Summation of rational series twisted by strongly B-multiplicative coefficients

Jean-Paul Allouche*

CNRS, Institut de Mathématiques de Jussieu-PRG
Université Pierre et Marie Curie, Case 247
F-75252 Paris Cedex 05, France

jean-paul.allouche@imj-prg.fr

Jonathan Sondow

209 West 97th Street
New York
NY 10025, U.S.A

jsondow@alumni.princeton.edu

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Abstract

We evaluate in closed form series of the type $\sum u(n)R(n)$, with $(u(n))_n$ a strongly B-multiplicative sequence and $R(n)$ a (well-chosen) rational function. A typical example is:

$$\sum_{n\geqslant 1}(-1)^{s_2(n)}\frac{4n+1}{2n(2n+1)(2n+2)} = -\frac{1}{4}$$

where $s_2(n)$ is the sum of the binary digits of the integer n. Furthermore closed formulas for series involving automatic sequences that are not strongly B-multiplicative, such as the regular paperfolding and Golay-Shapiro-Rudin sequences, are obtained; for example, for integer $d \geqslant 0$:

$$\sum_{n\geqslant 0}\frac{v(n)}{(n+1)^{2d+1}} = \frac{\pi^{2d+1}|E_{2d}|}{(2^{2d+2}-2)(2d)!}$$

where $(v(n))_n$ is the ± 1 regular paperfolding sequence and E_{2d} is an Euler number.

Keywords: summation of series; strongly B-multiplicative sequences; paperfolding sequence; Golay-Shapiro-Rudin sequence

1 Introduction

The problem of evaluating a series $\sum_n R(n)$ where R is a rational function with integer coefficients is classical: think of the values of the Riemann ζ function at integers. Such

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sums can also be "twisted", usually by a character (think of the L-functions), or by the usual arithmetic functions (e.g., the Möbius function μ).

Another possibility is to twist such sums by sequences related to the digits of n in some integer base. Examples can be found in [5] with, in particular, series $\sum \frac{u(n)}{n(n+1)}$, and in [7] with, in particular, series $\sum \frac{u(n)}{2n(2n+1)}$ (also see [9]): in both cases $u(n)$ counts the number of occurrences of a given block of digits in the B-ary expansion of the integer n, or is equal to $s_B(n)$, the sum of the B-ary digits of the integer n (B being an integer $\geqslant 2$). Two emblematic examples are (see [10, Problem B5, p. 682] and [12, 5] for the first one, and [14, 7] for the second one):

$$\sum_{n\geqslant 1} \frac{s_B(n)}{n(n+1)} = \frac{B}{B-1} \quad \text{and} \quad \sum_{n\geqslant 1} \frac{s_2(n)}{2n(2n+1)} = \frac{\gamma + \log \frac{4}{\pi}}{2}$$

where γ is the Euler-Mascheroni constant.

Similarly one can try to evaluate infinite products $\prod_n R(n)$, where $R(n)$ is a rational function, as well as twisted such products $\prod_n R(n)^{u(n)}$, where the sequence $(u(n))_{n\geqslant 0}$ is related to the digits of n in some integer base. An example can be found in [2] (also see [11] for the original problem):

$$\prod_{n\geqslant 1} \left(\frac{(4n+2)(4n+2)}{(4n+1)(4n+3)} \right)^{2z(n)} = \frac{4}{\pi}$$

where $z(n)$ is the sum of the number of 0's and the number of 1's in the binary expansion of n, i.e., the length of this expansion. Other examples can be found in [4], e.g.,

$$\prod_{n\geqslant 0} \left(\frac{(4n+2)(8n+7)(8n+3)(16n+10)}{(4n+3)(8n+6)(8n+2)(16n+11)} \right)^{u(n)} = \frac{1}{\sqrt{2}}$$

where $u(n) = (-1)^{a(n)}$ and $a(n)$ is equal to the number of blocks 1010 occurring in the binary expansion of n. The products studied in [4] (also see references therein) are of the form $\prod_n R(n)^{(-1)^{a(n)}}$ where $R(n)$ is a (well-chosen) rational function with integer coefficients, and $a(n)$ counts the number of occurrences of a given block of digits in the B-ary expansion of the integer n. The case where $a(n)$ counts the number of 1's occurring in the binary expansion of n is nothing but the case $a(n) = s_2(n)$. If $a(n) = s_B(n)$, the sequence $((-1)^{a(n)})_{n\geqslant 0}$ is strongly B-multiplicative: the more general evaluation of the product $\prod_n R(n)^{u(n)}$ where $(u(n))_{n\geqslant 0}$ is a strongly B-multiplicative sequence, is addressed in [8] (also see [13]). Recall that a *strongly B-multiplicative sequence* $(u(n))_{n\geqslant 0}$ satisfies $u(0) = 0$, and $u(Bn + j) = u(n)u(j)$ for all $j \in [0, B-1]$ and all $n \geqslant 0$. In particular, $(u(n))_{n\geqslant 0}$ is B-regular (or even B-automatic if it takes only finitely many values): recall that a sequence $(u(n))_{n\geqslant 0}$ is called *B-automatic* if its *B-kernel*, i.e., the set of subsequences $\{(u(B^a n + r))_{n\geqslant 0} \mid \ a \geqslant 0,\ 0 \leqslant r \leqslant B^a - 1\}$, is finite; a sequence $(u(n))_{n\geqslant 0}$ with values in $\mathbb{Z}$ is called *B-regular* if the $\mathbb{Z}$-module spanned by its B-kernel has finite type (for more on these notions, see, e.g., [6]).

Since $\log \prod_n R(n)^{u(n)} = \sum_n u(n) \log R(n)$, it is natural to look at "simpler" series of the form $\sum_n u(n)R(n)$ with R and u as previously. All the examples above involve sequences $(u(n))_{n \geqslant 0}$ that are B-regular or even B-automatic. Unfortunately we were not able to address the general case where $(u(n))_{n \geqslant 0}$ is any B-regular or any B-automatic sequence. The purpose of the present paper is to study the special case where, as in [8], the sequence $u(n)$ is strongly B-multiplicative and $R(n)$ is a well-chosen rational function. The paper can thus be seen as a companion paper to [8]. We will end with the evaluation of similar series where $(u(n))_{n \geqslant 0}$ is the regular paperfolding sequence or the Golay-Shapiro-Rudin sequence.

2 Preliminary definitions and results

This section quickly recalls definitions and results from [8].

Definition 1. Let $B \geqslant 2$ be an integer. A sequence of complex numbers $(u(n))_{n \geqslant 0}$ is *strongly B-multiplicative* if $u(0) = 1$ and, for all $n \geqslant 0$ and all $k \in \{0, 1, \ldots, B-1\}$,

$$u(Bn + k) = u(n)u(k).$$

Example 2. Let $B \geqslant 2$ be an integer and $s_B(n)$ be the sum of the B-ary digits of n. Then for every complex number $a \neq 0$ the sequence $(a^{s_B(n)})_{n \geqslant 0}$ is strongly B-multiplicative. This sequence is B-regular (see the introduction); it is B-automatic if and only if a is a root of unity.

The following lemma is a variation of Lemma 1 in [8].

Lemma 3. *Let $B > 1$ be an integer. Let $(u(n))_{n \geqslant 0}$ be a strongly B-multiplicative sequence of complex numbers different from the sequence $(1, 0, 0, \ldots)$. We suppose that $|u(n)| \leqslant 1$ for all $n \geqslant 0$ and that $|\sum_{0 \leqslant k < B} u(k)| < B$. Let f be a map from the set of nonnegative integers to the set of complex numbers such that $|f(n+1) - f(n)| = \mathcal{O}(n^{-2})$. Then the series $\sum_{n \geqslant 0} u(n) f(n)$ is convergent.*

Proof. Use [8, Lemma 1] to get the upper bound $|\sum_{0 \leqslant n < N} u(n)| < CN^{\alpha}$ for some positive constant C and some real number α in $(0, 1)$. Then use summation by parts. □

3 Main results

We state in this section some basic identities as well as first applications and examples. First we define δ_k, a special case of the Kronecker delta:

$$\delta_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4. *Let $B > 1$ be an integer. Let $(u(n))_{n \geqslant 0}$ be a strongly B-multiplicative sequence, and let f be a map from the nonnegative integers to the complex numbers, such that $(u(n))_{n \geqslant 0}$ and f satisfy the conditions of Lemma 3. Define the series $S_1(k, B, u, f)$, for $k = 0, 1, \ldots, B-1$, by*

$$S(k, B, u, f) := \sum_{n \geqslant 0} u(n) f(Bn + k).$$

Then the following linear relations hold:

$$\sum_{n \geqslant 0} u(n) f(n) = \sum_{0 \leqslant k \leqslant B-1} u(k) S(k, B, u, f)$$

and

$$\sum_{n \geqslant 0} u(n) \sum_{0 \leqslant k \leqslant B-1} f(Bn + k) = \sum_{0 \leqslant k \leqslant B-1} S(k, B, u, f).$$

In particular, define the series $S_1(k, B, u)$ and $S_2(k, B, u)$, for $k = 0, 1, \ldots, B-1$, by

$$S_1(k, B, u) := \sum_{n \geqslant \delta_k} \frac{u(n)}{Bn + k} \text{ and } S_2(k, B, u) := \sum_{n \geqslant \delta_k} \frac{u(n)}{(Bn + k)(Bn + k + 1)}.$$

Then the following linear relations hold:

$$(B-1) S_1(0, B, u) - \sum_{1 \leqslant k \leqslant B-1} u(k) S_1(k, B, u) = 0$$

and

$$\sum_{0 \leqslant k \leqslant B-1} (B - u(k)) S_2(k, B, u) = B - 1.$$

Proof. It follows from Lemma 3 that all the series in the theorem converge. To prove the first relation, we split $\sum_{n \geqslant 0} u(n) f(n)$, obtaining

$$\begin{aligned} \sum_{n \geqslant 0} u(n) f(n) &= \sum_{0 \leqslant k \leqslant B-1} \sum_{n \geqslant 0} u(Bn + k) f(Bn + k) = \sum_{0 \leqslant k \leqslant B-1} \sum_{n \geqslant 0} u(n) u(k) f(Bn + k) \\ &= \sum_{0 \leqslant k \leqslant B-1} u(k) \sum_{n \geqslant 0} u(n) f(Bn + k) = \sum_{0 \leqslant k \leqslant B-1} u(k) S(k, B, u, f). \end{aligned}$$

To prove the second relation, we write

$$\sum_{n \geqslant 0} u(n) \sum_{0 \leqslant k \leqslant B-1} f(Bn + k) = \sum_{0 \leqslant k \leqslant B-1} \sum_{n \geqslant 0} u(n) f(Bn + k) = \sum_{0 \leqslant k \leqslant B-1} S(k, B, u, f).$$

To prove the last part of the theorem, we make two choices for f. First we take f defined by $f(n) = 1/n$ for $n \neq 0$ and $f(0) = 0$. Then we take $f(n) = 1/n(n+1)$ if $n \neq 0$ and $f(0) = 0$. $\square$

Remark The formula $S_2(k,B,u) = S_1(k,B,u) - (S_1(k+1,B,u) - \delta_k)$ $(0 \leqslant k \leqslant B-2)$ holds. Nevertheless, the last two relations in Theorem 4 are independent, because $S_2(B-1,B,u)$ cannot be expressed in terms of the $S_1(k,B,u)$ for $k = 0,1,\ldots,B-1$.

Corollary 5. *If $(u(n))_{n\geqslant 0}$ is a strongly B-multiplicative sequence satisfying the conditions of Lemma 3, then*

$$\sum_{n\geqslant 1} u(n) \sum_{1\leqslant k\leqslant B-1} \left(\frac{1}{Bn} - \frac{u(k)}{Bn+k}\right) = \sum_{1\leqslant k\leqslant B-1} \frac{u(k)}{k}$$

and

$$\sum_{n\geqslant 1} u(n) \sum_{0\leqslant k\leqslant B-1} \frac{B-u(k)}{(Bn+k)(Bn+k+1)} = \sum_{1\leqslant k\leqslant B-1} \frac{u(k)}{k(k+1)}\cdot$$

Proof. This follows from the last part of Theorem 4 by substitution and manipulation. ☐

Recall that the nth harmonic number H_n and the nth alternating harmonic number H_n^* are defined by

$$H_n := \sum_{1\leqslant k\leqslant n} \frac{1}{k} \text{ and } H_n^* := \sum_{1\leqslant k\leqslant n} \frac{(-1)^{k-1}}{k}\cdot$$

Corollary 6. *If $N_{j,B}(n)$ is the number of occurrences of the digit $j \in \{0,1,\ldots,B-1\}$ in the B-ary expansion of n, then the following summations hold when $j \neq 0$:*

$$\sum_{n\geqslant 1} (-1)^{N_{j,B}(n)} \left(\frac{2}{Bn+j} + \frac{1}{Bn} \sum_{1\leqslant k\leqslant B-1} \frac{k}{Bn+k}\right) = H_{B-1} - \frac{2}{j}$$

and

$$\sum_{n\geqslant 1} (-1)^{N_{j,B}(n)} \left(\frac{B-1}{n(n+1)} + \frac{2B}{(Bn+j)(Bn+j+1)}\right) = B - 1 - \frac{2B}{j(j+1)}\cdot$$

Proof. It is not hard to see that, if $j \neq 0$, we can apply the last part of Theorem 4 to the sequence $u(n) := (-1)^{N_{j,B}(n)}$. Using Corollary 5 and the fact that $N_{j,B}(k) = \delta_{k,j}$ when $0 \leqslant k < B$, the result follows. ☐

Example 7. Taking $B = 2$ and $j = 1$, we get

$$\sum_{n\geqslant 1} (-1)^{N_{1,2}(n)} \frac{4n+1}{2n(2n+1)} = -1$$

and

$$\sum_{n\geqslant 1} (-1)^{N_{1,2}(n)} \frac{4n+1}{2n(2n+1)(2n+2)} = -\frac{1}{4}\cdot$$

Subtracting the second equation from the first, we multiply by 4 and obtain

$$\sum_{n\geqslant 1}(-1)^{N_{1,2}(n)}\frac{4n+1}{n(n+1)} = -3.$$

With $B = 3$ and $j = 1$ we get

$$\sum_{n\geqslant 1}(-1)^{N_{1,3}(n)}\frac{18n^2+21n+4}{3n(3n+1)(3n+2)} = -\frac{1}{2}$$

and

$$\sum_{n\geqslant 1}(-1)^{N_{1,3}(n)}\frac{6n^2+6n+1}{3n(3n+1)(3n+2)(3n+3)} = -\frac{1}{36}.$$

Corollary 8. *If $s_B(n)$ is the sum of the B-ary digits of n, then*

$$\sum_{n\geqslant 1}(-1)^{s_B(n)}\sum_{1\leqslant k\leqslant B-1}\left(\frac{1}{Bn}-\frac{(-1)^k}{Bn+k}\right) = -H^*_{B-1}$$

and

$$\sum_{n\geqslant 1}(-1)^{s_B(n)}\sum_{0\leqslant k\leqslant B-1}\frac{B-(-1)^k}{(Bn+k)(Bn+k+1)} = 1+\frac{(-1)^B}{B}-2H^*_{B-1}.$$

Proof. Setting $u(n) := (-1)^{s_B(n)}$, it is not hard to see that $u(2n+1) = -u(2n)$ for all $n \geqslant 0$. (Hint: look at the cases B even and B odd separately.) It follows that $(u(n))_{n\geqslant 0}$ satisfies the conditions of Lemma 3. Noting that $u(k) = (-1)^k$ when $0 \leqslant k < B$, the result follows from Corollary 5. □

Example 9. Taking $B = 2$ or 3 gives the same pair of series as those with that value of B in Example 1, since $s_2(n) = N_{1,2}(n)$ and $s_3(n) = N_{1,3}(n) + 2N_{2,3}(n)$. (We can also replace $s_3(n)$ with n, as $(-1)^{s_B(n)} = (-1)^n$ when B is odd.) With $B = 4$ we get

$$\sum_{n\geqslant 1}(-1)^{s_4(n)}\frac{128n^3+176n^2+76n+9}{4n(4n+1)(4n+2)(4n+3)} = -\frac{5}{12}$$

and

$$\sum_{n\geqslant 1}(-1)^{s_4(n)}\frac{128n^3+184n^2+80n+9}{4n(4n+1)(4n+2)(4n+3)(4n+4)} = -\frac{5}{12}.$$

4 More examples

Using Corollary 5 with sequences $(u(n))_{n\geqslant 0}$ taking complex values yields other examples of sums of series.

Example 10. We may let $u(n) := i^{s_2(n)}$ in Corollary 5. This gives the two summations

$$\sum_{n\geqslant 1}\left(\frac{i^{s_2(n)}}{2n}-\frac{i^{s_2(n)+1}}{2n+1}\right)=i=\sum_{n\geqslant 1}\frac{i^{s_2(n)}(3n+1)-i^{s_2(n)+1}n}{n(n+1)(2n+1)},$$

and by taking the imaginary and real parts we obtain the following result:

If χ is the non-principal Dirichlet character modulo 4, defined by

$$\chi(n) := \begin{cases} +1 & \text{if } n\equiv 1 \bmod 4,\\ -1 & \text{if } n\equiv 3 \bmod 4,\\ 0 & \text{otherwise,}\end{cases}$$

then

$$\sum_{n\geqslant 1}\left(\frac{\chi(s_2(n))}{2n}-\frac{\chi(s_2(n)+1)}{2n+1}\right)=1=\sum_{n\geqslant 1}\frac{(3n+1)\chi(s_2(n))-n\chi(s_2(n)+1)}{n(n+1)(2n+1)}$$

and

$$\sum_{n\geqslant 1}\left(\frac{\chi(s_2(n)+1)}{2n}-\frac{\chi(s_2(n)+2)}{2n+1}\right)=0=\sum_{n\geqslant 1}\frac{(3n+1)\chi(s_2(n)+1)-n\chi(s_2(n)+2)}{n(n+1)(2n+1)}.$$

Example 11. Generalizing Example 10 by replacing $i^{s_2(n)}$ with $e^{2i\pi s_2(n)/d}$, for integer $d\geqslant 2$, is straightforward, yielding the following summations (Example 10 is another formulation for the case $d=4$):

$$\sum_{n\geqslant 1}\left(\frac{\sin\frac{2\pi s_2(n)}{d}}{2n}-\frac{\sin\frac{2\pi(s_2(n)+1)}{d}}{2n+1}\right)=\sin\frac{2\pi}{d}=\sum_{n\geqslant 1}\frac{(3n+1)\sin\frac{2\pi s_2(n)}{d}-n\sin\frac{2\pi(s_2(n)+1)}{d}}{n(n+1)(2n+1)}$$

and

$$\sum_{n\geqslant 1}\left(\frac{\cos\frac{2\pi s_2(n)}{d}}{2n}-\frac{\cos\frac{2\pi(s_2(n)+1)}{d}}{2n+1}\right)=\cos\frac{2\pi}{d}=\sum_{n\geqslant 1}\frac{(3n+1)\cos\frac{2\pi s_2(n)}{d}-n\cos\frac{2\pi(s_2(n)+1)}{d}}{n(n+1)(2n+1)}.$$

5 The paperfolding and Golay-Shapiro-Rudin sequences

The results above involve sums $\sum u(n)R(n)$ where $(u(n))_{n\geqslant 0}$ is a strongly B-multiplicative sequence, which, in all of our examples except Example 2 with alpha not a root of unity, happens to take only finitely many values. This implies that $(u(n))_{n\geqslant 0}$ is B-*automatic* (see the introduction). One can then ask about more general sums $\sum u(n)R(n)$ where the sequence $(u(n))_{n\geqslant 0}$ is B-automatic. We give two cases where such series can be summed.

Theorem 12. *Let* $(v(n))_{n\geqslant 0}$ *be the* regular paperfolding sequence. *Its first few terms are given by (replacing* $+1$ *by* $+$ *and* -1 *by* $-$*)*

$$(v(n))_{n\geqslant 0} = +\ +\ -\ +\ +\ -\ -\ \ldots;$$

it can be defined by: $v(2n) = (-1)^n$ *and* $v(2n+1) = v(n)$ *for all* $n \geqslant 0$*. Then, for all integers* $d \geqslant 0$*, we have the relation*

$$\sum_{n\geqslant 0} \frac{v(n)}{(n+1)^{2d+1}} = \frac{\pi^{2d+1}|E_{2d}|}{(2^{2d+2}-2)(2d)!}$$

where the E_{2d}*'s are the Euler numbers defined by:*

$$\frac{1}{\cosh t} = \sum_{n\geqslant 0} \frac{E_{2n}}{(2n)!} t^{2n} \text{ for } |t| < \frac{\pi}{2}\cdot$$

Proof. First note that the series $\sum_{n\geqslant 0} \frac{v(n)}{(n+1)^s}$ converges for $\Re(s) > 0$: use the inequality $|\sum_{n<N} v(n)| = O(\log N)$ (see, e.g., [6, Exercise 28, p. 206]) and summation by parts; note that the sequence $(R_n)_{n\geqslant 1}$ in [6, Exercise 28, p. 206] is equal to the sequence $(v(n))_{n\geqslant 0}$ here. Now, Exercise 27 in [6, p. 205–206] asks to prove, for all complex numbers s with $\Re(s) > 0$, the equality (again with slightly different notation)

$$\sum_{n\geqslant 0} \frac{v(n)}{(n+1)^s} = \frac{2^s}{2^s-1} \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)^s}\cdot$$

This can be easily done by splitting the sum on the left into even and odd indexes. Recalling that the Dirichlet beta function is defined by $\beta(s) = \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)^s}$ for $\Re(s) > 0$, we thus have, for any nonnegative integer d,

$$\sum_{n\geqslant 0} \frac{v(n)}{(n+1)^{2d+1}} = \frac{2^{2d+1}}{2^{2d+1}-1}\beta(2d+1).$$

But, when s is an odd integer, the value of $\beta(s)$ can be expressed as a rational multiple of π (see, e.g., [1, 23.2.22, p. 807]):

$$\beta(2d+1) = \frac{(\pi/2)^{2d+1}}{2(2d)!}|E_{2d}|. \qquad \square$$

Example 13. Taking $d = 0$ in Theorem 12 yields a result due to F. von Haeseler (see [6, Exercise 27, p. 205–206])

$$\sum_{n\geqslant 0} \frac{v(n)}{n+1} = \frac{\pi}{2}\cdot$$

The second result we give in this section involves the Golay-Shapiro-Rudin sequence.

Theorem 14. *Let* $(r(n))_{n\geqslant 0}$ *be the* ± 1 Golay-Shapiro-Rudin sequence. *This sequence can be defined by* $r(n) = (-1)^{a(n)}$*, where* $a(n)$ *is the number of possibly overlapping occurrences of the block* 11 *in the binary expansion of* n*, so that (replacing* $+1$ *by* $+$ *and* -1 *by* -1*)*

$$(r(n))_{n\geqslant 0} = + \ + \ + \ - \ + \ + \ - \ + \ \ldots;$$

alternatively it can be defined by

$$r(0) = 1, \text{ and } r(2n) = r(n), \ r(2n+1) = (-1)^n r(n) \text{ for } n \geqslant 0.$$

Let $R(n)$ *be a function from the nonnegative integers to the complex numbers, such that* $|R(n+1) - R(n)| = \mathcal{O}(n^{-2})$*. Then we have the relation*

$$\sum_{n\geqslant 1} r(n)(R(n) - R(2n) + R(2n+1) - 2R(4n+1)) = R(1).$$

Proof. It is well known that $|\sum_{n<N} r(n)| < K\sqrt{n}$ for some positive constant K (actually more is known; see, e.g., [6, Theorem 3.3.2, p. 79] and the historical comments given in [6, 3.3, p. 121]). Thus, by summation by parts, the series $\sum_{n\geqslant 0} r(n)R(n)$ is convergent. Now we write

$$\begin{aligned}
\sum_{n\geqslant 0} r(n)R(n) &= \sum_{n\geqslant 0} r(2n)R(2n) + \sum_{n\geqslant 0} r(2n+1)R(2n+1) \\
&= \sum_{n\geqslant 0} r(n)R(2n) + \sum_{n\geqslant 0} (-1)^n r(n)R(2n+1) \\
&= \sum_{n\geqslant 0} r(n)R(2n) + \sum_{n\geqslant 0} r(2n)R(4n+1) - \sum_{n\geqslant 0} r(2n+1)R(4n+3) \\
&= \sum_{n\geqslant 0} r(n)(R(2n) + R(4n+1)) - \sum_{n\geqslant 0} r(2n+1)R(4n+3).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n\geqslant 0} r(n)(R(n) - R(2n) - R(4n+1)) &= -\sum_{n\geqslant 0} r(2n+1)R(4n+3) \\
&= -(\sum_{n\geqslant 0} r(n)R(2n+1) - \sum_{n\geqslant 0} r(2n)R(4n+1)) \\
&= -\sum_{n\geqslant 0} r(n)R(2n+1) + \sum_{n\geqslant 0} r(n)R(4n+1)
\end{aligned}$$

where the penultimate equality is obtained by splitting the sum $\sum_{n\geqslant 0} r(n)R(2n+1)$ into even and odd indices. Thus, finally

$$\sum_{n\geqslant 0} r(n)(R(n) - R(2n) + R(2n+1) - 2R(4n+1)) = 0,$$

hence

$$\sum_{n\geqslant 1} r(n)(R(n) - R(2n) + R(2n+1) - 2R(4n+1)) = R(1). \qquad \square$$

Example 15. Taking $R(n) = 1/n$ if $n \neq 0$ and $R(0) = 1$ in Theorem 14 above yields

$$\sum_{n \geqslant 1} r(n) \frac{8n^2 + 4n + 1}{2n(2n+1)(4n+1)} = 1.$$

Example 16. Taking R defined by $R(n) = \log n - \log(n+1)$ for $n \neq 0$ and $R(0) = 0$ in Theorem 14 above yields

$$\sum_{n \geqslant 1} r(n) \log \frac{(2n+1)^4}{(n+1)^2(4n+1)^2} = -\log 2.$$

Hence

$$\sum_{n \geqslant 0} r(n) \log \frac{(2n+1)^2}{(n+1)(4n+1)} = -\frac{1}{2} \log 2.$$

After exponentiating we obtain:

$$\prod_{n \geqslant 0} \left(\frac{(2n+1)^2}{(n+1)(4n+1)} \right)^{r(n)} = \frac{1}{\sqrt{2}}$$

thus recovering the value of an infinite product obtained in [3, Theorem 2, p. 148] (also see [4]).

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