On shuffling of infinite square-free words

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Abstract

In this paper we answer two recent questions from Charlier et al. (2014) and Harju (2013) about self-shuffling words. An infinite word w is called self-shuffling, if $w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i$ for some finite words U_i, V_i. Harju recently asked whether square-free self-shuffling words exist. We answer this question affirmatively. Besides that, we build an infinite word such that no word in its shift orbit closure is self-shuffling, answering positively a question of Charlier et al.

Keywords: infinite words, shuffling, square-free words, shift orbit closure, self-shuffling words

1 Introduction

A self-shuffling word, a notion which was recently introduced by Charlier et al. [2], is an infinite word that can be reproduced by shuffling it with itself. More formally, given

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two infinite words $x, y \in \Sigma^\omega$ over a finite alphabet Σ, we define $\mathscr{S}(x,y) \subseteq \Sigma^\omega$ to be the collection of all infinite words z for which there exists a factorization

$$z = \prod_{i=0}^{\infty} U_i V_i$$

with each $U_i, V_i \in \Sigma^*$ and with $x = \prod_{i=0}^{\infty} U_i$, $y = \prod_{i=0}^{\infty} V_i$. An infinite word $w \in \Sigma^\omega$ is *self-shuffling* if $w \in \mathscr{S}(w,w)$. Various well-known words, e.g., the Thue-Morse word or the Fibonacci word, were shown to be self-shuffling.

Harju [5] studied shuffles of both finite and infinite square-free words, i.e., words that have no factor of the form uu for some non-empty factor u. More results on square-free shuffles were obtained independently by Harju and Müller [6], and Currie and Saari [4]. However, the question about the existence of an infinite square-free self-shuffling word, posed in [5], remained open. We give a positive answer to this question in Sections 2 and 3.

The *shift orbit closure* S_w of an infinite word w can be defined, e.g., as the set of infinite words whose sets of factors are contained in the set of factors of w. In [2] it has been proved that each word has a non-self-shuffling word in its shift orbit closure, and the following question has been asked: Does there exist a word for which no element of its shift orbit closure is self-shuffling (Question 7.2)? In Section 4 we provide a positive answer to the question. More generally, we show the existence of a word such that for any three words x, y, z in its shift orbit closure, if x is a shuffle of y and z, then the three words are pairwise different. On the other hand, we show that for any infinite word there exist three different words x, y, z in its shift orbit closure such that $x \in \mathscr{S}(y,z)$ (see Proposition 7).

Apart from the usual concepts in combinatorics on words, which can be found for instance in the book of Lothaire [7], we make use of the following notations: For every $k \geqslant 1$, we denote the alphabet $\{0, 1, \ldots, k-1\}$ by Σ_k. For a word $w = uvz$ we say that u is a *prefix* of w, v is a *factor* of w, and z is a *suffix* of w. We denote these prefix- and suffix relations by $u \leqslant_p w$ and $v \leqslant_s w$, respectively. By $w[i,j]$ we denote the factor of w starting at position i and ending after position j. Note that we start numbering the positions with 0.

A *prefix code* is a set of words with the property that none of its elements is a prefix of another element. Similarly, a *suffix code* is a set of words where no element is a suffix of another one. A *bifix code* is a set that is both a prefix code and a suffix code. A morphism h is *square-free* if for all square-free words w, the image $h(w)$ is square-free.

2 A square-free self-shuffling word on four letters

Let $g : \Sigma_4^* \to \Sigma_4^*$ be the morphism defined as follows:

$$\begin{aligned} g(0) &= 0121, \\ g(1) &= 032, \\ g(2) &= 013, \\ g(3) &= 0302. \end{aligned}$$

We will show that the fixed point $w = g^\omega(0)$ is square-free and self-shuffling. Note that g is not a square-free morphism, that is, it does not preserve square-freeness, as $g(23) = 0130302$ contains the square 3030.

Lemma 1. *The word $w = g^\omega(0)$ contains no factor of the form $3u1u3$ for any $u \in \Sigma_4^*$.*

Proof. We assume that there exists a factor of the form $3u1u3$ in w, for some word $u \in \Sigma_4^*$. From the definition of g, we observe that u can not be empty. Furthermore, we see that every 3 in w is preceded by either 0 or 1. If $1 \leqslant_s u$, then we had an occurrence of the factor 11 in w, which is not possible by the definition of g, hence $0 \leqslant_s u$. Now, every 3 is followed by either 0 or 2 in w and 01 is followed by either 2 or 3. Since both $3u$ and $01u$ are factors of w, we must have $2 \leqslant_p u$. This means that the factor 012 appears at the center of $u1u$, which can only be followed by 1 in w, thus $21 \leqslant_p u$. However, this results in the factor 321 as a prefix of $3u1u3$, which does not appear in w, as seen from the definition of g. □

Lemma 2. *The word $w = g^\omega(0)$ is square-free.*

Proof. We first observe that $\{g(0), g(1), g(2), g(3)\}$ is a bifix code. Furthermore, we can verify that there are no squares uu with $|u| \leqslant 3$ in w. Let us assume now, that the square uu appears in w and that u is the shortest word with this property. If $u = 02u'$, then $u' = u''03$ must hold, since 02 appears only as a factor of $g(3)$, and thus uu is a suffix of the factor $g(3)u''g(3)u''03$ in w. As $w = g(w)$, also the shorter square $3g^{-1}(u'')3g^{-1}(u'')$ appears in w, a contradiction. The same desubstitution principle also leads to occurrences of shorter squares in w if $u = xu'$ and $x \in \{01, 03, 10, 12, 13, 21, 30, 32\}$.

If $u = 2u'$ then either $03 \leqslant_s u$ or $030 \leqslant_s u$ or $01 \leqslant_s u$, by the definition of g. In the last case, that is when $01 \leqslant_s u$, we must have $21 \leqslant_p u$, which is covered by the previous paragraph. If $u' = u''030$, then uu is followed by 2 in w and we can desubstitute to obtain the shorter square $g^{-1}(u'')3g^{-1}(u'')3$ in w. If $u = 2u'$ and $u' = u''03$, and uu is preceded by 03 or followed by 2 in w, we can desubstitute to $1g^{-1}(u'')1g^{-1}(u'')$ or $g^{-1}(u'')1g^{-1}(u'')1$, respectively. Therefore, assume that $u = 2u''03$ and as we already ruled out the case when $21 \leqslant_p u$, we can assume that uu is preceded by 030 and followed by 02 in w. This however means that we can desubstitute to get an occurrence of the factor $3g^{-1}(u'')1g^{-1}(u'')3$ in w, a contradiction to Lemma 1. □

We now show that $w = g^\omega(0)$ can be written as $w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i$ with $U_i, V_i \in \Sigma_4^*$.

Lemma 3. *The word $w = g^\omega(0)$ is self-shuffling.*

Proof. We use the notation $x = v^{-1}u$ meaning that $u = vx$ for finite words x, u, v. We are going to show that the self-shuffle is given by the following:

$$
\begin{array}{llll}
U_0 = g^2(0), \quad U_1 = 0, & \ldots, & U_{6i+2} = g^i(0^{-1}g(0)0), & U_{6i+3} = g^i(0^{-1}g(3)0), \\
& & U_{6i+4} = g^i(0^{-1}g(201)0), & U_{6i+5} = g^i(30), \\
& & U_{6i+6} = g^i(2g(03)), & U_{6i+7} = g^{i+1}(20), \\
V_0 = g(0)03, \quad V_1 = 2g(2)0, & \ldots, & V_{6i+2} = g^i(0^{-1}g(1)0), & V_{6i+3} = g^i(0^{-1}g(03)0), \\
& & V_{6i+4} = g^i(1), & V_{6i+5} = g^i(3), \\
& & V_{6i+6} = g^{i+1}(0), & V_{6i+7} = g^{i+1}(0^{-1}g(2)0).
\end{array}
$$

Now we verify that

$$w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i,$$

from which it follows that w is self-shuffling. It suffices to show that each of the above products is fixed by g. Indeed, straightforward computations show that

$$\prod_{i=0}^{\infty} U_i = g^2(0)g^2(121)g^3(121)\cdots ,$$

which is fixed by g:

$$
\begin{aligned}
g\left(\prod_{i=0}^{\infty} U_i\right) &= g\left(g^2(0)g^2(121)g^3(121)\cdots\right) = g^3(0)g^3(121)g^4(121)\cdots \\
&= g^2(0121)g^3(121)g^4(121)\cdots = g^2(0)g^2(121)g^3(121)\cdots = \prod_{i=0}^{\infty} U_i,
\end{aligned}
$$

hence $\prod_{i=0}^{\infty} U_i$ is fixed by g and thus $w = \prod_{i=0}^{\infty} U_i$. In a similar way we show that $w = \prod_{i=0}^{\infty} V_i = \prod_{i=0}^{\infty} U_i V_i$. □

3 Square-free self-shuffling words on three letters

We remark that we can immediately produce a square-free self-shuffling word over Σ_3 from $g^\omega(0)$: Charlier et al. [2] noticed that the property of being self-shuffling is preserved by the application of a morphism. Furthermore, Brandenburg [1] showed that the morphism $f : \Sigma_4^* \to \Sigma_3^*$, defined by

$$
\begin{aligned}
f(0) &= 010201202101210212, \\
f(1) &= 010201202102010212, \\
f(2) &= 010201202120121012, \\
f(3) &= 010201210201021012,
\end{aligned}
$$

is square-free. Therefore, the word $f(g^\omega(0))$ is a ternary square-free self-shuffling word, from which we can produce a multitude of others by applying square-free morphisms from Σ_3^* to Σ_3^*.

4 A word with non self-shuffling shift orbit closure

In this section we provide a positive answer to the question from [2] whether there exists a word for which no element of its shift orbit closure is self-shuffling.

The *Hall word* $\mathcal{H} = 012021012102\cdots$ is defined as the fixed point of the morphism $h(0) = 012, h(1) = 02, h(2) = 1$. Sometimes it is referred to as a *ternary Thue-Morse* word. It is well known that this word is square-free. We show that no word in the shift orbit closure $S_{\mathcal{H}}$ of the Hall word is self-shuffling. More generally, we show that if x is a shuffle of y and z for $x, y, z \in S_{\mathcal{H}}$, then they are pairwise different.

Proposition 4. *There are no words x, y in the shift orbit closure of the Hall word such that $x \in \mathscr{S}(y, y)$.*

Proof. Suppose the converse, i.e., there exist words $x, y \in S_{\mathcal{H}}$ such that

$$x = \prod_{i=0}^{\infty} U_i V_i, \qquad y = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i.$$

Define the set X of infinite words as follows:

$$X = \{x \in S_{\mathcal{H}} \mid x \in \mathscr{S}(y, y) \text{ for some } y \in S_{\mathcal{H}}\}.$$

In other words, X consists of words in $S_{\mathcal{H}}$ which can be introduced as a shuffle of some word y in $S_{\mathcal{H}}$ with itself. Now suppose, for the sake of contradiction, that X is non empty, and consider $x \in X$ with the first block U_0 of the smallest possible positive length. We remark that such x and corresponding y are not necessarily unique.

We can suppose without loss of generality that y starts with 0 or 10. Otherwise, we exchange 0 and 2, consider the morphism $0 \mapsto 1, 1 \mapsto 20, 2 \mapsto 210$, and the argument is symmetric.

It is not hard to see from the properties of the morphism h that removing every occurrence of 1 from x and y results in $(02)^\omega$. Hence the blocks in the factorizations of y after removal of 1 are of the form $(02)^i$ for some integer i. Thus the first letter of each block U_i and V_i that is different from 1 is 0, and the last letter different from 1 is 2.

Then, U_i and V_i are images by the morphism h of factors of the fixed point of h. Therefore, there are words $x', y' \in S_{\mathcal{H}}$ such that $x = h(x'), y = h(y'), U_i = h(U_i'), V_i = h(V_i')$, and $x' = \prod_{i=0}^{\infty} U_i' V_i'$, $y' = \prod_{i=0}^{\infty} U_i' = \prod_{i=0}^{\infty} V_i'$.

Notice that the first block U_0 cannot be equal to 1. Indeed, otherwise x starts with 11, which is impossible, since 11 is not a factor of the fixed point of h.

Clearly, taking the preimage decreases the lengths of blocks in the factorization (except for those equal to 1), and since $U_0 \neq 1$, the length of the first block in the preimage is smaller, i.e., $|U_0'| < |U_0|$. This is a contradiction with the minimality of $|U_0|$. □

Corollary 5. *There are no self-shuffling words in the shift orbit closure of $\mathcal{H}$.*

With a similar argument we can prove the following:

Proposition 6. *There are no words x, y in the shift orbit closure of $\mathcal{H}$ such that $x \in \mathscr{S}(x, y)$.*

Proof. First we introduce a notation $x \in \mathscr{S}_2(y, z)$, meaning that there exists a shuffle starting with the word z (i.e., $U_0 = \varepsilon$, $V_0 \neq \varepsilon$). Next, $x \in \mathscr{S}(x, y)$ implies that there exists z in the same shift orbit closure such that $z \in \mathscr{S}_2(z, y)$. Indeed, one can remove the prefix U_0 of x to get z, i.e., $z = (U_0)^{-1}x$, and keep all the other blocks U_i, V_i in the shuffle product.

Define the set Z of infinite words as follows:

$$Z = \{z \in S_{\mathcal{H}} \mid z \in \mathscr{S}_2(z, y) \text{ for some } y \in S_{\mathcal{H}}\}.$$

In other words, Z consists of words in $S_{\mathcal{H}}$ which can be introduced as a shuffle of some word y in $S_{\mathcal{H}}$ with z starting with the block V_0. Now consider $z \in Z$ with the first block V_0 of the smallest possible length. We remark that such z and a corresponding y are not necessarily unique.

As in the proof of Proposition 4, the shuffle cannot start with a block of length 1. Again, if we remove every occurrence of 1 in y (and in z), we get $(02)^\omega$ or $(20)^\omega$; moreover, since V_0 contains letters different from 1, the first letter different from 1 is the same in y and z. So, without loss of generality we assume that both y and z without 1 are $(02)^\omega$, and the blocks U_i and V_i without 1 are integer powers of 02. Then, U_i and V_i are images by the morphism h of factors of $\mathcal{H}$. Therefore, there are words $z', y' \in S_{\mathcal{H}}$ such that $z = h(z'), y = h(y'), U_i = h(U_i'), V_i = h(V_i')$, and $z' = \prod_{i=0}^{\infty}(U_i'V_i') = \prod_{i=0}^{\infty} V_i'$, $y' = \prod_{i=0}^{\infty} U_i'$ (i.e., $z' \in Z$).

As in the proof of Proposition 4, since $V_0 \neq 1$, the length of the first block in the preimage is smaller, i.e., $|V_0'| < |V_0|$. This is again a contradiction with the minimality of $|V_0|$. ☐

So, we proved that if there are three words x, y, z in the shift orbit closure of the fixed point of h such that $x \in \mathscr{S}(y, z)$, then they should be pairwise distinct. Now we are going to prove that for any infinite word there exist three different words in its shift orbit closure such that $x \in \mathscr{S}(y, z)$.

An infinite word x is called *recurrent*, if each of its prefixes occurs infinitely many times in it.

Proposition 7. *Let x be a recurrent infinite word. Then there exist two words y, z in the shift orbit closure of x such that $x \in \mathscr{S}(y, z)$.*

Proof. We build the shuffle inductively.

Start from any prefix U_0 of x. Since x is recurrent, each of its prefixes occurs infinitely many times in it. Find another occurrence of U_0 in x and denote its position by i_1. Put $V_0 = x[|U_0|, i_1 + |U_0| - 1]$.

At step k, suppose that the shuffle of the prefix of x is built:

$$x[0,\Sigma_{l=0}^{k-1}(|U_l|+|V_l|)-1]=\prod_{l=0}^{k-1}U_lV_l,$$
$$y[0,\Sigma_{l=0}^{k-1}|U_l|-1]=\prod_{l=0}^{k-1}U_l,$$
$$z[0,\Sigma_{l=0}^{k-1}|V_l|-1]=\prod_{l=0}^{k-1}V_l,$$

such that $\prod_{l=0}^{k-1}U_l$ is the suffix of $x[0,\Sigma_{l=0}^{k-1}(|U_l|+|V_l|)-1]=\prod_{i=0}^{k-1}U_lV_l$ starting at position i_{k-1}, and $\prod_{l=0}^{k-1}V_l$ is the suffix of $x[0,\Sigma_{l=0}^{k-1}(|U_l|+|V_l|)-1]=\prod_{i=0}^{k-1}U_lV_l$ starting at position j_{k-1}.

Find another occurrence of $\prod_{l=0}^{k-1}V_l$ in x at some position $j_k>j_{k-1}$. We can do it since x is recurrent. Put $U_k=x[\Sigma_{l=0}^{k-1}(|U_l|+|V_l|),j_k-1+\Sigma_{l=0}^{k-1}|V_l|]$. We note that $\prod_{l=0}^{k}U_l$ is a factor of x by the construction; more precisely, it occurs at position i_{k-1}.

Find an occurrence of $\prod_{l=0}^{k}U_l$ at some position $i_k>i_{k-1}$, put $V_k=x[\Sigma_{l=0}^{k-1}(|U_l|+|V_l|)+|U_k|,i_k-1+\Sigma_{l=0}^{k}|U_l|]$. As above, $\prod_{l=0}^{k}V_l$ is a factor of x by the construction since it occurs at position j_{k-1}. Moreover, both $\prod_{l=0}^{k}U_l$ and $\prod_{l=0}^{k}V_l$ are suffixes of $x[0,\Sigma_{l=0}^{k}(|U_l|+|V_l|)-1]=\prod_{i=0}^{k}U_lV_l$.

Continuing this line of reasoning, we build the required factorization. □

Since each infinite word contains a recurrent (actually, even a uniformly recurrent) word in its shift orbit closure, we obtain the following corollary:

Corollary 8. *Each infinite word w contains words x,y,z in its shift orbit closure such that $x\in\mathscr{S}(y,z)$.*

The following example shows that the recurrence condition in Proposition 7 cannot be omitted:

Example 9. Consider the word $3\mathcal{H}=3012021\cdots$ which is obtained from $\mathcal{H}$ by adding a letter 3 in the beginning. Then the shift orbit closure of $3\mathcal{H}$ consists of the shift orbit closure of $\mathcal{H}$ and the word $3\mathcal{H}$ itself. Assuming $3\mathcal{H}$ is a shuffle of two words in its shift orbit closure, one of them is $3\mathcal{H}$ (there are no other 3's) and the other one is something in the shift orbit closure of $\mathcal{H}$, we let y denote this other word. Clearly, the shuffle starts with 3, and cutting the first letter 3, we get $\mathcal{H}\in\mathscr{S}(\mathcal{H},y)$, a contradiction with Proposition 6.

There also exist examples where each letter occurs infinitely many times:

Example 10. The following word:

$$x=012001120001112\cdots0^k1^k2\cdots$$

does not have two words y, z in its shift orbit closure such that $x \in \mathscr{S}(y,z)$. The idea of the proof is that the shift orbit closure consists of words of the following form: 1^*20^ω, 0^*1^ω, x itself and all their right shifts. Shuffling any two words of those types, it is not hard to see that there exists a prefix of the shuffle which contains too many or too few occurrences of some letter compare to the prefix of x. We leave the details of the proof to the reader.

By Corollary 8, there are x, y, z in the shift orbit closure of $\mathcal{H}$ such that $x \in \mathscr{S}(y,z)$. To conclude this section, we give an explicit construction of two words in the shift orbit closure of $\mathcal{H}$ which can be shuffled to give $\mathcal{H}$. We remark though that this construction gives a shuffle different from the one given by Corollary 8. Let:

$$h : \begin{cases} 0 & \mapsto & 012 \\ 1 & \mapsto & 02 \\ 2 & \mapsto & 1 \end{cases} \quad \text{and} \quad h' : \begin{cases} 0 & \mapsto & 210 \\ 1 & \mapsto & 20 \\ 2 & \mapsto & 1. \end{cases}$$

By definition, the shift orbit closure of the Hall word is closed under h. Moreover this shift orbit closure is also closed under h'. One of the ways to see this is the following. It is well known that the Thue-Morse word, which is a fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 10$ starting with 0, is a morphic image of $\mathcal{H}$ under a morphism $0 \mapsto 011$, $1 \mapsto 01$, $2 \mapsto 0$. Therefore, the set of factors of the Hall word is closed under reversal: if $v \in F(\mathcal{H})$, then $v^R \in F(\mathcal{H})$ (here for a word $v = v_1 \cdots v_n$ its reversal v^R is given by $v^R = v_n \cdots v_1$). Now by induction we prove that for each word v one has $h'(v) = (h(v^R))^R$ (it is enough to prove this equality for letters and for the concatenation of two words). This implies that the shift orbit closure of the Hall word is closed h'.

The morphisms h and h' satisfy:

$$h' \circ h : \begin{cases} 0 & \mapsto & 210201 \\ 1 & \mapsto & 2101 \\ 2 & \mapsto & 20 \end{cases} \quad h \circ h' : \begin{cases} 0 & \mapsto & 102012 \\ 1 & \mapsto & 1012 \\ 2 & \mapsto & 02 \end{cases}$$

$$h^2 : \begin{cases} 0 & \mapsto & 012021 \\ 1 & \mapsto & 0121 \\ 2 & \mapsto & 02 \end{cases} \quad h'^2 : \begin{cases} 0 & \mapsto & 120210 \\ 1 & \mapsto & 1210 \\ 2 & \mapsto & 20. \end{cases}$$

Note that if w is an infinite word, then $2(h \circ h')(w) = (h' \circ h)(w)$ and $0h'^2(w) = h^2(w)$.

Theorem 11. $h^\omega(0) \in \mathscr{S}(h^2((h'^2)^\omega(1)), h'^3(h^\omega(0)))$.

Proof. Let

$$U_0 = 01,\ U_1 = h'(0),\ U_2 = h'(1),\ V_0 = h'(1),$$

and for every $i \geqslant 0$,

$$U_{i+3} = h'^2(h^i(1)) \text{ and } V_{i+1} = h'^2(h^i(1)).$$

Let furthermore

$$u = \prod_{i=0}^{\infty} U_i,\ v = \prod_{i=0}^{\infty} V_i,\ \text{and } w = \prod_{i=0}^{\infty} U_i V_i\,.$$

We show that $w = h^\omega(0)$, $u = h^2((h'^2)^\omega(1))$ and $v = h'^3(h^\omega(0))$.

Note that $2h(h'(h^\omega(0))) = h'(h^\omega(0))$, thus $h'(h^\omega(0)) = \prod_{i=0}^{\infty} h^i(2)$. Then we have

$$v = 20\prod_{i=0}^{\infty} h'^2(h^i(1)) = h'^2\left(\prod_{i=0}^{\infty} h^i(2)\right) = h'^3(h^\omega(0)).$$

Moreover,

$$\begin{aligned} u &= 0121020\prod_{i=0}^{\infty} h'^2(h^i(1)) = 01210h'^2\left(\prod_{i=0}^{\infty} h^i(2)\right) \\ &= 01210h'^3\left(h^\omega(0)\right) = 01h'(0h'^2(h^\omega(0))) = 01h'(h^\omega(0)) = 0h'(2h^\omega(0)). \end{aligned}$$

Since $h'^2(2h^\omega(0)) = 20h'^2(h^\omega(0)) = 2h^\omega(0)$, the word $2h^\omega(0)$ is the fixed point $(h'^2)^\omega(2)$ of h'^2, and then $h'(2h^\omega(0))$ is the fixed point $(h'^2)^\omega(1)$. Thus $u = 0(h'^2)^\omega(1) = h^2((h'^2)^\omega(1))$. Finally:

$$w = 0120210121020\prod_{i=0}^{\infty} h'^2(h^i(021)) = 012021h(021)h^2\left(\prod_{i=0}^{\infty} h^i(021)\right) = 012\prod_{i=0}^{\infty} h^i(021).$$

Applying the morphism h to the second expression for w, we get

$$h(w) = 012021h\left(\prod_{i=0}^{\infty} h^i(021)\right) = 012\prod_{i=0}^{\infty} h^i(021).$$

Thus $w = h^\omega(0)$ since h is injective. □

5 Conclusions

We showed that infinite square-free self-shuffling words exist. The natural question that arises now is whether we can find infinite self-shuffling words subject to even stronger avoidability constraints: For this we recall the notion of *repetition threshold* $RT(k)$, which is defined as the least real number such that an infinite word over Σ_k exists, that does not contain repetitions of exponent greater than $RT(k)$. Due to the collective effort of many researchers (see [3, 8] and references therein), the repetition threshold for all alphabet sizes is known and characterized as follows:

$$RT(k) = \begin{cases} \frac{7}{4} & \text{if } k = 3 \\ \frac{7}{5} & \text{if } k = 4 \\ \frac{k}{k-1} & \text{else.} \end{cases}$$

A word $w \in \Sigma_k^\omega$ without factors of exponent greater than $RT(k)$ is called a *Dejean word*. Charlier et al. showed that the Thue-Morse word, which is a binary Dejean word, is self-shuffling [2].

Question 12. Do there exist self-shuffling Dejean words over non-binary alphabets?

References

[1] F.-J. Brandenburg. Uniformly Growing kth Power-Free Homomorphisms. *Theor. Comput. Sci.* 23: 69–82, 1983.

[2] E. Charlier, T. Kamae, S. Puzynina, L. Q. Zamboni. Infinite self-shuffling words. *J. Comb. Theory, Ser. A,* 128: 1–40, 2014.

[3] J. Currie, N. Rampersad. A proof of Dejean's conjecture. *Math. Comput.* 80(274): 1063–1070, 2011.

[4] J. Currie, K. Saari. Square-free Words with Square-free Self-shuffles. *Electr. J. Comb.,* 21(1): #P1.9 (2014).

[5] T. Harju. A Note on Square-Free Shuffles of Words. *LNCS* 8079 (WORDS 2013): 154–160.

[6] T. Harju, M. Müller. Square-Free Shuffles of Words. CoRR abs/1309.2137 (2013)

[7] M. Lothaire. *Algebraic combinatorics on words.* Cambridge University Press, 2002.

[8] M. Rao. Last cases of Dejean's conjecture. *Theor. Comput. Sci.* 412(27): 3010–3018, 2011.