude that B is maximal. So (IM) holds and $M_T(G)$ is a matroid. □

Corollary 3.5. *Let (D, B_0) be a dimaze such that the underlying graph of D is a tree and B_0 is a vertex class of a bipartition of D with edges directed towards B_0. Then $M_L(D, B_0)$ is a matroid.*

[5]For a vertex $v \notin I$, $N^{\downarrow}(v) \setminus W^0$ may be empty.

[6]In order to see $\gamma \leqslant |V|$, fix a well-ordering of V and map each β to the least element in C^{β}.

Proof. By the theorem, we need only present $M_L(D, B_0)$ as a transversal matroid defined on a tree. Define a bipartite graph G with the vertex classes $(V \setminus B_0) \cup \{b' : b \in B_0\}$ and B_0 and the edge set $\{vw : (v, w) \in E(D)\} \cup \{bb' : b \in B_0\}$. Since $M_L(D, B_0) \cong M_T(G)$ and G is a tree, $M_L(D, B_0)$ is a matroid. □

Consider the infinitely branching rooted tree, i.e. a rooted tree such that each vertex has infinitely many children. Let B_0 consist of the vertices on alternate levels, starting from the root. Define $\mathcal{T}$ by directing all edges towards B_0. Corollary 3.5 shows that $M_L(\mathcal{T}, B_0)$ is a matroid. Clearly, this matroid does not contain any finite circuit. Moreover, as any finite set C^* misses a base obtained by adding finitely many vertices to $B_0 \setminus C^*$, any cocircuit must be infinite. We remark that this matroid is not dual to any legal transversal matroid, which is introduced in [2] to describe the duals of strict gammoids given by Theorem 2.5. With Lemma 3.1, we conclude the following.

Corollary 3.6. *Every dimaze that defines a strict gammoid isomorphic to $M_L(\mathcal{T}, B_0)$ contains an alternating comb.*

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